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The Riemann Hypothesis

For the aficionado and virtuoso alike.

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Dedication

For Pinot

- P.B.

For my parents, my lovely wife Shirley, my daughter Priscilla and son Matthew.

- S.C.

For my parents Tom and Katrea.

- B.R.

For my family.

- A.W.

Preface

This book is an introduction to the theory surrounding the Riemann Hypothesis. It is primarily a source book with the emphasis on the original papers that make up Part II; while Part I serves as a compendium of known results and as a primer for the material presented in the papers. The text is suitable for a graduate course or seminar or simply as a reference for anyone interested in this extraordinary conjecture.

We have divided the papers into two chapters. Chapter 11 consists of four expository papers on the Riemann Hypothesis, while Chapter 12 gathers original papers that develop the theory surrounding the Riemann Hypothesis.

The Riemann Hypothesis is difficult and perhaps none of the approaches to date will bear fruit. This translates into a difficulty in selecting appropriate papers. There is simply a lack of profound developments and attacks on the full problem. However, there is an intimate connection between the Prime Number Theorem and the Riemann Hypothesis. They are connected theoretically and historically and the Riemann Hypothesis may be thought of as a grand generalization of the Prime Number Theorem. There is a large body of theory on the Prime Number Theorem and a progression of solutions. Thus we have chosen several papers that give proofs of the Prime Number Theorem.

Since there have been no successful attacks on the Riemann Hypothesis, a large body of evidence has been generated in its support. This evidence is largely computational, and hence we have included several papers that focus on, or use computation of the zeta function. We have also included Weil's proof of the Riemann Hypothesis for function fields (Section 12.8), and the deterministic polynomial primality test of Agrawal et al (Section 12.20).

The material in Part I is organized (for the most part) into independent chapters. One could cover the material in any order; however, we would recommend starting with the four expository papers in Chapter 11. The reader who is unfamiliar with the basic theory and algorithms used in studying the Riemann zeta function may wish to begin with Chapters 2 and 3. The remaining chapters stand on their own quite nicely and can be covered in any order the reader fancies (obviously with our preference being first to last). We have tried to link the material to the original papers in order to facilitate in depth of study of the topics presented.

We would like to thank the community of authors, publishers and libraries for their kind permission and assistance in re-publishing the papers included in Part II. In particular: "On Newman's Quick Way to the Prime Number Theorem" and "Pair Correlation of Zeros and Primes in Short Intervals" are re-printed with kind permission of Springer Science and Business Media; "The Pair Correlation of Zeros of the Zeta Function" is re-printed with kind permission of the American Mathematical Society; and "On the Difference $\pi(x) - \mathrm{Li}(x)$" is re-printed with kind permission of the London Mathematical Society.

Contents

Notation

The notation in this book is standard. Specific symbols and functions are defined as needed throughout, and the standard meaning of basic symbols and functions is assumed. The following is a list of symbols that appear frequently in the text, and the meanings we intend them to convey.

$\Rightarrow$	"If ..., then ..." in natural language
$\in$	membership in a set
$:=$	defined to be
$x \equiv y \pmod p$	x is congruent to y modulo p
$[x]$	the integral part of x
$\{x\}$	the fractional part of x
$\lvert x\rvert$	the absolute value of x
$x!$	for $x \in \mathbb{N}$, $x! = x \cdot (x-1) \cdot \ldots \cdot 2 \cdot 1$
(n, m)	the greatest common divisor of n and m
$\phi(x)$	Euler's totient function evaluated at x
$\log(x)$	the natural logarithm, $\log_e(x) = \ln(x)$
$\det(A)$	the determinant of matrix A
$\pi(x)$	the number of prime numbers $p \leq x$
$\mathrm{Li}(x)$	the logarithmic integral of x, $\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}$
$\sum$	summation
$\prod$	product
$\to$	tends towards
x^+	towards x from the right
x^-	towards x from the left
$f'(x)$	the first derivative of $f(x)$ with respect to x
$\Re(x)$	the real part of x
$\Im(x)$	the imaginary part of x
$\arg(x)$	the argument of a complex number x

$\Delta_C \arg(f(x))$	the number of changes in the argument of $f(x)$ along the contour C
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	the set of integers
$\mathbb{Z}/p\mathbb{Z}$	the ring of integers modulo p
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{C}$	the set of complex numbers
$f(x) = O(g(x))$	$\lvert f(x)\rvert \leq A\lvert g(x)\rvert$ for some constant A and all values of $x > x_0$ for some x_0
$f(x) = o(g(x))$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
$f \ll g$	$\lvert f(x)\rvert \leq A\lvert g(x)\rvert$ for some constant A and all values of $x > x_0$ for some x_0
$f \ll_\varepsilon g$	$\lvert f(x)\rvert \leq A(\varepsilon)\lvert g(x)\rvert$ for some given function $A(\varepsilon)$ and all values of $x > x_0$ for some x_0
$f(x) = \Omega(g(x))$	$\lvert f(x)\rvert \geq A\lvert g(x)\rvert$ for some constant A and all values of $x > x_0$ for some x_0
$f \sim g$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

Part I

Introduction to the Riemann Hypothesis

1

Why This Book

One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation [124].

Bernhard Riemann, 1859

The above comment appears in Riemann's memoir to the Berlin Academy of Sciences (Section 12.2). It appears to be a passing thought, yet it has become, arguably, the most central problem in modern mathematics.

This book presents the Riemann Hypothesis, connected problems, and a taste of the related body of theory. The majority of the content is in Part II, while Part I contains a summary and exposition of the main results. It is targeted at the educated non-expert; most of the material is accessible to an advanced mathematics student, and much is accessible to anyone with some university mathematics.

Part II is a selection of original papers. This collection encompasses several important milestones in the evolution of the theory connected to the Riemann Hypothesis. It also includes some authoritative expository papers. These are the "expert witnesses" who offer the most informed commentary on the Riemann Hypothesis.

1.1 The Holy Grail

The Riemann Hypothesis has been the Holy Grail of mathematics for a century and a half. Bernard Riemann, one of the extraordinary mathematical talents of the 19th century, formulated the problem in 1859. The Hypothesis makes a very precise connection between two seemingly unrelated mathematical objects (namely prime numbers and the zeros of analytic functions). If solved, it would give us profound insight into number theory and, in particular, the nature of prime numbers.

Why is the Riemann Hypothesis so important? Why is it the problem that many mathematicians would sell their souls to solve? There are a number of great old unsolved problems in mathematics, but none of them have quite the stature of the Riemann Hypothesis. This stature can be attributed to a variety of causes ranging from mathematical to cultural. In common with the other old great unsolved problems, the Riemann Hypothesis is clearly very difficult. It has resisted solution for 150 years and has been attempted by many of the greatest minds in mathematics.

The problem was highlighted at the 1900 International Congress of Mathematicians; a conference held every 4 years and the most prestigious international mathematics meeting. David Hilbert, a pre-eminent mathematician of his generation, raised 23 problems that he thought would shape 20th century mathematics. This was somewhat self-fulfilling as solving a Hilbert problem guaranteed instant fame and perhaps local riches. Many of Hilbert's problems have now been solved. The most notable recent example is Fermat's Last Theorem, and was solved by Andrew Wiles in 1995.

Being one of Hilbert's 23 problems was enough to guarantee the Riemann Hypothesis centrality in mathematics for more than a century. Adding to interest in the hypothesis is a million dollar bounty in the form of a "Millennium Prize Problem" of the Clay Mathematics Institute. That the Riemann Hypothesis should be listed as one of seven such mathematical problems (each with a million dollar prize associated with its solution) indicates not only the contemporary importance of a solution, but also the importance of motivating a new generation of researchers to explore the hypothesis further.

Solving any of the great unsolved problems in mathematics is akin to the first ascent of Everest. It is a formidable achievement, but after the conquest there is sometimes nowhere to go but down. Some of the great problems have proven to be isolated mountain peaks, disconnected from their neighbours. The Riemann Hypothesis is quite different in this regard. There is a large body of mathematical speculation that becomes fact if the Riemann Hypothesis is solved. We know many statements of the form "if the Riemann Hypothesis, then the following interesting mathematical statement" and this is rather different from the solution of problems such as the Fermat problem.

