Maxwell's Equations in Terms of Differential Forms

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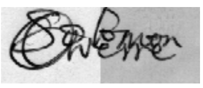
Abstract

Maxwell's equations, which depict classical electromagnetic theory, are pulled apart and brought together into a modern language of differential geometry. A background of vector fields and differential forms on a manifold is introduced, as well as the Hodge star operator, which eventually lead to the success of rewriting Maxwell's equations in terms of differential forms. In order to appreciate the beauty of differential forms, we first review these equations in covariant form which are shown afterwards to be consistent with the differential forms when expressed explicitly in terms of components.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



Solomon Akaraka Owerre, 20 May 2010

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1. Introduction

1.1 Introduction to Maxwell's Equations

The revolutionary work of Maxwell, published in 1865 took the individual and seemingly unconnected phenomena of electricity and magnetism and brought them into a coherent and unified theory. This unified theory of electricity and magnetism depicts the behaviour of two fields, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$, which depend on the electric charge density ρ and also on the electric current density $\mathbf{J}$.

Maxwell discussed his ideas in terms of a model in which vacuum was like an elastic solid. He tried to explain the meaning of his new equation in terms of the mathematical model. There was much reluctance to accept his theory, first because of the model, and second because there was at first no experimental justification. Today, we understand better that what counts are the equations themselves and not the model used to get them [FRS64].

Maxwell's equations have been generalized to other areas of physical interest. Our picture of the standard model consists of three forces: electromagnetism and the weak and strong nuclear forces are all gauge fields (invariant under gauge transformations), which means that they are described by equations closely modelled after Maxwell's equations. These equations have been written in different forms since their discovery. One of the advantages of rewriting Maxwell's equations is that it allows for greater generalization to other areas of physical interest.

The language of differential geometry has been an indispensable part of the study of theoretical physics. The first theory of physics to explicitly use differential geometry was Einstein's General Relativity, in which gravity is explained as the curvature of spacetime. The gauge theories of the standard model are of a very similar geometrical character (although quantized). But there is also a lot of differential geometry lurking in Maxwell's equations, which, after all, were the inspiration for both general relativity and gauge theory [BP94].

With the fact that the laws of physics must have the same form in all inertial frames, in our new mathematical language of differential geometry we need to work in a way that is independent of the choice of coordinates. This chapter will be devoted to the covariant form of Maxwell's equations.

1.2 Minkowski Spacetime

The geometrical structure of spacetime has a lot of topological structures embedded in it. The topological property of a space are those that are left unchanged under arbitrary smooth deformations of the space; topology tells us which points are the same, which are distinct, and which are in what neighbourhood. From experience, an event can be described by stating when (time) and where (space) it happened. We will suppose that a topology of spacetime is that Euclidean four-dimensional space ($\mathbb{R}^4$). This assumption means that we can describe points in spacetime by using four-dimensional coordinates, each point corresponding uniquely to the set of numbers (t, x_1, x_2, x_3). It is quite possible that the topology of spacetime is not Euclidean [Han76].

Relativistic spacetime contains only a single geometry which combines both space and time. As Minkowski expressed it: "Hence forth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality"[Han76].

The spacetime coordinates of a point x are denoted by a contravariant vector with four components x^α:

$$x^\alpha = (x^0, x^1, x^2, x^3) = (t, \mathbf{x}). \tag{1.1}$$

Throughout this chapter and subsequent chapters, we shall us the "Heaviside Lorentz " units where $\epsilon_0 = \mu_0 = c = 1$ and the Einstein's summation convention, according to which any repeated index appearing in a term of an equation is to be summed over. The covariant vector with four components x_β is given by

$$x_\beta = \eta_{\alpha\beta} x^\alpha, \tag{1.2}$$

where $\eta_{\alpha\beta}$ is a matrix with components

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.3}$$

which is called the *flat metric tensor* or the *flat Minkowski metric tensor*, $\alpha, \beta = 0, 1, 2, 3$. The metric tensor has the property that when acting on a contravariant index, $\eta_{\alpha\beta}$ converts the index to a covariant index and vice versa. So if we invert relation (1.2), we obtain:

$$x^\alpha = \eta^{\alpha\beta} x_\beta, \tag{1.4}$$

where $\eta^{\alpha\beta}$ is the matrix inverse to $\eta_{\alpha\beta}$.

Using relation (1.2) and (1.4) one can easily show that

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.5}$$

Thus,

$$\eta^{\alpha\beta} \eta_{\alpha\gamma} = \delta^\beta_\gamma, \tag{1.6}$$

where

$$\delta^\beta_\gamma = \begin{cases} 1 & \text{if } \gamma = \beta, \\ 0 & \text{if } \gamma \neq \beta. \end{cases} \tag{1.7}$$

Therefore, the position vector can be written either in the contravariant or in the covariant form.

The relativistically invariant spacetime distance element

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - dt^2, \tag{1.8}$$

can be written as

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = dx^\beta dx_\beta. \tag{1.9}$$

We can also define a covariant derivative in spacetime ∂_α as:

$$\frac{\partial}{\partial x^\alpha} := \partial_\alpha = (\partial_0, \partial_1, \partial_2, \partial_3) = \eta_{\alpha\beta}\partial^\beta, \tag{1.10}$$

which can be inverted by the use of (1.6) as

$$\frac{\partial}{\partial x_\beta} := \partial^\beta = \eta^{\beta\alpha}\partial_\alpha. \tag{1.11}$$

Generally, for any arbitrary contravariant vector C^α, we define a covariant component C_β by

$$C_\beta = \eta_{\beta\alpha}C^\alpha, \tag{1.12}$$

which gives

$$C^\alpha = \eta^{\alpha\beta}C_\beta, \tag{1.13}$$

on inversion.

1.3 Covariant Form of Maxwell's Equations

Maxwell's equations can be cast into covariant form. As Einstein expressed it: "The general laws of nature are to be expressed by equations which holds good for all systems of coordinates, that is are covariant with respect to any substitution whatever (generally covariant)" [BP94].

Maxwell's theory of electromagnetism is, alongside with Einstein's theory of gravitation, one of the most beautiful of classical field theories. Having chosen units in which $\mu_0 = \epsilon_0 = c = 1$, Maxwell's equations then take the form:

$$\nabla \cdot \mathbf{E} = \rho \tag{1.14}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \tag{1.15}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.16}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1.17}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, ρ and $\mathbf{J}$ are the charge and current densities.

Taking the divergence of equation (1.14) and substituting equation (1.15) into the resulting equation, we obtain the continuity equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{1.18}$$

Note that we have used the fact that for any vector $\mathbf{H}$ and scalar Ψ, the following identities hold:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 \tag{1.19}$$

$$\nabla \times (\nabla \Psi) = 0. \tag{1.20}$$

Also, since equation (1.16) always holds, this means that $\mathbf{B}$ must be a curl of a vector function, namely the vector potential $\mathbf{A}$:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{1.21}$$

Substituting equation (1.21) into equation (1.17), we obtain

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0, \tag{1.22}$$

which means that the quantity with vanishing curl in equation (1.22) can be written as the gradient of a scalar function, namely the scalar potential Φ:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{1.23}$$

The minus sign attached to the gradient is for technical convenience.

These Maxwell's equations can be written in covariant form by introducing the four-vector potential A^α and the electric current four-vector potential J^α defined by:

$$A^\alpha = \left(\Phi, A^1, A^2, A^3\right) = \left(A^0, A^1, A^2, A^3\right) \tag{1.24}$$

$$J^\alpha = \left(\rho, J^1, J^2, J^3\right) = \left(J^0, J^1, J^2, J^3\right). \tag{1.25}$$

Equations (1.21) and (1.23) can then be written out explicitly in component form, for example

$$B_2 = \frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1} = \frac{\partial A^1}{\partial x_3} - \frac{\partial A^3}{\partial x_1} = \partial^3 A^1 - \partial^1 A^3 \tag{1.26}$$

$$E_1 = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = \partial^0 A^1 - \partial^1 A^0. \tag{1.27}$$

It is evident that the $\mathbf{E}$ and $\mathbf{B}$ fields are element of the second-rank, antisymmetric, contravariant field-strength tensor $F^{\mu\nu}$ defined by

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha. \tag{1.28}$$

Explicitly, the field-strength is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{1.29}$$

where α corresponds to the rows and β corresponds to the column. The components of the fields in equations (1.21) and (1.23) can be easily identified as

$$E_i = F^{0i}, \tag{1.30}$$

$$B_i = \frac{1}{2}\epsilon^{ijk}F^{jk}, \quad i, j, k = 1, 2, 3, \tag{1.31}$$

where the Levi-Civita symbol
$$\epsilon^{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3),\ (3,1,2),\ (2,3,1), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (3,2,1),\ (2,1,3), \\ 0 & \text{otherwise.} \end{cases} \tag{1.32}$$

It is very easy to show that the covariant field tensor defined by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \tag{1.33}$$

has components

$$F_{\alpha\beta} = \eta_{\alpha\gamma}\eta_{\beta\lambda}F^{\gamma\lambda} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{1.34}$$

The homogeneous Maxwell's equations (1.16) and (1.17) correspond to the Jacobi identities:

$$\partial^\gamma F^{\alpha\beta} + \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} = 0, \tag{1.35}$$

where α, β, γ are any of the three integers $0, 1, 2, 3$. For instance, if $\gamma = 1, \alpha = 2, \beta = 3$ we have from equations (1.29) and (1.35),

$$\partial^1 F^{32} + \partial^2 F^{13} + \partial^3 F^{21} = -\left(\partial^1 B_1 + \partial^2 B_2 + \partial^3 B_3\right) = -\left(\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3\right) = 0, \tag{1.36}$$

which indeed corresponds to equation (1.16).