The Riemann Hypothesis can be formulated in many diverse and seemingly unrelated ways; this is one of its beauties. The most common formulation is that certain numbers, the zeros of the "Riemann Zeta function", all lie on a certain line (precise definitions later). This formulation can, to some extent, be verified numerically.

In one of the largest calculations done to date, it was checked that the first ten trillion of these zeros lie on the correct line. So there are ten trillion pieces of evidence indicating that the Riemann Hypothesis is true and not a single piece of evidence indicating that it is false. A physicist might be overwhelmingly pleased with this much evidence in favour of the Hypothesis; but to the mathematician this is hardly evidence at all. However, it is interesting ancillary information.

In order to prove the Riemann Hypothesis it is required to show that all of these numbers lie in the right place, not just the first trillion. Until such a proof is provided the Riemann Hypothesis cannot be incorporated into the body of mathematical facts and accepted as true by mathematicians (Even though it is probably true!). This is not just pedantic fussiness. Certain mathematical phenomena that appear true, and that can be tested in part computationally, are false, but only false past computational range (This is seen in papers 12.12, 12.9 and 12.14).

Accept for a moment that the Riemann Hypothesis is the greatest unsolved problem in mathematics and that the greatest achievement any young graduate student could aspire to is to solve it. Why isn't it better known? Why hasn't it permeated public consciousness? (In the way black holes and unified field theory have, at least to some extent.) Part of the reason for this is it is hard to state rigourously, or even unambiguously. Some undergraduate mathematics is required in order for one to be familiar enough with the objects involved to even be able to state the Hypothesis accurately. Our suspicion is that a large proportion of professional mathematicians could not precisely state the Riemann Hypothesis if asked.

> *If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?*
>
> Attributed to **David Hilbert**

1.2 Riemann's Zeta and Liouville's Lambda

The Riemann zeta function is defined, for $\Re(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.2.1}$$

The Riemann Hypothesis is usually given as: the nontrivial zeros of the Riemann zeta function lie on the line $\Re(s) = \frac{1}{2}$. There is already, of course, the problem that the above series doesn't converge on this line, so one is already talking about an analytic continuation.

Our immediate goal is to give as simple an (equivalent) statement of the Riemann Hypothesis as we can. Loosely the statement is "the number of integers with an even number of prime factors is the same as the number of integers with an odd number of prime factors." This is made precise in terms of the Liouville Function.

The Liouville Function gives the parity of the number of prime factors.

Definition 1.1. *The Liouville Function is defined by*

$$\lambda(n) = (-1)^{\omega(n)}$$

where $\omega(n)$ is the number of, not necessarily distinct, prime factors in n, with multiple factors counted multiply.

So $\lambda(2) = \lambda(3) = \lambda(5) = \lambda(7) = \lambda(8) = -1$ and $\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1$ and $\lambda(x)$ is completely multiplicative (i.e., $\lambda(xy) = \lambda(x)\lambda(y)$ for any $x, y \in \mathbb{N}$) taking only values ± 1. (Alternatively one can define λ as the completely multiplicative function with $\lambda(p) = -1$ for any prime p.)

The connections between the Liouville function and the Riemann Hypothesis were explored by Landau in his doctoral thesis of 1899.

Theorem 1.2. *The Riemann Hypothesis is equivalent to the statement that for every fixed $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{\lambda(1) + \lambda(2) + \ldots + \lambda(n)}{n^{\frac{1}{2}+\varepsilon}} = 0.$$

This translates to the following statement. The Riemann Hypothesis is equivalent to the statement that an integer has equal probability of having an odd number or an even number of distinct prime factors (in the precise sense given above). This formulation has inherent intuitive appeal.

We can translate the equivalence once again. The sequence

$$\{\lambda(i)\}_{i=1}^{\infty} = \{1, -1, -1, 1, -1, 1, -1, 1, 1, 1, -1, 1, 1, -1, 1, 1, 1, 1, 1, -1, -1, \ldots\}$$

behaves more or less like a random sequence of 1's and -1's in that the difference between the number of 1's and -1's is not much larger than the square root of the number of terms.

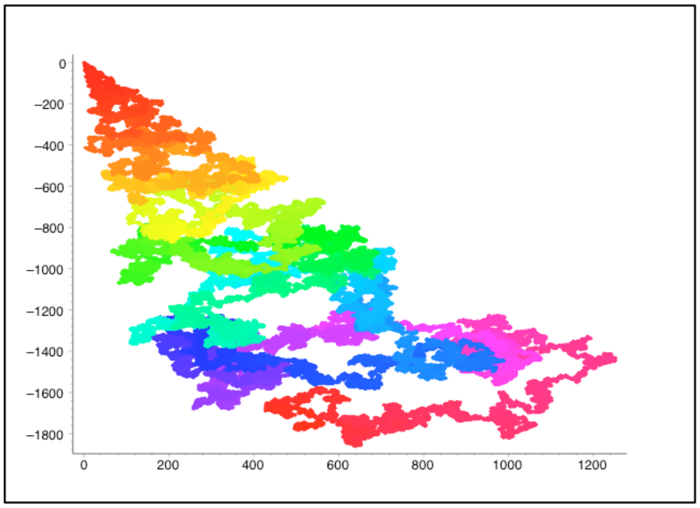

Fig. 1.1. $\{\lambda(i)\}_{i=1}^{\infty}$ plotted as a "random walk," by walking $(\pm 1, \pm 1)$ through pairs of points of the sequence.

This is an elementary and intuitive statement of the Riemann Hypothesis. It is an example of the many diverse re-formulations given in Chapter 5. It is relatively easy to formulate, and lends itself to a pretty picture, which makes it a useful instructive tool.

1.3 The Prime Number Theorem

The Prime Number Theorem is a jewel of mathematics. It states that the number of primes less than or equal to n is approximately $n/\log n$, and was conjectured by Gauss in 1792, on the basis of substantial computation and insight. One can view the Riemann Hypothesis as a precise form of the Prime Number Theorem, where the rate of convergence is made specific (see Section 5.1).

Theorem 1.3 (The Prime Number Theorem). *Let $\pi(n)$ denote the number of primes less than or equal to n then*

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log(n)} = 1.$$

As with the Riemann Hypothesis, the Prime Number Theorem can be formulated in terms of the Liouville Lambda function. A result also due to Landau in his doctoral thesis of 1899.

Theorem 1.4. *The Prime Number Theorem is is equivalent to the statement that*

$$\lim_{n\to\infty} \frac{\lambda(1) + \lambda(2) + \ldots + \lambda(n)}{n} = 0.$$

So the Prime Number Theorem is a relatively weak statement of the fact that an integer has equal probability of having an odd number or an even number of distinct prime factors

The Prime Number Theorem was first proved independently by de la Vallée Poussin (see Section 12.4) and Hadamard (see Section 12.3) around 1896, although Chebyshev came close in 1852 (see Section 12.1). The first and easiest proofs are analytic and exploit the rich connections between number theory and complex analysis. It has resisted trivialization and no really easy proof is known. This is especially true for the so called elementary proofs which use little or no complex analysis, just considerable ingenuity and dexterity.

The primes arise sporadically and, apparently, relatively randomly, at least in the sense that there is no easy way to find a large prime number with no obvious congruences. So even the amount of structure implied by the Prime Number Theorem is initially surprising.

Included in the original papers are a variety of proofs of the Prime Number Theorem. Korevaar's expository paper (see Section 12.16) is perhaps the most accessible of these. We focus on the Prime Number Theorem for its intrinsic interest but also because it represents a high point in the quest for resolution of the Riemann Hypothesis. Now that we have some notion of what the Riemann Hypothesis is, we can move on to the precise analytic formulation.

2

Analytic Preliminaries

The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics [59].

G. H. Hardy, 1967

In this chapter we develop some of the more important, and beautiful, results in the classical theory of the zeta function. The material is mathematically sophisticated; however, our presentation should be accessible to the reader with a first course in complex analysis. At the very least, the results should be meaningful even if the details are elusive.

We first develop the functional equation for the Riemann zeta function from Riemann's seminal paper, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (see 12.2), as well as some basic properties of the zeta function. Then we present part of de la Vallée Poussin's proof of the Prime Number Theorem (see 12.4), in particular we prove that $\zeta(\frac{1}{2}+it) \neq 0$ for $t \in \mathbb{R}$. We also develop the main idea necessary to verify the Riemann Hypothesis, namely we prove that $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$, where $N(T)$ is the number of zeros of $\zeta(s)$ up to height T. Finally we give a proof of Hardy's result that there are infinitely many zeros of $\zeta(s)$ on the critical line, from 12.5. The original papers are included in Part II, and are the best entry point to the material.

We begin with precise definitions of the functions under consideration.

2.1 The Riemann Zeta Function

The Riemann Hypothesis is a precise statement, and in one sense what it means is clear, but what it's connected with, what it implies, where it comes from, can be very unobvious [129].

M. Huxley

Defining the Riemann zeta function is of itself a non-trivial undertaking. The function, while easy enough to formally define, is of sufficient complexity that such a statement would be unenlightening. Instead we will "build" the Riemann zeta function in small steps. We will, by and large, follow the historic development of the zeta function from Euler to Riemann. This development sheds some light on the deep connection between the zeta function and the prime numbers. We begin with the following example of a Dirichlet series.

Let $s = \sigma + it$ be a complex number. We consider the Dirichlet series,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots.$$

This series will be the first building block of the Riemann zeta function. Notice that if we set $s = 1$ we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots,$$

the well-known harmonic series, which diverges. Also notice that whenever $\Re(s) \leq 1$ the series will diverge. It is also easy to see that this series converges whenever $\Re(s) > 1$ (this follows from the integral test). So this Dirichlet series defines an analytic function in the region $\Re(s) > 1$. We will initially define the Riemann zeta function to be

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2.1.1}$$

for $\Re(s) > 1$.

Euler was the first to give any substantial analysis of this Dirichlet series. However, Euler confined his analysis to the real line. He was the first to evaluate, to high precision, the values of the series for $s = 2, 3, \ldots, 15, 16$. For example, Euler established the formula,

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

Riemann was the first to make an intensive study of this series as a function of a complex variable.

Euler's most important contribution to the theory of the zeta function is the Euler Product Formula. This formula demonstrates explicitly the connection between prime numbers and the zeta function. Euler noticed, by the Fundamental Theorem of Arithmetic, that every positive integer can be uniquely written as a product of powers of different primes. Thus, for any $n \in \mathbb{N}$, we may write

$$n = \prod_{p_i} p_i^{e_i},$$

where the p_i range over all primes, and the e_i are non-negative integers. The exponents e_i will vary as n varies, but it is clear that if we consider each $n \in \mathbb{N}$ we will use every possible combination of exponents $e_i \in \mathbb{N}$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right),$$

where the infinite product is over all the primes. On examining the convergence of both the infinite series and the infinite product, we easily find

Theorem 2.1 (Euler Product Formula). *For $s = \sigma + it$ and $\sigma > 1$, we have*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2.1.2}$$

The Euler product formula is also called the analytic form of the Fundamental Theorem of Arithmetic. It demonstrates how the Riemann zeta function encodes information on the prime factorization of integers and the distribution of primes.

We can also re-cast one of our earlier observations in terms of the Euler Product Formula. Since convergent infinite products never vanish, it then follows from the formula that,

Theorem 2.2. *For all $s \in \mathbb{C}$ with $\Re(s) > 1$, we have $\zeta(s) \neq 0$.*

We have seen that the Dirichlet series (2.1.1) diverges for any s with $\Re(s) \leq 1$. In particular, when $s = 1$ the series is the harmonic series. Consequently, the Dirichlet series (2.1.1) does not define the Riemann zeta function outside the region $\Re(s) > 1$. We will continue to build the zeta function, as promised. However, first we note that our definition of the zeta function, valid for $\Re(s) > 1$, actually forces the values of $\zeta(s)$ for all $s \in \mathbb{C}$. This is a consequence of the fact that $\zeta(s)$ is analytic for $\Re(s) > 1$, and continues analytically to the entire plane, with one exceptional point, as explained below.

Here we recall that analytic continuation allows us to "continue" an analytic function on one domain to an analytic function of a larger domain, uniquely, under certain conditions. (Specifically, given functions f_1, analytic on domain D_1, and f_2, analytic on domain D_2, such that $D_1 \cap D_2 \neq \phi$ and $f_1 = f_2$ on $D_1 \cap D_2$ then $f_1 = f_2$ on $D_1 \cup D_2$.) So if we can find a function, analytic on $\mathbb{C}\backslash\{1\}$, that agrees with our Dirichlet series on D, then we have succeeded in defining $\zeta(s)$ for all $s \in \mathbb{C}\backslash\{1\}$.

In his 1859 memoir, Riemann proves that the function $\zeta(s)$ can be continued analytically to a meromorphic function over the whole complex plane, with the exception of $s = 1$. At $s = 1$, $\zeta(s)$ has a simple pole, with residue 1.

We now define the Riemann zeta function. Following convention, we write $s = \sigma + it$ when $s \in \mathbb{C}$.

Definition 2.3. *The Riemann zeta function $\zeta(s)$ is the analytic continuation of the Dirichlet series* (2.1.1) *to the whole complex plane, minus the point $s = 1$.*

Defining the zeta function in this way is concise and accurate, but its properties are quite unclear. We now continue to build the zeta function by finding the analytic continuation of $\zeta(s)$ explicitly. To start with, when $\Re(s) > 1$, we can write

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \\ &= s \sum_{n=1}^{\infty} n \int_n^{n+1} x^{-s-1} dx.\end{aligned}$$

Let $x = [x] + \{x\}$, where $[x]$ and $\{x\}$ are the integral and fractional parts of x respectively. Since $[x]$ is always a constant n for any x in the interval $[n, n+1)$ we have

$$\zeta(s) = s \sum_{n=1}^{\infty} \int_n^{n+1} [x] x^{-s-1} dx = s \int_1^{\infty} [x] x^{-s-1} dx.$$

By writing $[x] = x - \{x\}$, we obtain

$$\begin{aligned}\zeta(s) &= s \int_1^{\infty} x^{-s} dx - s \int_1^{\infty} \{x\} x^{-s-1} dx \\ &= \frac{s}{s-1} - s \int_1^{\infty} \{x\} x^{-s-1} dx, \qquad \sigma > 1. \qquad (2.1.3)\end{aligned}$$

We now observe that since $0 \leq \{x\} < 1$, the improper integral in (2.1.3) converges when $\sigma > 0$ because the integral $\int_1^{\infty} x^{-\sigma-1} dx$ converges. Thus the improper integral in (2.1.3) defines an analytic function of s in the region $\Re(s) > 0$. Therefore, the meromorphic function at the right hand side of (2.1.3) gives the analytic continuation of $\zeta(s)$ to the region $\Re(s) > 0$ and the term $\frac{s}{s-1}$ gives the simple pole of $\zeta(s)$ at $s = 1$ with residue 1.

Equation (2.1.3) only extends the definition of the Riemann zeta function to the larger region $\Re(s) > 0$. However, Riemann used a similar argument to obtain the analytic continuation to the whole complex plane. He started from the classical definition of the gamma function.

We recall that the gamma function extends the factorial function to the entire complex plane with the exception of the non-positive integers. For $s \in \mathbb{C}$ we define $\Gamma(s)$ as

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt.$$

The Γ function is analytic on the entire complex plane with the exception of $s = 0, -1, -2, \ldots$, and the residue of $\Gamma(s)$ at $s = -n$ is $\frac{(-1)^n}{n!}$. Note that for $s \in \mathbb{N}$ we have $\Gamma(s) = (s-1)!$.