The inhomogeneous Maxwell's equations (1.15) and (1.14) can be written as

$$\partial_\beta F^{\alpha\beta} = J^\alpha, \tag{1.37}$$

for instance, if $\alpha = 0$, we have from equations (1.29) and (1.37)

$$\partial_0 F^{00} + \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \rho, \tag{1.38}$$

which indeed agrees with equation (1.15).

Notice that the four Maxwell's equations have been reduced to a set of two equations (1.35) and (1.37). The continuity equation (1.18) was obtained from the inhomogeneous equations (1.15) and (1.14), similarly, the continuity equation in covariant form can obtained from (1.37) by operating ∂_μ on both sides of equation (1.37). Thus,

$$\partial_\alpha J^\alpha = \partial_\alpha \partial_\beta F^{\alpha\beta} = 0, \tag{1.39}$$

since $\partial_\alpha\partial_\beta$ is symmetric in α and β while $F^{\alpha\beta}$ is antisymmetric in α and β. The expression (1.39) is the conservation of electric charge whose underlying symmetry is gauge invariance.

1.4 Gauge Transformation

Equations (1.21) and (1.23) show that $\mathbf{A}$ determines $\mathbf{B}$, as well as part of $\mathbf{E}$. Notice that $\mathbf{B}$ is left invariant by the transformation

$$\mathbf{A} \longrightarrow \mathbf{A}^{'} = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{1.40}$$

for any arbitrary scalar function $\chi(x,t)$. But the theory of electricity and magnetism is a unified theory, so the electric field $\mathbf{E}$ in equation (1.23) must be invariant as well though not with the same transformation as the magnetic field $\mathbf{B}$.

The invariance of $\mathbf{E}$ is accomplished by the transformation

$$\Phi \longrightarrow \Phi^{'} = \Phi - \frac{\partial \chi}{\partial x^0}. \tag{1.41}$$

The transformations (1.40) and (1.41) are called g*auge transformation*, and the invariance of the fields under such transformations is called *gauge invariance* [Jac62].

In the language of covariance, we see that four-potential $A^\alpha = (\Phi, \mathbf{A})$ is not unique, the same electromagnetic field tensor $F^{\alpha\beta}$ can be obtained from the potential

$$A^\alpha = \left(\Phi - \frac{\partial \chi}{\partial x^0}, \mathbf{A} + \boldsymbol{\nabla}\chi\right). \tag{1.42}$$

Substituting (1.42) into (1.28) we obtain

$$\begin{aligned}
F^{\alpha\beta} &= \partial^\alpha A^\beta + \partial^\alpha \partial^\beta \chi - \partial^\beta A^\alpha - \partial^\beta \partial^\alpha \chi \\
&= \partial^\alpha A^\beta - \partial^\beta A^\alpha + \left[\partial^\alpha, \partial^\beta\right] \chi \\
&= \partial^\alpha A^\beta - \partial^\beta A^\alpha,
\end{aligned}$$

using the fact that $\left[\partial^\alpha, \partial^\beta\right] = 0$.

The transformation $A^\alpha \longrightarrow A'^\alpha = A^\alpha + \partial^\alpha \chi$ is a gauge transformation.

2. Vector Fields and Differential Forms

2.1 Vector Fields

This chapter is devoted to the basic concepts of vector fields and differential forms on manifolds. It is assumed that the reader already knows about topological spaces, topological manifolds and smooth manifolds. We are familiar with a vector $\mathbf{v} \in \mathbb{R}^n$ as a coordinate of n-tuples $(v^1, \ldots, v^n)$, with n components, v^j. Given the standard basis $\mathbf{e_1}, \ldots, \mathbf{e_n}$, a vector $\mathbf{v}$ has a unique expansion,

$$\mathbf{v} = v^j \mathbf{e_j}, \tag{2.1}$$

where the n real numbers, v^j, are the components of $\mathbf{v}$ with respect to the standard basis.

However, this picture is not visible on manifolds; how should one define a simultaneous basis vector, for two vectors at different points on a sphere, for example? If these two vectors are thought of as little arrows tangent to the sphere, they lie entirely in different planes, which makes defining basis vectors and coordinates cumbersome. Thus, we demand a definition of vector fields that is independent of coordinate choice. The trick of defining vector fields on manifolds is to note that given a field of arrows, one can differentiate a function in the direction of the arrows, with the directional derivatives.

If $\mathbf{v}$ is a vector field and f is a function on $\mathbb{R}^n$, the directional derivative of f in the direction of $\mathbf{v}$ is defined by

$$\mathbf{v}(f) :\, = D_{\mathbf{v}}(f) = v^j \partial_j f, \tag{2.2}$$

where $j = 1, 2, \ldots, n$. Since (2.2) is true for all f, we can extract f from both sides and write

$$\mathbf{v} = v^j \partial_j. \tag{2.3}$$

The vector field $\mathbf{v}$ should not be identified with components v^j but instead with the operator $v^j \partial_j$. Given a function f on a manifold; at each point of the manifold, we shall take a derivative of f in the direction of $\mathbf{v}$ giving us a new function $v^j \partial_j f$. That is, the combination of a vector field with a function gives another function on a manifold, which is related to the derivative of the original function.

The space of all infinitely differentiable (smooth) functions on a manifold will be denoted by $C^\infty(M)$ and the set of all vectors on the manifold M will be denoted by $\mathfrak{F}(M)$.

Definition 2.1.1. *A vector field $\mathbf{v}$ on a manifold M is defined as a function from $C^\infty(M)$ to $C^\infty(M)$ which satisfies the following axioms:*

i. $\mathbf{v}(\alpha f + \beta g) = \alpha \mathbf{v}(f) + \beta \mathbf{v}(g)$ *(Linearity)*

ii. $\mathbf{v}(fg) = g\mathbf{v}(f) + f\mathbf{v}(g)$ *(Leibniz law)*

for all $f, g \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$.

It is obvious from the above definition, that the vector fields are independent of the coordinate system used.

Proposition 2.1.2. *The vector fields $\{\partial_j\}$ form a basis of $\mathfrak{F}(\mathbb{R}^n)$.*

Proof. Let $\pi^i\colon \mathbb{R}^n \to \mathbb{R}, \quad \pi^i \in C^\infty(\mathbb{R}^n)$. Suppose $v^j\partial_j = 0$ then, the action on the coordinate function π^i gives $v^j\partial_j\pi^i = v^j\delta^i_j = v^i = 0$ for each $i \le n$, where

$$\partial_j\pi^i = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.4}$$

Therefore the vector fields $\{\partial_j\}$ are linearly independent. To show that they also form a basis, note that for fixed x_0, a first-order Taylor expansion yields

$$f(x) = f(x_0) + \partial_j f(x_0) \cdot (x^j - x^j_0) + \epsilon_{x_0} \cdot \|x - x_0\|$$

where $\epsilon_{x_0}(x) \longrightarrow 0$ as $x \longrightarrow x_0$.

Also $v(\alpha) = 0$, where α is a constant,

because, $v(\alpha) = \alpha v(1) = \alpha v(1 \cdot 1) = 2\alpha v(1) = 2v(\alpha)$.

Therefore,

$$\begin{aligned} v(f)(x_0) &= v(\pi^j)(x_0)\partial_j f(x_0) + v(\epsilon_{x_0})(x_0) \cdot \|x_0 - x_0\| + \epsilon_{x_0}(x_0) \cdot v(\|x_0 - x_0\|) \\ &= v(\pi^j)(x_0)\partial_j f(x_0). \end{aligned}$$

Hence $v = v(\pi^j)\partial_j$. This clearly shows that every vector field $\mathbf{v} \in \mathbb{R}^n$ has a unique expansion as a linear combination $v^j\partial_j$, therefore the vector fields $\{\partial_j\}$ form a basis of $\mathfrak{F}(\mathbb{R}^n)$. □

2.1.1 Tangent Vectors

In calculus, we usually regard vectors as arrows characterized by their direction and length. In the same spirit, it is good to think of a vector field on a manifold M as an arrow assigned to each point of M. This kind of assignment of arrows is called a *tangent vector field.*

A concrete understanding of a tangent vector at $p \in M$ comes from the realization that a tangent vector should allow us to take directional derivatives at the point p. For example, given a vector field $\mathbf{v}$ in an open neighbourhood U of a point $p \in M$, we can take the directional derivative $\mathbf{v}(f)$ of any function $f \in C^\infty(M)$ and then evaluate the function $\mathbf{v}(f)$ at $p \in M$. In other words, the result $\mathbf{v}(f)(p)$ may be interpreted as the differentiation of f in the direction of $\mathbf{v}_p$ at the point $p \in M$.

Definition 2.1.3. *A tangent vector $\mathbf{v}_p$ at a point $p \in M$ is a linear map from the algebra of smooth functions to the real numbers,*

$$\mathbf{v}_p\colon C^\infty(M) \to \mathbb{R}$$

such that the following properties are satisfied:

i. $\mathbf{v}_p(\alpha f + \beta g) = \alpha\mathbf{v}_p(f) + \beta\mathbf{v}_p(g)$ *(Linearity)*

ii. $\mathbf{v}_p(fg) = g\mathbf{v}_p(f) + f\mathbf{v}_p(g)$ *(Leibniz law)*

for all $f, g \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$.

The collection of all tangent vectors at a point $p \in M$ is a *tangent space* at p and shall be denoted by $T_p(M)$. A vector field $\mathbf{v} \in M$ determines tangent vector $\mathbf{v}_p \in T_p(M)$ at each point $p \in M$. Given two tangent vectors $\mathbf{v}_p, \mathbf{w}_p \in T_p(M)$ and a constant $\beta \in \mathbb{R}$, we can define new tangent vectors at p by

i. $(\mathbf{v}_p + \mathbf{w}_p)(f) = \mathbf{v}_p(f) + \mathbf{w}_p(f)$

ii. $(\beta \mathbf{v}_p)(f) = \beta(\mathbf{v}_p)(f)$.