We now continue following Riemann. We have

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} t^{\frac{s}{2}-1} dt$$

for $\sigma > 0$. On setting $t = n^2 \pi x$, we observe that

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2 \pi x} dx.$$

Hence, with some care on exchanging summation and integration, for $\sigma > 1$,

$$\begin{aligned} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1} \left(\sum_{n=1}^\infty e^{-n^2 \pi x} \right) dx \\ &= \int_0^\infty x^{\frac{s}{2}-1} \left(\frac{\vartheta(x) - 1}{2} \right) dx \end{aligned}$$

where

$$\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}$$

is the Jacobi theta function. The functional equation (also due to Jacobi) for $\vartheta(x)$ is

$$x^{\frac{1}{2}}\vartheta(x) = \vartheta(x^{-1}),$$

and is valid for $x > 0$. This equation is far from obvious; however, the proof lies beyond our focus. The standard proof proceeds using Poisson summation, and can be found in Chapter 2 of [21].

Finally, using the functional equation of $\vartheta(x)$, we obtain

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}} \right) \cdot \left(\frac{\vartheta(x)-1}{2} \right) dx \right\}. \tag{2.1.4}$$

Due to the exponential decay of $\vartheta(x)$, the improper integral in (2.1.4) converges for *every* $s \in \mathbb{C}$ and hence defines an entire function in $\mathbb{C}$. Therefore, (2.1.4) gives the analytic continuation of $\zeta(s)$ to the whole complex plane, with the exception of $s = 1$.

Theorem 2.4. *The function*

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}} \right) \cdot \left(\frac{\vartheta(x)-1}{2} \right) dx \right\}$$

is meromorphic with a simple pole at $s = 1$ with residue 1.

We have now succeeded in our goal of continuing the Dirichlet series that we started with, to $\zeta(s)$, a meromorphic function on $\mathbb{C}$. We can now consider all complex numbers in our search for the zeros of $\zeta(s)$. We are interested in these zeros as they encode information about the prime numbers. However, not all of the zeros of $\zeta(s)$ are of interest to us. Surprisingly we can find, with relative ease, an infinite number of zeros, all lying outside of the region $0 \leq \Re(s) \leq 1$. We will refer to these zeros as the *trivial zeros* of $\zeta(s)$ and we will exclude them from the statement of the Riemann Hypothesis.

Before discussing the zeros of $\zeta(s)$ we will develop a functional equation for it. Riemann noticed that formula (2.1.4) not only gives the analytic continuation of $\zeta(s)$, but can also used to derive a functional equation for $\zeta(s)$. He observed that the term $\frac{1}{s(s-1)}$ and the improper integral in (2.1.4) are invariant under the substitution of s by $1 - s$. Hence we have

Theorem 2.5 (The Functional Equation). *For any s in $\mathbb{C}$,*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

For convenience, and clarity, we will define the xi function as

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{2.1.5}$$

In view of (2.1.4), $\xi(s)$ is an entire function and satisfies the simple functional equation

$$\xi(s) = \xi(1-s). \tag{2.1.6}$$

This shows that $\xi(s)$ is symmetric around the vertical line $\Re(s) = \frac{1}{2}$.

We now have developed the zeta function sufficiently to begin considering its various properties; in particular, the location of its zeros. There are a few assertions we can make based on the elementary theory we have already presented.

We begin our discussion by isolating the trivial zeros of $\zeta(s)$. Recall that the only poles of $\Gamma(s)$ are simple and situated at $s = 0, -1, -2, \ldots$. It follows from (2.1.4) that $\zeta(s)$ has simple zeros at $s = -2, -4, \ldots$ (the pole $s = 0$ of $\Gamma(\frac{s}{2})$ is cancelled by the term $\frac{1}{s(s-1)}$). These zeros, arising from the poles of the Gamma function, are termed the *trivial zeros*. From the functional equation and Theorem 2.2, all other zeros, the *non-trivial zeros*, lie in the vertical strip $0 \leq \Re(s) \leq 1$. In view of (2.1.5), the non-trivial zeros of $\zeta(s)$ are precisely the zeros of $\xi(s)$ and hence they are symmetric about the vertical line $\Re(s) = \frac{1}{2}$. Also, in view of (2.1.4), they are symmetric about the real axis, $t = 0$. We summarize these results in the following theorem.

Theorem 2.6. *The zeros of $\zeta(s)$ satisfy:*

1. *$\zeta(s)$ has no zero for $\Re(s) > 1$;*
2. *the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;*
3. *$\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;*
4. *the non-trivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$, and the real axis $\Im(s) = 0$;*
5. *the zeros of $\xi(s)$ are precisely the non-trivial zeros of $\zeta(s)$.*

The strip $0 \leq \Re(s) \leq 1$ is called the *critical strip* and the vertical line $\Re(s) = \frac{1}{2}$ is called the *critical line*.

Riemann commented on the zeros of $\zeta(s)$ in his memoir (see the statement at the start of Chapter 1). From his statements the Riemann Hypothesis was formulated.

Conjecture 2.7 (The Riemann Hypothesis). *All non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

Riemann's eight page memoir has legendary status in mathematics. It not only proposed the Riemann Hypothesis, but also accelerated the development of analytic number theory. Riemann conjectured the asymptotic formula for the number, $N(T)$, of zeros of $\zeta(s)$ in the critical strip with $0 \le \gamma < T$ to be

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

(proved by von Mangoldt in 1905). Additionally, he conjectured the product representation of $\xi(s)$ to be

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \tag{2.1.7}$$

where A, B are constants and ρ runs over all the non-trivial zeros of $\zeta(s)$ (proved by Hadamard in 1893).

2.2 Zero-free Region

An approach to the Riemann Hypothesis is to expand the zero-free region as much as possible. However, the proof that the zero-free region includes the vertical line $\Re(s) = 1$ (i.e., $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$) is already non-trivial. In fact, this statement is equivalent to the Prime Number Theorem, namely $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$ (a problem that required a century of mathematics to solve). Since we wish to focus our attention here on the analysis of $\zeta(s)$, we refer the reader to proofs of this equivalence in 12.3, 12.4 and 12.16.

Theorem 2.8. *For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.*

Proof. In order to prove this result we follow the 1899 approach of de la Vallée Poussin (see 12.4). Recall that when $\sigma > 1$, the zeta function is defined as a Dirichlet series and that the Euler product formula gives us

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{2.2.1}$$

where $s = \sigma + it$. Taking logarithms of each side of (2.2.1), we obtain

$$\log \zeta(s) = -\sum_{p} \log \left(1 - \frac{1}{p^s}\right).$$

Using the Taylor expansion of $\log(1-x)$ at $x = 0$, we have

$$\begin{aligned}\log\zeta(s) &= \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-sm}\\ &= \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}p^{-imt}\\ &= \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}e^{-imt\log p}.\end{aligned} \tag{2.2.2}$$

It follows that the real part of $\log\zeta(s)$ is

$$\Re(\log\zeta(s)) = \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}\cos(mt\log p). \tag{2.2.3}$$

Note that, by (2.2.3),

$$\begin{aligned}&3\Re(\log\zeta(\sigma)) + 4\Re(\log\zeta(\sigma+it)) + \Re(\log\zeta(\sigma+2ti))\\ &\quad= 3\sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m} + 4\sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}\cos(mt\log p)\\ &\qquad+ \sum_p \sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}\cos(2mt\log p)\\ &\quad= \sum_p \sum_{m=1}^{\infty} \frac{1}{m}\frac{1}{p^{\sigma m}}\Big(3 + 4\cos(mt\log p) + \cos(2mt\log p)\Big).\end{aligned}$$

Using $\log W = \log|W| + i\arg(W)$ and the elementary inequality

$$2(1+\cos\theta)^2 = 3 + 4\cos\theta + \cos(2\theta) \geqslant 0, \tag{2.2.4}$$

valid for any $\theta\in\mathbb{R}$, we obtain

$$3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2ti)| \geqslant 0,$$

or equivalently,

$$|\zeta(\sigma)|^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geqslant 1. \tag{2.2.5}$$

Since $\zeta(\sigma)$ has a simple pole at $\sigma = 1$ with residue 1, the Laurent series of $\zeta(\sigma)$ at $\sigma = 1$ is

$$\zeta(\sigma) = \frac{1}{1-\sigma} + a_0 + a_1(\sigma-1) + a_2(\sigma-1)^2 + \cdots = \frac{1}{1-\sigma} + g(\sigma),$$

where $g(\sigma)$ is analytic at $\sigma = 1$. Hence, for $1 < \sigma \le 2$, we have $|g(\sigma)| \le A_0$, for some $A_0 > 0$, and

$$|\zeta(\sigma)| = \frac{1}{1-\sigma} + A_0.$$

Now we will show that $\zeta(1+it) \neq 0$ by using inequality (2.2.5). To obtain a contradiction, suppose that there is a zero on the line $\sigma = 1$. So $\zeta(1+it) = 0$ for some $t \in \mathbb{R}$, $t \neq 0$. Then by the mean-value theorem,

$$\begin{aligned} |\zeta(\sigma+it)| &= |\zeta(\sigma+it) - \zeta(1+it)| \\ &= |\sigma - 1||\zeta'(\sigma_0 + it)|, \qquad 1 < \sigma_0 < \sigma \\ &\le A_1(\sigma - 1), \end{aligned}$$

where A_1 is a constant depending only on t. Also, when σ approaches 1 we have $|\zeta(\sigma+2it)| < A_2$, where A_2 again depends only on t. Note that in (2.2.5) the degree of the term $\sigma - 1$, which is 4, is greater than that of the term $\frac{1}{\sigma-1}$, which is 3. So for fixed t, as σ tends to 1^+, we have

$$\begin{aligned} &\lim_{\sigma \to 1^+} |\zeta(\sigma)|^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)| \\ &\qquad \le \lim_{\sigma \to 1^+} \left(\frac{1}{\sigma - 1} + A_0 \right)^3 {A_1}^4 (\sigma-1)^4 A_2 \\ &\qquad = 0. \end{aligned}$$

This contradicts (2.2.5). Hence we conclude that $\zeta(1+it) \neq 0$ for any $t \in \mathbb{R}$. □

This result gives us a critical part of the proof of the Prime Number Theorem, by extending the zero-free region of $\zeta(s)$ to include the line $\Re(s) = 1$. It would seem intuitive that by extending the zero-free region even further we could conclude other powerful results on the distribution of the primes. In fact, it can be proved more explicitly that the asymptotic formula,

$$\pi(x) = \mathrm{Li}(x) + O(x^{\Theta} \log x),$$

is equivalent to

$$\zeta(\sigma + it) \neq 0, \text{ for } \sigma > \Theta, \tag{2.2.6}$$

where $\frac{1}{2} \le \Theta \le 1$. In particular, the Riemann Hypothesis (that is $\Theta = 1/2$ in (2.2.6)) is equivalent to the statement,

$$\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}} \log x).$$

This formulation gives an immediate method by which to expand the zero-free region. However, we are still unable to improve the zero-free region in the form (2.2.6) for any $\Theta < 1$. The best result to date, proved by Vinogradov and Korobov independently in 1958, is that $\zeta(s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{c(\alpha)}{(\log |t| + 1)^{\alpha}}$$

for any $\alpha > 2/3$ [80, 156].

2.3 Counting the Zeros of $\zeta(s)$

Proving that the non-trivial zeros of $\zeta(s)$ have real part $\frac{1}{2}$, and proving that the non-trivial zeros of $\zeta(s)$ lie outside a prescribed region, are both, presumably, extremely difficult. However we can still glean valuable heuristic evidence for the Riemann Hypothesis by counting non-trivial zeros. We can develop tools that allow us to count the number of zeros in the critical strip with imaginary part $|\Im(s)| < T$ for any finite, positive, real T. Once we know how many zeros should lie in a given region, we can verify the Riemann Hypothesis in that region computationally. In this section we develop the theory that will allow us to make this argument.

We begin with the argument principle. The argument principle in complex analysis gives a very useful tool to count the zeros or the poles of a meromorphic function inside a specified region. For a proof of this well-known result in complex analysis see §79 of [31].

The Argument Principle. *Let f be meromorphic in a domain interior to a positively oriented simple closed contour C such that f is analytic and non-zero on C. Then,*

$$\frac{1}{2\pi} \Delta_C \arg(f(z)) = Z - P$$

where Z is the number of zeros, P is the number of poles of $f(z)$ inside C, counting multiplicities, and $\Delta_C \arg(f(s))$ counts the changes in the argument of $f(s)$ along the contour C.

We apply this principle to count the number of zeros of $\zeta(s)$ within the rectangle $\{\sigma + it \in \mathbb{C} : 0 < \sigma < 1, 0 \leq t < T\}$. We denote this number by,

$$N(T) := |\{\sigma + it : 0 < \sigma < 1, 0 \leq t < T, \zeta(\sigma + it) = 0\}|.$$

As previously mentioned, the following theorem was conjectured by Riemann [125] and proved by von Mangoldt [159]. We follow the exposition of Davenport from [40]. We present the proof in more detail than is typical in this book, as it exemplifies the general form of arguments in this field. Such arguments can be relatively complicated and technical; the reader is forewarned.

Theorem 2.9. *With $N(T)$ as defined above, we have*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Proof. Instead of working directly with $\zeta(s)$, we will make use of the $\xi(s)$ function, defined in (2.1.5). Since $\xi(s)$ has the same zeros as $\zeta(s)$ in the critical strip, and $\xi(s)$ is an entire function, we can apply the argument principle to $\xi(s)$ instead of $\zeta(s)$. Let R be the positively oriented rectangular contour with vertices $-1, 2, 2 + iT$ and $-1 + iT$. By the argument principle we have,

$$N(T) = \frac{1}{2\pi} \Delta_R \arg(\xi(s)).$$

We now divide R into three sub-contours. Let L_1 be the horizontal line segment from -1 to 2. Let L_2 be the contour consisting of the vertical line segment from 2 to $2+iT$ and then the horizontal line segment from $2+iT$ to $\frac{1}{2}+iT$. Finally, let L_3 be the contour consisting of the horizontal line segment from $\frac{1}{2} + iT$ to $-1 + iT$ and then the vertical line segment from $-1 + iT$ to -1. Now,

$$\Delta_R \arg(\xi(s)) = \Delta_{L_1} \arg(\xi(s)) + \Delta_{L_2} \arg(\xi(s)) + \Delta_{L_3} \arg(\xi(s)). \qquad (2.3.1)$$

We wish to trace the argument change of $\xi(s)$ along each contour.

To begin with, there is no argument change along L_1 since the values of $\xi(s)$ here are real, and hence all arguments of $\xi(s)$ are zero. Thus,

$$\Delta_{L_1} \arg(\xi(s)) = 0. \qquad (2.3.2)$$

From the functional equation for $\xi(s)$ we have,

$$\xi(\sigma + it) = \xi(1 - \sigma - it) = \overline{\xi(1 - \sigma + it)}.$$

So the argument change of $\xi(s)$ as s moves along L_3 is the same as the argument change of $\xi(s)$ as s moves along L_2. Hence, in conjunction with (2.3.2), (2.3.1) becomes,

$$N(T) = \frac{1}{2\pi} 2\Delta_{L_2} \arg(\xi(s)) = \frac{1}{\pi}\Delta_{L_2} \arg(\xi(s)). \tag{2.3.3}$$

From the definition (2.1.5), and the basic relation $z\Gamma(z) = \Gamma(z+1)$, we have

$$\xi(s) = (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s). \tag{2.3.4}$$

We next work out the argument changes, along L_2, of these four factors of the right hand side of (2.3.4) separately.

We begin by considering $\Delta_{L_2} \arg(s-1)$,

$$\begin{aligned}
\Delta_{L_2} \arg(s-1) &= \arg\left(-\frac{1}{2}+iT\right) - \arg(1) \\
&= \arg\left(-\frac{1}{2}+iT\right) \\
&= \frac{\pi}{2} + \arctan\left(\frac{1}{2T}\right) \\
&= \frac{\pi}{2} + O(T^{-1})
\end{aligned}$$

because $\arctan\left(\frac{1}{2T}\right) = O(T^{-1})$.