With this definition, it is evident that for each point $p \in M$, the corresponding tangent space $T_p(M)$ at that point is a vector space. On the other hand, there is no natural way of adding two tangent vectors $\mathbf{v}_p \in T_p(M)$ and $\mathbf{w}_q \in T_q(M)$ on a manifold at different points unless p and q are equal.

It can be shown that the tangent vectors $\{\partial_j|_p\}$ on $\mathbb{R}^n$ form a basis at $T_p(\mathbb{R}^n)$.

2.2 The Space of 1-Forms

The electric field, the magnetic field, the electromagnetic field on spacetime, the current — all these are examples of differential forms. The gradient, the curl, and the divergence can all be thought of as different aspects of single operator d that acts on the differential forms. The fundamental theorem of calculus, Stoke's theorem, and Gauss' theorem are all special cases of a single theorem about differential forms [BP94]. The concept of gradient of a function can be generalized to functions on arbitrary manifold. Notice that the directional derivative of a function f on $\mathbb{R}^n$ (see (2.2)) is simply

$$\boldsymbol{\nabla} f \cdot \mathbf{v} = \mathbf{v}(f), \tag{2.5}$$

with the following properties

i. $\boldsymbol{\nabla} f \cdot (\mathbf{v} + \mathbf{w}) = \boldsymbol{\nabla} f \cdot \mathbf{v} + \boldsymbol{\nabla} f \cdot \mathbf{w}$

ii. $\boldsymbol{\nabla} f \cdot (g\mathbf{v}) = g(\boldsymbol{\nabla} f \cdot \mathbf{v})$

where $g \in C^\infty(\mathbb{R}^n)$ and $\mathbf{v}, \mathbf{w} \in \mathfrak{F}(\mathbb{R}^n)$.

On a manifold, there exists a function called df which is required to do the similar job as $\boldsymbol{\nabla} f$ on any manifold.

Definition 2.2.1. *Let $f\colon M \to \mathbb{R}$ be a real-valued C^∞ function. We define the differential or 1-form df of the function as a linear map $df\colon \mathfrak{F}(M) \to C^\infty(M)$ defined by*

$$df(\mathbf{v}) = \mathbf{v}(f),$$

for all $\mathbf{v} \in \mathfrak{F}(M)$.

Since df is a linear map, it has to satisfy the following properties

i. $df(\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{w})f = (\mathbf{v})f + (\mathbf{w})f = df(\mathbf{v}) + df(\mathbf{w})$

ii. $df(g\mathbf{v}) = g\mathbf{v}(f) = gdf(\mathbf{v})$.

Let $\Omega^1(M)$ represent the space of all 1-forms on a manifold M.

Definition 2.2.2. *The exterior derivative of a function f is a linear map*

$$d\colon C^\infty(M) \to \Omega^1(M),$$

that maps each function to its differential (1-form) df.

Proposition 2.2.3. *The exterior derivative of a function satisfies the following properties*

i. $d(\alpha f + \beta g) = \alpha df + \beta dg$

ii. $d(fg) = gdf + fdg$

for any $f, g \in C^\infty(M)$ *and any* $\alpha, \beta \in \mathbb{R}$.

Proof. i. $d(\alpha f + \beta g)\mathbf{v} = \mathbf{v}(\alpha f + \beta g) = \alpha \mathbf{v}(f) + \beta \mathbf{v}(f) = \alpha df(\mathbf{v}) + \beta dg(\mathbf{v})$

ii. $d(fg)\mathbf{v} = \mathbf{v}(fg) = g\mathbf{v}(f) + f\mathbf{v}(g) = gdf(\mathbf{v}) + fdg(\mathbf{v})$.

□

One of the most perplexing ideas in calculus is that of differential. In the study of calculus the differential of the dependent variable $y = f(x)$ in terms of the differential of the independent variable is given by $dy = f^{'}(x)dx$. The major problem of this expression is the quantity dx. What does "dx" mean? Physicists often think of dx as an infinitesimal change in position, a term introduced by Newton for an idea that could not be expounded at that time. Differentials are being thought of as 1-forms.

We have seen that the vector fields $\{\partial_j\}$ form a basis of $\mathfrak{F}(\mathbb{R}^n)$. Similarly, the 1-forms $\{dx^i\}$ form a basis of $\Omega^1(\mathbb{R}^n)$ such that

$$dx^i(\partial_j) = \partial_j x^i = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.6}$$

Proposition 2.2.4. *The* 1*-forms* $\{dx^i\}$ *are linearly independent, that is, if*

$$w = w_i dx^i = 0,$$

then all the functions w_i *are zero, and form a basis of* $\mathfrak{F}(\mathbb{R}^n)$*.*

Proof. If $w = w_i dx^i = 0$,

then, by multiplying both sides by ∂_j we have:

$w_i dx^i(\partial_j) = w_i \partial_j x^i = w_i \delta^i_j = w_j = 0.$

Similarly, one can easily show from the proof of Proposition (2.1.2), that the 1-forms $\{dx^i\}$ form a basis of $\mathfrak{F}(\mathbb{R}^n)$. □

This clearly shows that any arbitrary 1-form w can be expanded as a linear combination

$$w = w_1 dx^1 + \cdots + w_n dx^n, \tag{2.7}$$

where the components w_i are C^∞ functions.

Proposition 2.2.5. *Suppose* $f = f(x^1, \ldots, x^n)$ *is a smooth function on* $\mathbb{R}^n$*, then the differential (exterior derivative)* df *is given by the expression*

$$df = \partial_i f \, dx^i.$$

Proof.

$$\begin{aligned} df(\mathbf{v}) &= \mathbf{v}(f) = v^i \partial_i f \\ &= v^j \delta_i^j \partial_i f = v^j \partial_i f dx^i(\partial_j) \\ &= \partial_i f dx^i(v^j \partial_j) = \partial_i f dx^i(\mathbf{v}) \end{aligned}$$

□

2.2.1 Cotangent Vectors

A vector field on a manifold leads to a tangent vector at each point $p \in M$, similarly, a 1-form on M leads to a vector at each point called *cotangent vector*.

1-form fields assign to each point of a manifold an element of the dual tangent space. Differential p-froms are defined pointwise as the exterior products of 1-forms fields [HO03].

Definition 2.2.6. *A cotangent vector w at a point $p \in M$ is a linear map from the tangent space $T_p(M)$ to the real number $\mathbb{R}$*

$$w\colon T_p(M) \to \mathbb{R}.$$

Let $T_p^*(M)$ denote the space of cotangent vectors at point p. The set of linear functionals (maps) on a vector space is called the dual vector space. It can be shown from linear algebra, that the dual of a vector space is also a vector space of the same dimension. Thus, the space $T_p^*(M)$ of all 1-forms at $p \in M$ is a vector space which is dual to the tangent space $T_p(M)$.

On $\mathbb{R}^n$, the set of differential forms $\{dx^i|_p\}$ constitutes a basis of cotangent space which is dual to basis $\{\partial_j|_p\}$ of the tangent space.

2.3 Change of Coordinates

We shall show how the components of a vector field transform upon a change of coordinates. The reason why we have avoided coordinates in our definitions is that the world does not come equipped with coordinates. Moreover, our brains cannot function without coordinates, they are what we impose when we want to describe where things are. They are indispensable in many applications of physical interest.

Unfortunately, different people might pick different coordinates, so it is advisable to know how the components of a vector field or 1-form transform upon this change of coordinates. We shall first of all describe how one can use coordinates locally on any manifold to work with vector fields and differential forms. Given an n-dimensional manifold M, a chat is a *diffeomorphism* ϕ from an open set U of M to $\mathbb{R}^n$.

$$\phi\colon U \to \mathbb{R}^n. \tag{2.8}$$

In other words, we can turn calculations on U to calculations on $\mathbb{R}^n$. The coordinates x^j on U are *called the local coordinate*s of U, and any function on U can be written as $f(x^1, \ldots, x^n)$ of these local coordinates.

The coordinate vector fields ∂_j that form a basis on $\mathbb{R}^n$ can be push forward by ϕ^{-1} to a basis of the vector fields on U which can also be denoted by ∂_j. These are called *coordinate vector fields* associated with the local coordinates x^j on U.

Thus, we can express any vector field $\mathbf{v}$ on U as

$$\mathbf{v} = v^j \partial_j. \tag{2.9}$$

Similarly, the 1-forms dx^j are a basis of 1-forms on $\mathbb{R}^n$, which may be pulled back by ϕ to obtain a basis of 1-forms on U. These are called the *coordinate* 1-*forms* associated with the local coordinates x^j, which are written as dx^j.