Next we consider $\Delta_{L_2} \arg(\pi^{-\frac{s}{2}})$,

$$\begin{aligned}
\Delta_{L_2} \arg(\pi^{-\frac{s}{2}}) &= \Delta_{L_2} \arg\left(\exp\left(-\frac{s}{2}\log\pi\right)\right) \\
&= \arg\left(\exp\left(-\frac{1}{2}\left(-\frac{1}{2}+iT\right)\log\pi\right)\right) \\
&= \arg\left(\exp\left(\frac{1-2iT}{4}\log\pi\right)\right) \\
&= -\frac{T}{2}\log\pi.
\end{aligned}$$

Now we use Stirling's formula to give an asymptotic estimate for $\Gamma(s)$, (see (5) of Chapter 10 in [40]),

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O(|s|^{-1}), \tag{2.3.5}$$

which is valid when $|s|$ tends to ∞ and the argument satisfies $-\pi+\delta < \arg(s) < \pi-\delta$ for any fixed $\delta > 0$. So we have,

$$\begin{aligned}\Delta_{L_2} \arg\left(\Gamma\left(\frac{1}{2}s+1\right)\right) &= \Im\left(\log \Gamma\left(\frac{5}{4}+\frac{iT}{2}\right)\right) \\ &= \Im\left\{\left(\frac{3}{4}+i\frac{T}{4}\right)\log\left(\frac{5}{4}+i\frac{T}{2}\right)-\frac{5}{4}-\frac{iT}{2}+\frac{1}{2}\log 2\pi + O(T^{-1})\right\} \\ &= \frac{1}{2}T\log\frac{T}{2}-\frac{1}{2}T+\frac{3}{8}\pi+O(T^{-1}).\end{aligned}$$

Putting all these together we obtain, from (2.3.3),

$$\begin{aligned}N(T) &= \frac{1}{\pi}\Delta_{L_2}\arg\xi(s) \\ &= \frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+\mathcal{S}(T)+O(T^{-1}),\end{aligned} \tag{2.3.6}$$

where

$$\mathcal{S}(T) = \frac{1}{\pi}\Delta_{L_2}\arg\zeta(s) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right). \tag{2.3.7}$$

It now remains to estimate $\mathcal{S}(T)$.

Taking the logarithmic derivative of the Hadamard product representation (2.1.7), we obtain

$$\begin{aligned}\frac{\xi'(s)}{\xi(s)} &= B+\sum_\rho \frac{\frac{-1}{\rho}}{1-\frac{s}{\rho}}+\sum_\rho\frac{1}{\rho} \\ &= B+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).\end{aligned} \tag{2.3.8}$$

Since $\xi(s)$ is alternatively defined by (2.1.5), we also have

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s}+\frac{1}{s-1}+\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}s\right)+\frac{\zeta'(s)}{\zeta(s)}. \tag{2.3.9}$$

Now combining (2.3.8) and (2.3.9) we have,

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1}-B-\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}s+1\right)-\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Given $t \geq 2$ and $1 \leq \sigma \leq 2$, by Stirling's formula for $\Gamma(s)$,

$$\left|\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}s+1\right)\right| \le A_3 \log t,$$

so we have

$$-\Re\left(\frac{\zeta'}{\zeta}(s)\right) \le A_4 \log t - \sum_\rho \Re\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \tag{2.3.10}$$

If $s=\sigma+it$, $2\le t$ and $1<\sigma\le 2$, then

$$-\Re\left(\frac{\zeta'}{\zeta}(s)\right) < A_4 \log t - \sum_\rho \Re\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \tag{2.3.11}$$

Since $\frac{\zeta'}{\zeta}(s)$ is analytic at $s=2+iT$,

$$-\Re\left(\frac{\zeta'}{\zeta}(2+iT)\right) \le A_5 \tag{2.3.12}$$

for some positive absolute constant A_5. If $\rho=\beta+i\gamma$ and $s=2+iT$, then

$$\begin{aligned}
\Re\left(\frac{1}{s-\rho}\right) &= \Re\left(\frac{1}{2-\beta+i(T-\gamma)}\right)\\
&= \frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2}\\
&\ge \frac{1}{4+(T-\gamma)^2}\\
&\gg \frac{1}{1+(T-\gamma)^2}
\end{aligned}$$

as $0<\beta<1$. Also we have $\Re(\frac{1}{\rho})=\frac{\beta}{\beta^2+\gamma^2}\ge 0$, so by (2.3.11) and (2.3.12),

$$\sum_\rho \frac{1}{1+(T-\gamma)^2} \ll \sum_\rho \Re\left(\frac{1}{s-\rho}\right) \ll \log T.$$

We have proven that for $T\ge 1$,

$$\sum_\rho \frac{1}{1+(T-\gamma)^2} = O(\log T). \tag{2.3.13}$$

It immediately follows that

$$\begin{aligned}\Big|\Big\{\rho = \beta + i\gamma : 0 < \beta < 1, T \le \gamma \le T+1, \zeta(\rho) = 0\Big\}\Big| \\ &\le 2 \sum_{T\le\gamma\le T+1} \frac{1}{1+(T-\gamma)^2} \\ &\le 2 \sum_{\rho} \frac{1}{1+(T-\gamma)^2} \ll \log T. \qquad (2.3.14)\end{aligned}$$

For large t and $-1 \le \sigma \le 2$,

$$\frac{\zeta'}{\zeta}(s) = O(\log t) + \sum_{\rho} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right). \qquad (2.3.15)$$

When $|\gamma - t| > 1$, we have

$$\left|\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right| = \frac{2-\sigma}{|(s-\rho)(2+it-\rho)|} \le \frac{3}{(\gamma-t)^2},$$

and therefore,

$$\left|\sum_{|\gamma-t|>1} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right)\right| \le \sum_{|\gamma-t|>1} \frac{3}{(\gamma-t)^2} \ll \sum_{\rho} \frac{1}{1+(\gamma-t)^2} \ll \log t.$$

This, combined with (2.3.15), gives

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma-t|\le 1} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) + O(\log t). \qquad (2.3.16)$$

Next, from (2.3.7), we have

$$\pi \mathcal{S}(T) = \arg \zeta\left(\frac{1}{2} + iT\right) = \int_{\frac{1}{2}+iT}^{2+iT} \frac{\zeta'}{\zeta}(s) ds = \log \zeta(s)\Big|_{\frac{1}{2}+iT}^{2+iT}. \qquad (2.3.17)$$

Since $\log \omega = \log|\omega| + i \arg \omega$,

$$\begin{aligned}-\int_{\frac{1}{2}+iT}^{2+iT} \Im\left(\frac{\zeta'}{\zeta}(s)\right) ds &= -\arg(\zeta(s))\Big|_{\frac{1}{2}+iT}^{2+iT} \\ &= -\arg(\zeta(2+iT)) + \arg\left(\zeta\left(\frac{1}{2}+iT\right)\right).\end{aligned}$$

Therefore we have,

$$-\int_{\frac{1}{2}+iT}^{2+iT} \Im \frac{\zeta'}{\zeta}(s)ds = O(1) + \pi \mathcal{S}(T).$$

Thus, with (2.3.16) and (2.3.17),

$$\mathcal{S}(T) \ll \sum_{|\gamma - t| < 1} \int_{\frac{1}{2}+iT}^{2+iT} \Im \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) ds + \log T,$$

and we have

$$\int_{\frac{1}{2}+iT}^{2+iT} \Im \left(\frac{1}{s-\rho} \right) ds = \Im \log(s-\rho) \Big|_{\frac{1}{2}+iT}^{2+iT} = \arg(s-\rho) \Big|_{\frac{1}{2}+iT}^{2+iT} \ll 1.$$

By (2.3.14) we obtain that,

$$\mathcal{S}(T) \ll \sum_{|\gamma - T| < 1} 1 + \log T \ll \log T,$$

and finally that,

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

This completes the proof.

□

We now have the theoretical underpinnings for verifying the Riemann Hypothesis computationally. We can adapt these results to give us a feasible method of counting the number of zeros of $\zeta(s)$ in the critical strip up to any desired height. Combined with efficient methods of computing values of $\zeta(\frac{1}{2} + it)$ we can gain valuable heuristic evidence in favour of the Riemann Hypothesis and, in part, justify over a century of belief in its truth. For a detailed development of these ideas, see Chapter 3.