Thus, we can express any 1-form ω on U as

$$\omega = \omega_j dx^j. \tag{2.10}$$

Given the coordinate functions $\{x^j\}$ on $\mathbb{R}^n$ with a basis $\{\partial_j\}$ of $\mathfrak{F}(\mathbb{R}^n)$ and a vector field $\mathbf{v} \in \mathbb{R}^n$, one can uniquely express this vector field as

$$\mathbf{v} = v^j \partial_j. \tag{2.11}$$

Suppose there exist another coordinate functions $\left\{x'^k\right\}$ on $\mathbb{R}^n$ such that $\left\{\partial'_k\right\}$ form a basis of $\mathfrak{F}(\mathbb{R}^n)$. Then the vector field can be expressed as

$$\mathbf{v} = v'^k \partial'_k. \tag{2.12}$$

The same vector is expressed in two ways, it follows that

$$v'^k \partial'_k = v^j \partial_j, \tag{2.13}$$

from chain rule of differentiation:

$$\partial_j = \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x'^k}\frac{\partial x'^k}{\partial x^j} = \frac{\partial x'^k}{\partial x^j}\partial'_k, \tag{2.14}$$

equation (2.13) becomes

$$v'^k \partial'_k = v^j \partial_j = v^j \frac{\partial x'^k}{\partial x^j}\partial'_k \tag{2.15}$$

$$\implies v'^k = \frac{\partial x'^k}{\partial x^j} v^j. \tag{2.16}$$

With this coordinate change, it follows that

$$\begin{aligned} D'_{\mathbf{v}}(f) &= v'^k \frac{\partial f}{\partial x'^k} \\ &= v^j \left(\frac{\partial x'^k}{\partial x^j}\right)\frac{\partial f}{\partial x'^k} \\ &= v^j \frac{\partial f}{\partial x^j} = D_{\mathbf{v}}(f) \end{aligned}$$

These expressions hold for any manifold M since we can define a chart (diffeomorphism) in (2.8), similarly, we can do the same transformation for 1-forms.

Thus, whenever we define something by use of local coordinates, if we wish the definition to have intrinsic significance we must check that it has the same meaning in all coordinate systems [Fra97]. This emphasizes the point that the physical laws of nature must be the same in all coordinate systems (generally covariant).

2.4 The Space of p-Forms

We shall now generalize 1-forms to p-forms on a manifold. Differential forms allows one to generalize cross products to any number of dimensions. This kind of product is called *wedge product or exterior product* and is denoted by $\wedge$.

Suppose V is a real vector space, we know that the cross product of any two vectors in V is antisymmetric, that is

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}. \tag{2.17}$$

Let ΛV be the exterior algebra over V. In other words, the wedge product of any number of vectors in V belongs to ΛV. Therefore for all $v, w \in V$, we have

$$v \wedge w = -w \wedge v. \tag{2.18}$$

In a general context, if V is an n-dimensional vector space, we define $\Lambda^p V$ to be the subspace of ΛV which is spanned by the linear combination of the p-linear alternating fold product of vectors in V, that is

$$v_1 \wedge v_2 \wedge \cdots \wedge v_p.$$

In particular,

$$\Lambda^1 V = V, \tag{2.19}$$

and by convention

$$\Lambda^0 V = \mathbb{R}. \tag{2.20}$$

If p is bigger than n, the dimension of V, $\Lambda^p V$ is zero,

$$\Lambda^p V = 0 \quad \text{for } p > n, \tag{2.21}$$

because any p-tuple of vectors $(v_1, \ldots, v_p)$ are linearly dependent. The direct sum of these subspaces is given by

$$\Lambda V = \bigoplus_{p=0}^{n} \Lambda^p V. \tag{2.22}$$

Therefore, dim $\Lambda^p V = \binom{n}{p}$ and dim $\Lambda V = 2^n$.

The generalization of the cross product of vector fields to the wedge (or exterior) products of 1-forms (or cotangent vectors) on a manifold M comes from the replacement of the real numbers with smooth functions $C^\infty(M)$ on M and vector space V with the 1-forms $\Omega^1(M)$. The space of differential forms on M will be denoted by $\Omega(M)$ which is an algebra generated by $\Omega^1(M)$, that is, they are comprised of linear combinations of wedge products of 1-forms with functions as coefficients.

Definition 2.4.1. *The linear combinations of wedge products of p 1-forms are called p-forms and the space of all p-forms on M denoted by $\Omega^p(M)$ is a vector space of dimension $\binom{n}{p}$ over the space of smooth functions. Thus*

$$\Omega(M) = \bigoplus_{p=0}^{n} \Omega^p(M),$$

where n= dim M and $\Omega^0(M) = C^\infty(M)$.

Suppose our manifold M is $\mathbb{R}^n$, let $\{x^i\}$ be the coordinates on $\mathbb{R}^n$ with a basis $\{dx^i\}$. We can express $w \in \Omega^p(\mathbb{R}^n)$ as

$$w = w_{i_1,\cdots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \quad \text{summed over} \quad 1 \le i_1 < \cdots < i_p \le n \tag{2.23}$$

$$\text{or} \quad w = \frac{1}{p!} w_{i_1,\cdots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}; \quad \text{summed over all } i_p's \text{ from } 1 \text{ to } n. \tag{2.24}$$

Notice that the wedge operator is skew symmetric, that is,

$$dx^{i_1} \wedge dx^{i_2} = -\, dx^{i_2} \wedge dx^{i_1} \tag{2.25}$$

$$dx^{i_1} \wedge dx^{i_1} = -\, dx^{i_1} \wedge dx^{i_1} = 0; \tag{2.26}$$

also, the functions $w_{i_1,\ldots,i_p}$ are assumed to be antisymmetric. Analytically speaking, differential forms (p-form) are antisymmetric tensors; pictorially speaking, they are intersecting stacks of surfaces.

Proposition 2.4.2. *For any manifold M, $\Omega(M)$ is graded commutative, that is, if $\omega \in \Omega^p(M)$ and $\mu \in \Omega^q(M)$, then*

$$\omega \wedge \mu = (-1)^{pq} \mu \wedge \omega.$$

Proof. Let $\{x^i\}$ be the coordinate on some open subset U of M with a basis $\{dx^i\}$, we can express $\omega \in \Omega^p(M)$ and $\mu \in \Omega^q(M)$ as,

$$\omega = \omega_{i_1,\cdots,i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

and

$$\mu = \mu_{j_1,\cdots,j_q} dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_q}$$

$$\begin{aligned}
\omega \wedge \mu &= (\omega_{i_1,\cdots,i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}) \wedge (\mu_{j_1,\cdots,j_q} dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_q}) \\
&= (-1)^p \mu_{j_1,\cdots,j_q} dx^{j_1} \wedge \omega_{i_1,\cdots,i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_2} \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_q} \\
&= (-1)^{2p} \mu_{j_1,\cdots,j_q} dx^{j_1} \wedge dx^{j_2} \wedge w_{i_1,\cdots,i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_q} \\
&\vdots \\
&= (-1)^{pq} \mu_{j_1,\cdots,j_q} dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \wedge \cdots \wedge dx^{j_q} \wedge \omega_{i_1,\cdots,i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \\
&= (-1)^{pq} \mu \wedge \omega.
\end{aligned}$$

□

Wedge product is much more powerful than cross product, in that it can be computed in any dimensions.

2.5 Exterior Derivatives

Having defined the operator $d\colon \Omega^0(M) \to \Omega^1(M); f \mapsto df$ that gives the differential of functions, we shall now extend this to a map,

$$d\colon \Omega^p(M) \to \Omega^{p+1}(M), \quad \text{for all p} \tag{2.27}$$

which takes the derivative of a p-form to produce a $(p+1)$-form. This operator turns out to have marvellous algebraic properties that will generalize the gradient, curl and divergence of vectors.

Definition 2.5.1. *The exterior derivative is the unique set of maps*

$$d\colon \Omega^p(M) \to \Omega^{p+1}(M),$$

which satisfies the following properties:

i. $d\colon \Omega^0(M) \to \Omega^1(M),$ is the ordinary derivative

ii. $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu,$ for $\mu \in \Omega^q(M)$ and $\omega \in \Omega^p(M)$

iii. $d(\alpha\omega + \beta\mu) = \alpha d\omega + \beta d\mu$ for all $\omega, \mu \in \Omega^p(M)$ and $\alpha, \beta \in \mathbb{R}$

iv. $d(d\omega) = 0$ for all $\omega \in \Omega^p(M)$.

These properties are easily proved but we shall skip them for convenience.

Definition 2.5.2. *A form ω is closed if $d\omega = 0$. It is exact if $\omega = d\phi$ for some ϕ (of degree one less than ω).*

3. The Metric and Star Operator

3.1 The Metric

In this chapter, we shall embark on the concept of a metric tensor and the Hodge star operator, as well as the process of converting vector fields into 1-forms on a manifold.

A metric tensor introduces the length of a vector and an angle between every two vectors. The components of the metric are defined by the values of the scalar products of the basis vectors [HO03].

Definition 3.1.1. *A real vector space V is called a metric vector space if on V, a scalar product is defined as a bilinear, symmetric, and non-degenerate map*

$$g\colon V \times V \to \mathbb{R},$$

such that for all $u, v, w \in V$ and $\lambda \in \mathbb{R}$ the following properties are satisfied:

(i) $g(\lambda u + v, w) = \lambda g(u, w) + g(v, w)$ *(bilinear)*

(ii) $g(u, v) = g(v, u)$ *(symmetric)*

(iii) $g(u, v) = 0 \quad \forall\, u \in V$ *iff* $v = 0$ *(non-degenerate).*

If $\{e_\alpha\}$ is the orthonormal basis in V, then,

$$g(e_\alpha, e_\beta) = g_{\alpha\beta} = \pm\delta_{\alpha\beta} = \begin{cases} \pm 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{3.1}$$

The signature of the metric is identified by the number of $+1's$ and $-1's$ usually denoted by (p, q). The idea of a metric can be extended to the space $\mathfrak{F}(M)$ of all vector fields and the space $\Omega^1(M)$ of all 1-forms on a manifold M. On a smooth manifold M, a metric g assigns to each point $p \in M$ a metric g_p on the tangent space T_pM in a smooth varying way

$$g_p\colon T_pM \times T_pM \to \mathbb{R},$$

which satisfies the above properties with λ replaced by $f \in C^\infty(M)$, $g(u, v)$ is a function on M whose value at p is $g_p(u_p, v_p)$, where $u, v \in \mathfrak{F}(M)$ and $u_p, v_p \in T_p(M)$.