2.4 Hardy's Theorem

Hardy's Theorem is one of the first important results in favour of the Riemann Hypothesis. It establishes a fact that is the most basic necessary condition for the Riemann Hypothesis to be true. Consequently it gives the first critical result for belief in the Hypothesis. We will now present and prove Hardy's Theorem. For Hardy's original paper, see 12.5, we follow the exposition of Edwards [48].

Theorem 2.10 (Hardy's Theorem). *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

Proof. The idea of the proof is to apply Mellin's inversion to $\xi(s)$, namely if

$$F(s) = \int_{-\infty}^{\infty} f(x)x^{-s}dx,$$

then,

$$f(s) = \int_{a-i\infty}^{a+i\infty} F(z)z^{s-1}dz.$$

For an introduction to Mellin's inversion see §2.7 of [146].

Recall equation (2.1.4),

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)} + \int_1^{\infty}\left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}}\right)\cdot\left(\frac{\vartheta(x)-1}{2}\right)dx\right\},$$

and (2.1.5),

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

After some substitution and simplification, we realize

$$\frac{2\xi(s)}{s(s-1)} = \int_0^{\infty} u^{-s}\left(\vartheta(u^2) - 1 - \frac{1}{u}\right)du$$

for $0 < \Re(s) < 1$. Application of Mellin's inversion formula gives

$$\vartheta(z^2) - 1 - \frac{1}{z} = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{2\xi(s)}{s(s-2)}z^{s-1}ds. \tag{2.4.1}$$

We first note that the function $\vartheta(z^2) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 z^2}$ is not only defined for $z \in \mathbb{R}$, but also for $z \in W$ where $W := \{z \in \mathbb{C} : -\frac{\pi}{4} < arg(z) < \frac{\pi}{4}\}$. We claim that $\vartheta(z^2)$ and all its derivatives approach zero as s approaches $e^{i\pi/4}$ (e.g., along the circle $|z| = 1$ in W). In fact, using the functional equation $z^{\frac{1}{2}}\vartheta(z) = \vartheta(z^{-1})$, we have

$$\begin{aligned}
\vartheta(z^2) &= \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi n^2 (z^2 - i)} \\
&= -\vartheta(z^2 - i) + 2\vartheta(4(z^2 - i)) \\
&= -\frac{1}{(z^2 - i)^{1/2}} \vartheta\left(\frac{1}{z^2 - i}\right) + \frac{1}{(z^2 - i)^{1/2}} \vartheta\left(\frac{1}{4(z^2 - i)}\right) \\
&= \frac{1}{(z^2 - i)^{1/2}} \sum_{n \ odd} e^{-\frac{\pi n^2}{4(z^2 - i)}}.
\end{aligned}$$

Since $u^k e^{-1/u}$ approaches zero as $u \longrightarrow 0^+$ for any integer k, $\vartheta(z^2)$ and all its derivatives approach zero as z approaches $e^{i\pi/4}$ in the wedge W.

Consider now the integral of the right hand side of (2.4.1). In view of (2.1.3), we have

$$\begin{aligned}
\left|\zeta\left(\frac{1}{2} + it\right)\right| &= \left|\frac{1/2 + it}{-1/2 + it} - (1/2 + it) \int_1^{\infty} \{x\} x^{-\frac{1}{2} - it - 1} dx\right| \\
&\leq 1 + |1/2 + it| \int_1^{\infty} x^{-3/2} dx \\
&\ll |t|
\end{aligned}$$

as $t \longrightarrow \pm\infty$. Also, from formula (2.3.5) for the Gamma function, we have

$$\log \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right) = \left(-\frac{1}{4} + i\frac{t}{2}\right) \log\left(\frac{1}{4} + i\frac{t}{2}\right) - \frac{1}{2} - i\frac{t}{2} + \frac{1}{2}\log 2\pi + O(|t|^{-1}).$$

It follows that

$$\left|\Gamma\left(\frac{1}{4} + i\frac{t}{2}\right)\right| \ll e^{-\frac{\pi}{4}|t|}$$

as $t \longrightarrow \pm\infty$. Thus, we have

$$\left|\xi\left(\frac{1}{2} + it\right)\right| \ll |t|^3 e^{-\frac{\pi}{4}|t|}$$

which shows that the integral in the right hand side of (2.4.1) converges in the wedge W, and hence is analytic in W.

Formula (2.4.1) takes a more simple form if the operator $z(d^2/dz^2)z$ is applied to both sides to give

$$H(z) = \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} 2\xi(s) z^{s-1} ds,$$

where

$$H(z) = z\frac{d^2}{dz^2}z\vartheta(z^2).$$

This shows, similarly to $\vartheta(z^2)$, that $H(z)$ and all of its derivatives approach zero as z approaches $e^{i\pi/4}$. Next, by using the Taylor series of the exponential function, we obtain

$$\begin{aligned}
z^{1/2}H(z) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\xi\left(\frac{1}{2}+it\right)z^{it}dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\xi\left(\frac{1}{2}+it\right)\sum_{n=0}^{\infty}\frac{(it\log z)^n}{n!}dt \\
&= \sum_{n=0}^{\infty}c_n(i\log z)^n
\end{aligned}$$

where

$$c_n = \frac{1}{\pi n!}\int_{-\infty}^{\infty}\xi\left(\frac{1}{2}+it\right)t^n dt.$$

Since $\xi(s)$ is symmetric about the critical line, $c_{2n+1} = 0$ for all n.

Suppose Hardy's theorem is false. Then $\xi\left(\frac{1}{2}+it\right)$ has a finite number of zeros, and $\xi\left(\frac{1}{2}+it\right)$ does not change sign for all large $t \in \mathbb{R}$. For example, if $\xi\left(\frac{1}{2}+it\right)$ is positive for $t \geq T$, then

$$\begin{aligned}
c_{2n}\pi(2n)! &= 2\int_{0}^{\infty}\xi\left(\frac{1}{2}+it\right)t^{2n}dt \\
&\geq 2\int_{0}^{T+2}\xi\left(\frac{1}{2}+it\right)t^{2n}dt \\
&\geq 2\left\{-\int_{0}^{T}\left|\xi\left(\frac{1}{2}+it\right)\right|T^{2n}dt + \int_{T+1}^{T+2}\xi\left(\frac{1}{2}+it\right)(T+1)^{2n}dt\right\} \\
&\geq A_6(T+1)^{2n} - A_7T^{2n}
\end{aligned}$$

for some positive constants A_6 and A_7. Hence, $c_{2n} > 0$ for all sufficiently large n. Similarly, if $\xi\left(\frac{1}{2}+it\right)$ is negative for $t \geq T$, then $c_{2n} < 0$ for all sufficiently large n. Thus, if the function $z^{1/2}H(z)$ is differentiated sufficiently many times with respect to $i\log z$, then the right hand side becomes an even power series in which every term has the same sign. Consequently, as z approaches $e^{i\pi/4}$, the value of this even power series does not approach zero. However, it must approach zero as $d/d(i \log z) = -izd/dz$, and differentiating repeatedly carries $z^{1/2}H(z)$ to a function which approaches zero as z approaches $e^{i\pi/4}$. This contradiction proves Hardy's theorem.

$\square$

As we noted previously, Hardy's Theorem establishes a minimal necessary condition for the truth of the Riemann Hypothesis. Mathematicians have continued to attack the Riemann Hypothesis from this angle, by proving necessary conditions of increasing difficulty. Selberg proved that a positive proportion of the zeros of $\zeta(s)$ lie on the critical line (a result that, in part, earned him the Fields Medal in 1950) [133]. This was improved by Levinson to the result that $\frac{1}{3}$ of the zeros lie on the critical line [89]. The best and most recent result in this vein is due to Conrey who proved that $\frac{2}{5}$ of the zeros lie on the critical line [33].

3

Algorithms for Calculating $\zeta(s)$

The Riemann Hypothesis has been open since its "proposal" in 1859. The most direct form of attack is to compute: to search for a counter-example, and also to search computationally for insight. Mathematicians from the time of Riemann have developed a large body of computational techniques, and evidence in support of the Riemann Hypothesis. The advent of modern computing empowered mathematicians with new tools, and motivated the development of more efficient algorithms for the computation of $\zeta(s)$.