If the signature of g is $(n, 0)$, n being the dimension of M, we say that g is a Riemannian metric, while if g has the signature of $(n-1, 1)$, we say that g is Lorentzian. A manifold equipped with a metric will be called a semi-Riemannian manifold denoted by (M, g).

Setting $\tilde{g}(u)(v) = g(u, v)$, we obtain an isomorphism

$$\tilde{g}\colon T_p(M) \to T_p^*(M), \tag{3.2}$$

which can be proved by using the non-degeneracy property and the fact that dim $T_p(M)$ =dim $T_p^*(M)$. Alternatively, we may write $\tilde{g}(u) = g(u, \cdot)$.

Let $\{\partial_\alpha\}$ be a basis of a vector field on an open neighbourhood U of a point in M, then, the components of the metric are given by

$$g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta). \tag{3.3}$$

If the dimension of M is n, then, $g_{\alpha\beta}$ is an $n \times n$ matrix. The non-degeneracy property shows that $g_{\alpha\beta}$ is invertible, we shall denote the inverse by $g^{\alpha\beta}$, also $g_{\alpha\gamma}g^{\alpha\beta} = \delta^{\beta}_{\gamma}$. This leads to the raising and lowering of indices which is a process of converting vector fields to 1-forms. With the help of (3.2), one can easily convert between tangent vectors and cotangent vectors.

Example 3.1.2. *If $u = u^\alpha \partial_\alpha$ is a vector field on a chart, then, the corresponding 1-form can be calculated as follows:*

$$\begin{aligned} \tilde{g}(u)(v) &= g(u,v) = g(u^\alpha\partial_\alpha, v^\beta\partial_\beta) = g_{\alpha\beta}u^\alpha v^\beta \\ &= g_{\alpha\gamma}\delta^\gamma_\beta u^\alpha v^\beta = g_{\alpha\gamma}u^\alpha v^\beta dx^\gamma(\partial_\beta) \\ &= (g_{\alpha\gamma}u^\alpha dx^\gamma)\, v^\beta\partial_\beta = (u_\gamma dx^\gamma)\, v \\ \implies \tilde{g}(u) &= u_\gamma dx^\gamma, \text{where} \quad g_{\alpha\gamma}u^\alpha = u_\gamma. \end{aligned}$$

Conversely, if $\xi = \xi_\beta dx^\beta$ is a 1-form on a chart, then, the corresponding vector field can be calculated as follows:

$$\begin{aligned} \tilde{g}^{-1}(\xi)(\eta) &= g^{-1}(\xi,\eta) = g^{-1}(\xi_\beta dx^\beta, \eta_\gamma dx^\gamma) = g^{\beta\gamma}\xi_\beta\eta_\gamma \\ &= g^{\beta\alpha}\delta^\gamma_\alpha \xi_\beta\eta_\gamma = g^{\beta\alpha}\xi_\beta\eta_\gamma dx^\gamma(\partial_\alpha) \\ &= \left(g^{\beta\alpha}\xi_\beta\partial_\alpha\right)\eta_\gamma dx^\gamma = (\xi^\alpha\partial_\alpha)\,\eta \\ \implies \tilde{g}^{-1}(\xi) &= \xi^\alpha\partial_\alpha, \text{where} \quad g^{\beta\alpha}\xi_\beta = \xi^\alpha. \end{aligned}$$

In the general context,

$$S^{n_1\cdots n_k}{}_{l_1\cdots l_j} = g^{n_1 m_1\cdots n_k m_k} S_{m_1\cdots m_k, l_1\cdots l_j}, \qquad S^{n_1\cdots n_k}{}_{l_1\cdots l_j} = g_{l_1 m_1\cdots l_j m_k} S^{n_1\cdots n_k, m_1\cdots m_j}.$$

Using the fact that we can switch from 1-form to vector fields and vice versa with the help of a metric, we define the inner product of two 1-forms ξ, η as

$$\langle \xi, \eta\rangle = g^{\beta\gamma}\xi_\beta\eta_\gamma. \tag{3.4}$$

If $\theta^1 \wedge \cdots \wedge \theta^p$ and $\sigma^1 \wedge \cdots \wedge \sigma^p$ are orthonormal basis of p-forms, then,

$$\left\langle \theta^1 \wedge \cdots \wedge \theta^p, \sigma^1 \wedge \cdots \wedge \sigma^p \right\rangle = \det\left[g(\theta^\alpha, \sigma^\beta)\right], \tag{3.5}$$

we define

$$\left\langle \theta^\alpha, \theta^\beta \right\rangle = \begin{cases} \varepsilon_\beta = \pm 1, & \text{if } \alpha = \beta\ , \\ 0, & \text{if } \alpha \neq \beta\ . \end{cases} \tag{3.6}$$

Definition 3.1.3. *Let M be an n-dimensional manifold, we define the volume form on a chat (U_α, ϕ_α) as*

$$\mathfrak{Y} = dx^1 \wedge \cdots \wedge dx^n;$$

if the manifold is a semi-Riemannian manifold we have,

$$\mathfrak{Y} = \sqrt{|\det g_{\alpha\beta}|}\,dx^1 \wedge \cdots \wedge dx^n.$$

3.2 The Hodge Star Operator

The binomial coefficient $\binom{n}{p}$ which represents the dimension of $\Omega^p(M)$ is the number of ways of selecting p(unordered) objects from a collection of n objects. It is evident that

$$\binom{n}{p} = \binom{n}{n-p}, \tag{3.7}$$

which means that there are as many p-forms as $(n-p)$-forms. In other word, there should be a way of converting p-forms to $(n-p)$-forms, for instance, 3-forms on 4-dimension can be converted to 1-forms and vice versa. The operator that does this conversion is called the *Hodge Star Operator*.

Definition 3.2.1. *The Hodge star operator is the unique linear map on a semi-Riemmanian manifold from p-forms to $(n-p)$-forms defined by*

$$\star \colon \Omega^p(M) \to \Omega^{(n-p)}(M),$$

such that for all $\xi, \eta \in \Omega^p(M)$,

$$\xi \wedge \star\eta = \langle \xi, \eta \rangle \mathfrak{Y}.$$

This is an isomorphism between p-forms and $(n-p)$-forms, and $\star\eta$ is called the dual of η. Suppose that $dx^1, \cdots, dx^n$ are positively oriented orthonormal basis of 1-forms on some chart (U_α, ϕ_α) on a manifold M. In particular, $\mathfrak{Y} = dx^1 \wedge \cdots \wedge dx^n$.

Let $1 \leq i_1 <, \ldots, < i_p \leq n$ be an ordered distinct increasing indices and let $j_1 <, \ldots, < j_{n-p}$ be their complement in the set $\{1, \ldots, n\}$, then,

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} = \mathrm{sgn}(I) dx^1 \wedge \cdots \wedge dx^n, \tag{3.8}$$

where, $\mathrm{sgn}(I)$ is the sign of the permutation $i_1 <, \ldots, < i_p$ in $\{1, \ldots, n\}$. In other words, the wedge products of p-forms and $(n-p)$-forms yield the volume form up to a sign.

We claim that

$$\star \left(dx^{i_1} \wedge \cdots \wedge dx^{i_p} \right) = \mathrm{sgn}(I) \varepsilon_{i_1} \cdots \varepsilon_{i_p} dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}}, \tag{3.9}$$

where $\mathrm{sgn}(I)\varepsilon_{i_1} \cdots \varepsilon_{i_p} = \pm 1$.

One can easily verify that (3.9) satisfies the Definition (3.2.1) by setting $\xi, \eta = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. Using the result of Proposition (2.4.2) we obtain

$$\begin{aligned} dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} &= (-1)^{p(n-p)} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} \\ \implies \mathrm{sgn(J)} dx^1 \wedge \cdots \wedge dx^n &= (-1)^{p(n-p)} \mathrm{sgn}(I) dx^1 \wedge \cdots \wedge dx^n \\ \implies \mathrm{sgn(I)}\mathrm{sgn}(J) &= (-1)^{p(n-p)} \mathrm{sgn}(I)\mathrm{sgn}(I) = (-1)^{p(n-p)}, \end{aligned}$$

since $\mathrm{sgn}(I)\mathrm{sgn}(I) = 1$, but $\mathrm{sgn}(I)$ is not always equal to $\mathrm{sgn}(J)$ which is the sign of the permutation $j_1 < \cdots < j_{n-p}$ in $\{1, \cdots, n\}$. Applying $\star$ once more on (3.9), we obtain

$$\star\star \left(dx^{i_1} \wedge \cdots \wedge dx^{i_p} \right) = \mathrm{sgn}(I) \varepsilon_{i_1} \cdots \varepsilon_{i_p} \star \left(dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} \right). \tag{3.10}$$

Notice that the $\star$ on the right hand side will take us from $(n-p)$-forms to $n-(n-p)$-forms $= p$-forms, therefore we have,

$$\begin{aligned}\star\star\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right) &= \text{sgn}(I)\varepsilon_{i_1}\cdots\varepsilon_{i_p}\text{sgn}(J)\varepsilon_{j_1}\cdots\varepsilon_{j_{n-p}}\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\\ &= \text{sgn}(I)\text{sgn}(J)\prod_{k=1}^{n}\varepsilon_k\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\\ \implies \star^2 &= (-1)^{p(n-p)+s}\\ \text{where}\quad \prod_{k=1}^{n}\varepsilon_k &= (-1)^s \text{ and } s \text{ is the signature of the metric.}\end{aligned}$$

The signature of the metric $s=0$ for Riemannian manifold and $s=1$ for Lorentzian manifold, thus,

$$\star^2 = \begin{cases}(-1)^{p(n-p)} & \text{for Riemannian manifold}\\ (-1)^{p(n-p)+1} & \text{for Lorentzian manifold .}\end{cases} \tag{3.11}$$

We could rewrite (3.9) by introducing the totally antisymmetric Levi-Civita permutation symbol defined by

$$\epsilon_{i_1,\ldots,i_p} = \begin{cases}+1 & \text{if } (i_1,\ldots,i_p) \text{ is an even permutation of } (1,\ldots,n),\\ -1 & \text{if } (i_1,\ldots,i_p) \text{ is an odd permutation of } (1,\ldots,n),\\ 0 & \text{otherwise.}\end{cases} \tag{3.12}$$

The Levi-Civita symbol of all the indices up is equal to the permutation with all the indices down on Riemannian manifold,

$$\epsilon_{i_1,\ldots,i_p} = \epsilon^{i_1,\ldots,i_p}, \tag{3.13}$$

since the Riemannian metric which is positive definite is used to raise or lower indices. However, this is not the case in Minkowski (Lorentzian manifold) 4-dimensional spacetime where index raising and lowering is done with Minkowski metric $\eta_{\mu\nu}$. Thus, in Minkowski 4-dimensional spacetime

$$\epsilon_{i_0 i_1 i_2 i_3} = -\epsilon^{i_0 i_1 i_2 i_3}, \qquad (\epsilon_{0123}=1), \tag{3.14}$$

since $\eta_{00}=-1$ (by the convention we have chosen in (1.3)) . The indices i_0,i_1,i_2,i_3 are any of the integers $0,1,2,3$. The important thing to note is that raising or lowering the index 0 introduces a negative sign.

Using (3.12) we obtain

$$\star\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right) = \frac{1}{(n-p)!}\epsilon^{i_1\cdots i_p}{}_{j_1\cdots j_{n-p}}dx^{j_1}\wedge\cdots\wedge dx^{j_{n-p}}. \tag{3.15}$$

Example 3.2.2. *Suppose dx^1,dx^2,dx^3 are a basis of 1-forms on some chart (U_α,ϕ_α) on 3-dimensional Riemannain manifold. Then, using (3.12) and (3.15) we obtain*

$$\left.\begin{aligned}\star dx^1 &= \frac{1}{2!}\epsilon^1_{j_1j_2}dx^{j_1}\wedge dx^{j_2}\\ &= \frac{1}{2}(\epsilon^1_{23}dx^2\wedge dx^3+\epsilon^1_{32}dx^3\wedge dx^2)\\ &= \epsilon_{123}dx^2\wedge dx^3 = dx^2\wedge dx^3\\ \star dx^2 &= \frac{1}{2}\epsilon^2_{j_1j_2}dx^{j_1}\wedge dx^{j_2} = -dx^1\wedge dx^3\\ \star dx^3 &= \frac{1}{2}\epsilon^3_{j_1j_2}dx^{j_1}\wedge dx^{j_2} = dx^1\wedge dx^2\end{aligned}\right\} \tag{3.16}$$

conversely,

$$\left.\begin{aligned} \star(dx^2 \wedge dx^3) &= \frac{1}{1!}\epsilon_1^{23} dx^1 = dx^1 \\ \star(-dx^1 \wedge dx^3) &= -\frac{1}{1!}\epsilon_2^{13} dx^2 = dx^2 \\ \star(dx^1 \wedge dx^2) &= \frac{1}{1!}\epsilon_3^{12} dx^3 = dx^3. \end{aligned}\right\} \tag{3.17}$$

In fact, a close look at the first set of equations shows that they are related to

$$\mathbf{i} = \mathbf{j} \times \mathbf{k}, \quad \mathbf{j} = -\mathbf{i} \times \mathbf{k}, \quad \mathbf{k} = \mathbf{i} \times \mathbf{j},$$

if 1 *is a* 0*-form or function. Then,*

$$\star 1 = \frac{1}{3!}\epsilon_{j_1 j_2 j_3} dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} = dx^1 \wedge dx^2 \wedge dx^3, \quad \textit{conversely} \quad \star\left(dx^1 \wedge dx^2 \wedge dx^3\right) = \frac{1}{0!}\epsilon^{123} = 1.$$

It follows that

$$\star^2 = 1 \tag{3.18}$$

Let dx^0, dx^1, dx^2, dx^3 *be a basis of* 1*-forms on some chart* (U_α, ϕ_α) *on* 4*-dimensional Minkowski spacetime then,*

$$\left.\begin{aligned} \star\left(dx^1 \wedge dx^0\right) &= \frac{1}{2!}\epsilon^{10}_{j_1 j_2} dx^{j_1} \wedge dx^{j_2} \\ &= \frac{1}{2}\left(\epsilon^{10}_{23} dx^2 \wedge dx^3 + \epsilon^{10}_{32} dx^3 \wedge dx^2\right) \\ &= \frac{1}{2}\left(-\epsilon_{1023} dx^2 \wedge dx^3 - \epsilon_{1032} dx^3 \wedge dx^2\right) \\ &= \epsilon_{0123} dx^2 \wedge dx^3 = dx^2 \wedge dx^3 \\ \star\left(dx^2 \wedge dx^0\right) &= \frac{1}{2!}\epsilon^{20}_{j_1 j_2} dx^{j_1} \wedge dx^{j_2} = \epsilon_{0123} dx^3 \wedge dx^1 \\ &= dx^3 \wedge dx^1 \\ \star\left(dx^3 \wedge dx^0\right) &= \frac{1}{2!}\epsilon^{30}_{j_1 j_2} dx^{j_1} \wedge dx^{j_2} = \epsilon_{0123} dx^1 \wedge dx^2 \\ &= dx^1 \wedge dx^2 \end{aligned}\right\} \tag{3.19}$$

conversely,

$$\left.\begin{aligned} \star(dx^2 \wedge dx^3) &= \frac{1}{2!}\epsilon^{23}_{j_1 j_2} dx^0 \wedge dx^1 = -\epsilon_{0123} dx^1 \wedge dx^0 \\ &= -dx^1 \wedge dx^0 \\ \star(dx^3 \wedge dx^1) &= \frac{1}{2!}\epsilon^{31}_{j_1 j_2} dx^{j_1} \wedge dx^{j_2} = \epsilon_{3120} dx^2 \wedge dx^0 \\ &= -dx^2 \wedge dx^0, \\ \star\left(dx^1 \wedge dx^2\right) &= \frac{1}{2!}\epsilon^{12}_{j_1 j_2} dx^{j_1} \wedge dx^{j_2} = -dx^3 \wedge dx^0. \end{aligned}\right\} \tag{3.20}$$

Notice something interesting in the above example, in 4*-dimensional Minkowski spacetime, the dual (Hodge star operator) of a* 2*-form is also a* 2*-form that is,*

$$\star\colon \Omega^2(M) \to \Omega^2(M), \quad \textit{with} \quad \star^2 = -1. \tag{3.21}$$

The dual of a 3-form in 4-dimensional Minkowski space-time is given by

$$\left.\begin{aligned} \star(dx^1\wedge dx^2\wedge dx^3) &= \epsilon_0^{123}dx^0 = \epsilon_{1230}dx^0 = -dx^0 \\ \star(dx^0\wedge dx^1\wedge dx^3) &= \epsilon_2^{013}dx^2 = -\epsilon_{0132}dx^2 = dx^2 \\ \star(dx^0\wedge dx^2\wedge dx^3) &= \epsilon_1^{023}dx^1 = -\epsilon_{0231}dx^1 = -dx^1 \\ \star(dx^0\wedge dx^1\wedge dx^2) &= \epsilon_3^{012}dx^3 = -\epsilon_{0123}dx^2 = -dx^3 \end{aligned}\right\} \tag{3.22}$$

conversely,

$$\left.\begin{aligned} \star dx^0 &= \frac{1}{3!}\epsilon^0_{j_1j_2j_3}dx^{j_1}\wedge dx^{j_2}\wedge dx^{j_3} = -\epsilon_{0123}dx^1\wedge dx^2\wedge dx^3 = -dx^1\wedge dx^2\wedge dx^3 \\ \star dx^1 &= \frac{1}{3!}\epsilon^1_{j_1j_2j_3}dx^{j_1}\wedge dx^{j_2}\wedge dx^{j_3} = \epsilon_{1023}dx^0\wedge dx^2\wedge dx^3 = -dx^0\wedge dx^2\wedge dx^3 \\ \star dx^2 &= \frac{1}{3!}\epsilon^2_{j_1j_2j_3}dx^{j_1}\wedge dx^{j_2}\wedge dx^{j_3} = \epsilon_{2013}dx^0\wedge dx^1\wedge dx^3 = dx^0\wedge dx^1\wedge dx^3 \\ \star dx^3 &= \frac{1}{3!}\epsilon^3_{j_1j_2j_3}dx^0\wedge dx^1\wedge dx^2 = \epsilon_{3012}dx^0\wedge dx^1\wedge dx^2 = -dx^0\wedge dx^1\wedge dx^2. \end{aligned}\right\} \tag{3.23}$$

The Exterior derivative and Hodge star operator on $\mathbb{R}^3$ yield the known classical operators, curl, divergence and gradient of vectors, as we now show.

Suppose f is a 0-form on $\mathbb{R}^3$. Then

$$df = \partial_1 f dx^1 + \partial_2 f dx^2 + \partial_3 f dx^3, \tag{3.24}$$

if the coordinates are Cartesian, then the components are the components of the gradient of f. Thus,

$$df = \boldsymbol{\nabla} f\cdot d\mathbf{x}. \tag{3.25}$$

Let $A = A_1dx^1 + A_2dx^2 + A_3dx^3$ be a 1-form on $\mathbb{R}^3$. Then,

$$\left.\begin{aligned} dA &= \partial_2A_1dx^2\wedge dx^1 + \partial_3A_1dx^3\wedge dx^1 + \partial_1A_2dx^1\wedge dx^2 \\ &= \partial_3A_2dx^3\wedge dx^2 + \partial_1A_3dx^1\wedge dx^3 + \partial_2A_3dx^2\wedge dx^3 \\ &= (\partial_1A_2-\partial_2A_1)dx^1\wedge dx^2 + (\partial_1A_3-\partial_3A_1)dx^1\wedge dx^3 + (\partial_2A_3-\partial_3A_2)dx^2\wedge dx^3 \\ \star dA &= (\partial_2A_3-\partial_3A_2)dx^1 - (\partial_1A_3-\partial_3A_1)dx^2 + (\partial_1A_2-\partial_2A_1)dx^3, \end{aligned}\right\} \tag{3.26}$$

if the components are Cartesian, then the components are that of the curl a vector $\mathbf{A}$. That is

$$\star dA = (\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{x}. \tag{3.27}$$

Notice that

$$\left.\begin{aligned} \star A &= A_1dx^2\wedge dx^3 + A_2dx^3\wedge dx^1 + A_3dx^1\wedge dx^2 \\ d\star A &= (\partial_1A_1+\partial_2A_2+\partial_3A_3)dx^1\wedge dx^2\wedge dx^3 \\ \star d\star A &= \partial_1A_1+\partial_2A_2+\partial_3A_3 = \boldsymbol{\nabla}\cdot\mathbf{A} \quad \text{in Cartesian coordinate.} \end{aligned}\right\} \tag{3.28}$$

Also, $d(df) = 0$ corresponds to $\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = 0$ and $d(dA) = 0$ corresponds to $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A})$ =0.

4. Differential Form of Maxwell's Equations

4.1 The Homogeneous Maxwell's Equations

Having developed the mathematical language of differential forms, we hereby apply it to Maxwell's equations. First, consider the homogeneous Maxwell's equations (1.16) and (1.17), notice that in the language of differential forms, the divergence of a vector has been shown to be the exterior derivative of a 2-form on $\mathbb{R}^3$ (see (3.28)). The curl of a vector has also been shown to be the exterior derivative of 1-form on $\mathbb{R}^3$ (see (3.26) and (3.27)). Thus, instead of treating the magnetic field as a vector $\mathbf{B} = (B_1, B_2, B_3)$ we will treat it as a 2-form

$$B = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2. \tag{4.1}$$

Similarly, instead of treating the electric field as a vector $\mathbf{E} = (E_1, E_2, E_3)$, we will treat it as a 1-form

$$E = E_1 dx^1 + E_2 dx^2 + E_3 dx^3. \tag{4.2}$$

Next we shall consider the electric and magnetic fields as the inhabitants of spacetime and assume that the manifold M to be a semi-Riemannian manifold equipped with the Minkowski metric, in other words, as a 4-dimensional Lorentzian manifold or spacetime. Furthermore, we shall assume that the spacetime M can be split into a 3-dimensional manifold S, 'space', with a Riemannian metric and another space $\mathbb{R}$ for time. Then,

$$M = \mathbb{R} \times S.$$

Let x^i $(i = 1, 2, 3)$ denote local coordinates on an open subset $U \subseteq S$, and let x^0 denote the coordinate on $\mathbb{R}$, then the local coordinates on $\mathbb{R} \times U \subseteq M$ will be those given in (1.1) with the metric defined by (1.3).

We can then combine the electric and magnetic fields into a unified electromagnetic field F, which is a 2-form on $\mathbb{R} \times U \subseteq M$ defined by

$$F = B + E \wedge dx^0. \tag{4.3}$$

In component form we have

$$F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta \tag{4.4}$$

where $F_{\alpha\beta}$ is given by (1.34).

Explicitly, we have

$$\begin{aligned} F = {} & E_1 dx^1 \wedge dx^0 + E_2 dx^2 \wedge dx^0 + E_3 dx^3 \wedge dx^0 \\ & + B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2. \end{aligned} \tag{4.5}$$

Taking the exterior derivative of (4.3) we obtain

$$dF = d(B + E \wedge dx^0) = dB + dE \wedge dx^0. \tag{4.6}$$

In general, for any differential form η on spacetime, we have

$$\eta = \eta_I dx^I, \tag{4.7}$$

where I ranges over $\{i_1, \ldots, i_p\}$ and η_I is a function of spacetime.

Taking the exterior derivative of (4.7), we obtain

$$\begin{aligned} d\eta &= \partial_1\eta_I dx^1 \wedge dx^I + \partial_2\eta_I dx^2 \wedge dx^I + \partial_2\eta_I dx^3 \wedge dx^I + \partial_0\eta_I dx^0 \wedge dx^I \\ &= \partial_i\eta_I dx^i \wedge dx^I + \partial_0\eta_I dx^0 \wedge dx^I, \qquad i = 1, 2, 3 \\ &= d_s\eta + dx^0 \wedge \partial_0\eta \\ \implies d &= d_s + dx^0 \wedge \partial_0, \end{aligned}$$

where d_s is the exterior derivative of space and $(x^0 = t)$.

Since B and E are differential forms on a spacetime, we shall split the exterior derivative into spacelike part and timelike part. Using the identity above, we obtain the following from (4.6)

$$\begin{aligned} dF &= d_sB + dx^0 \wedge \partial_0 B + (d_sE + dx^0 \wedge \partial_0 E) \wedge dx^0 \\ &= d_sB + (d_sE + \partial_0 B) \wedge dx^0 + dx^0 \wedge dx^0 \wedge \partial_0 E \\ &= d_sB + (d_sE + \partial_0 B) \wedge dx^0. \end{aligned}$$

Now, $dF = 0$ is the same as

$$d_sB = 0 \tag{4.8}$$
$$d_sE + \partial_0 B = 0. \tag{4.9}$$

The equations (4.8) and (4.9) are exactly the same as (1.16) and (1.17).

In order to be fully convinced that this is true, let's do the calculation explicitly in component form.

Taking the exterior derivative of F in (4.5), we obtain:

$$\begin{aligned} dF = &\partial_2 E_1 dx^2 \wedge dx^1 \wedge dx^0 + \partial_3 E_1 dx^3 \wedge dx^1 \wedge dx^0 + \partial_1 E_2 dx^1 \wedge dx^2 \wedge dx^0 \\ &+ \partial_3 E_2 dx^3 \wedge dx^2 \wedge dx^0 + \partial_1 E_3 dx^1 \wedge dx^3 \wedge dx^0 + \partial_2 E_3 dx^2 \wedge dx^3 \wedge dx^0 \\ &+ \partial_1 B_1 dx^1 \wedge dx^2 \wedge dx^3 + \partial_0 B_1 dx^0 \wedge dx^2 \wedge dx^3 + \partial_2 B_2 dx^2 \wedge dx^3 \wedge dx^1 \\ &+ \partial_0 B_2 dx^0 \wedge dx^3 \wedge dx^1 + \partial_3 B_3 dx^3 \wedge dx^1 \wedge dx^2 + \partial_0 B_3 dx^0 \wedge dx^1 \wedge dx^2, \end{aligned}$$

collecting terms and using the antisymmetric property of wedge product, we obtain

$$\begin{aligned} dF = &(\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3) dx^1 \wedge dx^2 \wedge dx^3 \\ &(\partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1) dx^0 \wedge dx^2 \wedge dx^3 \\ &(\partial_3 E_1 - \partial_1 E_3 + \partial_0 B_2) dx^0 \wedge dx^3 \wedge dx^1 \\ &(\partial_1 E_2 - \partial_2 E_1 + \partial_0 B_3) dx^0 \wedge dx^1 \wedge dx^2. \end{aligned}$$

Note that $dF = 0$ is the same as

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0 \tag{4.10}$$

$$\left.\begin{aligned} \partial_2 E_3 - \partial_3 E_2 + \partial_0 B_1 = 0 \\ \partial_3 E_1 - \partial_1 E_3 + \partial_0 B_2 = 0 \\ \partial_1 E_2 - \partial_2 E_1 + \partial_0 B_3 = 0 \end{aligned}\right\}. \tag{4.11}$$

The equations (4.10) and (4.11) are exactly the same as (1.16) and (1.17). Hence, the homogeneous Maxwell's equations correspond to the closed form $dF = 0$ which is similar to the Jacobi identities (1.35).

4.2 The Inhomogeneous Maxwell's Equations

Notice that in the old-fashioned formulation of Maxwell's equations (see (1.15) — (1.17)), the homogeneous and the inhomogeneous versions are somehow related by reversing the role of $\mathbf{E}$ and $\mathbf{B}$. In the language of differential forms, this reversal relationship will lead to treating E as a 2-form and B as a 1-form.

Interestingly, the Hodge star operator does this work efficiently since one can easily convert a 1-form in 3-dimensional space to a 2-form and vice versa. Starting from (4.5) and using the results established in (3.19) and (3.20), we obtain:

$$\begin{aligned} \star F = &-B_1 dx^1 \wedge dx^0 - B_2 dx^2 \wedge dx^0 - B_3 dx^3 \wedge dx^0 \\ &+ E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2, \end{aligned} \tag{4.12}$$

or

$$\star F = \frac{1}{2}(\star F)_{\alpha\beta} dx^\alpha \wedge dx^\beta \tag{4.13}$$

where

$$(\star F)_{\alpha\beta} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}. \tag{4.14}$$

A close look at (4.5) and (4.12) shows that the effect of the dual operator on F amounts to the exchange

$$E_i \mapsto -B_i \quad \text{and} \quad B_i \mapsto E_i, \qquad i = 1, 2, 3$$

in (1.34).

This is the main difference between the homogeneous and the inhomogeneous Maxwell's equations. Another difference is that the inhomogeneous version contains ρ and $\mathbf{J}$. In the language of differential forms, we shall use the fact that the metric allows us to convert a vector field into a 1-form. Combining the charge density ρ and current density $\mathbf{J}$ into a unified vector field on Minkowski spacetime, we obtain

$$\mathbf{J} = J^\alpha \partial_\alpha = \rho \partial_0 + J^1 \partial_1 + J^2 \partial_2 + J^3 \partial_3. \tag{4.15}$$

Using the result of Example (3.1.2), with Minkowski metric (1.3), we obtain the 1-form

$$J = J_\beta dx^\beta = J^1 dx^1 + J^2 dx^2 + J^3 dx^3 - \rho dx^0, \tag{4.16}$$

where

$$J_\beta = \eta_{\alpha\beta} J^\alpha. \tag{4.17}$$

Let $\star_s$ denote the Hodge star operator on space, using (3.17) we can easily see that (4.12) is the same as

$$\star F = \star_s E - \star_s B \wedge dx^0 \tag{4.18}$$

which amounts to the exchange

$$E \mapsto -\star_s B \quad \text{and} \quad B \mapsto \star_s E,$$

in (4.3), taking the exterior derivative of (4.18), we obtain

$$d\star F = d_s \star_s E + \partial_0 \star_s E \wedge dx^0 - d_s \star_s B \wedge dx^0. \tag{4.19}$$

Applying the Hodge star operator, we obtain

$$\star d \star F = - \star_s d_s \star_s E \wedge dx^0 - \partial_0 E + \star_s d_s \star_s B, \tag{4.20}$$

if we set $\star d \star F = J$ and equate components, we obtain

$$\left.\begin{aligned} \star_s d_s \star_s E &= \rho \\ -\partial_0 E + \star_s d_s \star_s B &= J^i dx^i \end{aligned}\right\}, i = 1, 2, 3 \tag{4.21}$$

which is exactly the inhomogeneous Maxwell's equations as can be shown explicitly by taking the exterior derivative of (4.12),

$$\begin{aligned} d\star F = &-\partial_2 B_1 dx^2 \wedge dx^1 \wedge dx^0 - \partial_3 B_1 dx^3 \wedge dx^1 \wedge dx^0 - \partial_1 B_2 dx^1 \wedge dx^2 \wedge dx^0 \\ &- \partial_3 B_2 dx^3 \wedge dx^2 \wedge dx^0 - \partial_2 B_3 dx^2 \wedge dx^3 \wedge dx^0 - \partial_1 B_3 dx^1 \wedge dx^3 \wedge dx^0 \\ &+ \partial_0 E_1 dx^0 \wedge dx^2 \wedge dx^3 + \partial_1 E_1 dx^1 \wedge dx^2 \wedge dx^3 + \partial_0 E_2 dx^0 \wedge dx^3 \wedge dx^1 \\ &+ \partial_2 E_2 dx^2 \wedge dx^3 \wedge dx^1 + \partial_0 E_3 dx^0 \wedge dx^1 \wedge dx^2 + \partial_3 E_3 dx^3 \wedge dx^1 \wedge dx^2. \end{aligned}$$

Collecting terms, we obtain

$$\left.\begin{aligned} d\star F = &(\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3) dx^1 \wedge dx^2 \wedge dx^3 \\ &+ (\partial_3 B_2 - \partial_2 B_3 + \partial_0 E_1)\, dx^0 \wedge dx^2 \wedge dx^3 \\ &(\partial_3 B_1 - \partial_1 B_3 - \partial_0 E_2)\, dx^0 \wedge dx^1 \wedge dx^3 \\ &(\partial_2 B_1 - \partial_1 B_2 + \partial_0 E_3)\, dx^0 \wedge dx^1 \wedge dx^2 \end{aligned}\right\}. \tag{4.22}$$

Taking the dual of (4.22) and using the result of (3.22), we obtain

$$\begin{aligned} \star d \star F = &- (\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3)\, dx^0 + (\partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1) dx^1 \\ &+ (\partial_3 B_1 - \partial_1 B_3 - \partial_0 E_2)\, dx^2 + (\partial_1 B_2 - \partial_2 B_1 - \partial_0 E_3) dx^3 \end{aligned} \tag{4.23}$$

Now, $\star d \star F = J$ corresponds to

$$\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \rho \tag{4.24}$$

$$\left.\begin{aligned} \partial_2 B_3 - \partial_3 B_2 - \partial_0 E_1 &= J^1 \\ \partial_3 B_1 - \partial_1 B_3 - \partial_0 E_2 &= J^2 \\ \partial_1 B_2 - \partial_2 B_1 - \partial_0 E_3 &= J^3 \end{aligned}\right\}. \tag{4.25}$$

Notice that (4.24) and (4.25) are exactly the same as (1.15) and (1.14) also, $\star d \star F = J$ is similar to (1.37). Thus, the newfangled Maxwell's equations correspond to

$$\boxed{\begin{aligned} dF &= 0 \\ \star d \star F &= J \end{aligned}}$$

The continuity equation in covariant form (1.39) was derived from (1.37), in a similar way the continuity equation in differential form will be derived from $\star d \star F = J$.

Taking the dual of both sides, we obtain,

$$\star^2 d\star F = \star J \tag{4.26}$$

$$\implies \pm d\star F = \star J. \tag{4.27}$$

The sign in (4.27) depends on the value of $\star^2$ in (3.11) for Lorentzian manifold, but in this case $\star^2 = 1$ since $d\star F$ is a 3-form, therefore

$$d\star F = \star J. \tag{4.28}$$

Taking the exterior derivative of both sides in (4.28) and using property (iv) in Definition (2.5.1), we obtain

$$d\star J = dd\star F = 0, \tag{4.29}$$

which is the differential form of (1.39), as we will now show.

Taking the star of both sides of (4.16) and using the result of (3.23), we obtain,

$$\begin{aligned}\star J = \rho dx^1 \wedge dx^2 \wedge dx^3 - J^1 dx^0 \wedge dx^2 \wedge dx^3 \\ + J^2 dx^0 \wedge dx^1 \wedge dx^3 - J^3 dx^0 \wedge dx^1 \wedge dx^2.\end{aligned} \tag{4.30}$$

Operating the exterior derivative on (4.30), we obtain,

$$\begin{aligned}d\star J = \partial_0 \rho\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 - \partial_1 J^1 dx^1 \wedge dx^0 \wedge dx^2 \wedge dx^3 \\ + \partial_2 J^2 dx^2 \wedge dx^0 \wedge dx^1 \wedge dx^3 - \partial_3 J^3 dx^3 \wedge dx^0 \wedge dx^1 \wedge dx^2.\end{aligned} \tag{4.31}$$

Using the property of wedge product, we obtain,

$$d\star J = (\partial_0 \rho + \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{4.32}$$

Therefore $d\star J = 0$ corresponds to

$$\partial_0 \rho + \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 = 0, \tag{4.33}$$

which is exactly the continuity equation (1.18). This shows that the differential forms of Maxwell's equations are exactly the same as the covariant forms when expressed in terms of components.

4.3 The Vacuum Maxwell's Equations

In free space (vacuum), Maxwell's equations, in the old-fashioned formulation, correspond to $\rho, \mathbf{J} = 0$, which can be identified in the modern language of differential form as

$$dF = 0, \tag{4.34}$$

and

$$d\star F = 0 \quad (J = 0), \tag{4.35}$$

which amounts to the exchange

$$F \mapsto \star F.$$

We say that $F \in \Omega^2(M)$ is *self-dual* if $\star F = F$, and *anti-self-dual* if $\star F = -F$. In 3-dimensional Riemannian manifold, it was shown that $\star^2 = 1$. This implies that the Hodge star operator has eigenvalues of $\star = \pm 1$, therefore, we can consider any $F \in \Omega^2(M)$ as a sum of self-dual and anti-self-dual:

$$F = F_+ + F_-, \quad \text{where, } \star F_\pm = \pm F_\pm, \tag{4.36}$$

which can be easily shown by taking $F_\pm = \frac{1}{2}(F \pm \star F)$.

However, in the Lorentzian case $\star^2 = -1$, which implies that the eigenvalues are $\pm i$. If we consider complex-valued differential forms on M, it follows that, for any $F \in \Omega^2(M)$, we have,

$$F = F_+ + F_-, \quad \text{where, } \star F_\pm = \pm i F_\pm. \tag{4.37}$$

In both cases, if F is a self-dual or an anti-self-dual 2-form satisfying (4.34), automatically it satisfies (4.35). Certainly, F is complex-valued in the Lorentzian case but we can always split the real and imaginary part and obtain a real solution using the fact that Maxwell's equations are linear. Thus, the four vacuum Maxwell's equations correspond to either (4.34) or (4.35) in the language of differential forms.

4.4 Conclusion

Interestingly, Maxwell's equations have been drastically reduced into a language of differential geometry. These four sets of equations which perfectly describe the theory of electromagnetism have been reduced to a set of two equations which lay the foundations of most new theories in the physical world today.

The most revolutionary quantum leap in the history of theoretical physics is the birth of general relativity and quantum field theory (the standard model of elementary particle). These theories describe nature better than any physicist ever had at hand, although they have not been unified into a coherent picture of the world. One of the main ingredients of these theories is differential geometry. Euclidean geometry was abandoned in favour of differential geometry and classical field theories had to be quantized.

Maxwell's equations in the language of differential geometry lead to a generalization to these new theories, and these equations are a special case of Yang-Mills equations (beyond the scope of this essay), which is also gauge invariant and describe not only electromagnetism but also the strong and weak nuclear forces. This essay is nothing but the tip of the iceberg.

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