This chapter presents a basic sketch of the analytic ideas on which these algorithms are based, the three main algorithms used to compute values of $\zeta(s)$, and the analysis required to verify the Riemann Hypothesis within large regions of the critical strip. For additional details, which are sometimes formidable, the reader is directed to the sources cited within.

3.1 Euler-Maclaurin Summation

The zeta function is difficult to evaluate, especially in the critical strip. For this reason sophisticated methods have been developed and applied in order to perform these evaluations. Euler-Maclaurin summation was one of the first methods used to compute values of $\zeta(s)$. Euler used it in the computation of $\zeta(n)$ for $n = 2, 3, \ldots, 15, 16$. It was also used by Gram, Backlund and Hutchinson in order to verify the Riemann Hypothesis for $t \leq 50$, $t \leq 200$ and $t \leq 300$ respectively ($s = \frac{1}{2} + it$).

Euler-Maclaurin Evaluation of $\zeta(s)$. *For $N \geq 1$ we have,*

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{N-1} n^{-s} + \int_N^\infty x^{-s}dx + \frac{1}{2}N^{-s} - s\int_N^\infty B_1(\{x\})x^{-s-1}dx \\ &= \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} + \frac{B_2}{2}sN^{-s-1} + \cdots \\ &\qquad \cdots + \frac{B_{2v}}{(2v)!}s(s+1)\cdots(s+2v-2)N^{-s-2v+1} + R_{2v}\end{aligned}$$

where

$$R_{2v} = -\frac{s(s+1)\cdots(s+2v-1)}{(2v)!}\int_N^\infty B_{2v}(\{x\})x^{-s-2v}dx$$

Here $B_i(x)$ denotes the i^{th} Bernoulli polynomial, B_i denotes the i^{th} Bernoulli number, and $\{x\}$ denotes the fractional part of x.

Definition 3.1. *[48] The Bernoulli Numbers, B_i, are defined by the generating function,*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

The n^{th} Bernoulli Polynomial, $B_n(x)$, is given by

$$B_n(x) := \sum_{j=0}^{n} \binom{n}{j} B_{n-j}x^j.$$

If N is at all large, say N is approximately the same size as $|s|$, then the terms of the series decrease quite rapidly at first and it is natural to expect that the remainder R_{2v} will be very small. In fact, we have that

$$|R_{2v}| \leq \left| \frac{s(s+1)\cdots(s+2v+1)B_{2(v+1)}N^{-\sigma-2v-1}}{2(v+1)!(\sigma+2v+1)} \right|.$$

The application of Euler-Maclaurin summation to $\zeta(s)$ is discussed at length in chapter 6 of [48].

3.2 Backlund

Around 1912, Backlund developed a method of determining the number of zeros of $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$ up to a given height [48, p. 128]. This is part of proving that the first N zeros of $\zeta(s)$ all lie on the critical line.

Recall from Chapter 2 that $N(T)$ denotes the number of zeros (counting multiplicities) of $\zeta(s)$ in R, the rectangle $\{0 \leq \Re(s) \leq 1,\ 0 \leq \Im(s) \leq T\}$, and ∂R is the boundary of R, oriented positively. In [126] Riemann observed that,

$$N(T) = \frac{1}{2\pi i}\int_{\partial R} \frac{\xi'(s)}{\xi(s)} ds$$

and proposed the estimate,

$$N(T) \sim \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}.$$

This estimate was first proved by von Mangoldt in 1905 [159]. In [10] Backlund obtained the specific estimate,

$$\left| N(T) - \left(\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}\right)\right| < 0.137\log T + 0.443\log\log T + 4.350$$

for all $T \geq 2$.

This estimate can be deduced by using good estimates to R_{2v} as presented in the preceeding section. Current computational methods have superceded this estimate.

3.3 Hardy's Function

One way of computing the zeros of any real-valued function is to find small intervals in which that function changes sign. This method cannot be applied naively to arbitrary complex-valued functions. So, in order to calculate zeros of $\zeta(s)$ which lie on the critical line we would like to find a real-valued function whose zeros are exactly the zeros of $\zeta(s)$ on the critical line. This is achieved by considering the function $\xi(s)$. In particular, we recall that $\xi(s)$ satisfies $\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ (see (2.1.5)). Since $\xi(s)$ is a real-valued function on the critical line, we will find the zeros of $\xi(s)$ by determining where the function changes sign. We develop Hardy's function following [48, p. 119] (also see Section 2.1): for $s = \frac{1}{2} + it$, we have

$$\begin{aligned}
\xi(s) &= \left(\frac{1}{4} + \frac{it}{2}\right)\left(-\frac{1}{2} + it\right)\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\zeta\left(\frac{1}{2} + it\right) \\
&= -\frac{1}{2}\left(\frac{1}{4} + t^2\right)\pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\zeta\left(\frac{1}{2} + it\right) \\
&= \left[e^{\Re\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)} \cdot \pi^{-\frac{1}{4}}\left(\frac{-t^2}{2} - \frac{1}{8}\right)\right] \times \left[e^{i\Im\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)} \cdot \pi^{-\frac{it}{2}}\zeta\left(\frac{1}{2} + it\right)\right].
\end{aligned}$$

Now since the factor in the first set of brackets is always a negative real number, the sign of $\xi(\frac{1}{2}+it)$ depends entirely on the sign of the second factor. We define Hardy's Z-function to be

$$Z(t) := e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right)$$

where $\theta(t)$ is given by

$$\theta(t) := \Im \log \Gamma\left(\frac{1}{4}+\frac{it}{2}\right) - \frac{t}{2}\log \pi.$$

Now it is clear that the sign of $\xi(\frac{1}{2}+it)$ is opposite that of $Z(t)$.

At this point it seems that since $\zeta(\frac{1}{2}+it)$ appears in the formula for $Z(t)$ we have not gained any interesting information. However, the Riemann-Siegel formula which appears in the next section allows for efficient computation of $Z(t)$ to determine the zeros of $\zeta(\frac{1}{2}+it)$.

3.4 The Riemann-Siegel Formula

The Riemann-Siegel Formula was discovered amongst Riemann's private papers by Siegel in 1932 [135]. It gives an improvement over Euler-Maclaurin summation in approximating values of $\zeta(s)$. It aided computation, and added to the empirical evidence for the Riemann Hypothesis. However, it does not give enough information by itself to be used for direct verification of the Hypothesis. This is because the formula only gives a fast method of finding zeros of $\zeta(s)$ for which $\Re(s) = \frac{1}{2}$.

We begin with the approximate functional equation for the Riemann zeta function.

Theorem 3.2 (Approximate Functional Equation). *Let $x, y \in \mathbb{R}^+$ with $2\pi xy = |t|$; then for $s = \sigma + it$ with $0 \le \sigma \le 1$, we have*

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} + \chi(s)\sum_{n \le y}\frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}),$$

where $\chi(s)$ is given by

$$\chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s).$$

The approximate functional equation can be used to calculate $Z(t)$ in the following way. First we set $x = y = \sqrt{|t|/2\pi}$, this yields,

$$\zeta(s) = \sum_{n=1}^{\lfloor x \rfloor} \frac{1}{n^s} + \chi(s) \sum_{n=1}^{\lfloor x \rfloor} \frac{1}{n^{1-s}} + E_m(s).$$

Here the error term $E_m(s)$ satisfies $E_m(s) = O(|t|^{-\sigma/2})$. Now we substitute $s = \frac{1}{2} + it$ and multiply by $e^{i\theta(t)}$ to obtain,

$$\begin{aligned} Z(t) &= e^{i\theta(t)} \sum_{n=1}^{\lfloor x \rfloor} \frac{1}{n^{\frac{1}{2}+it}} + e^{-i\theta(t)} \sum_{n=1}^{\lfloor x \rfloor} \frac{1}{n^{\frac{1}{2}-it}} + O(t^{-\frac{1}{4}}) \\ &= 2 \sum_{n=1}^{\lfloor x \rfloor} \frac{\cos(\theta(t) - t \log n)}{n^{\frac{1}{2}}} + O(t^{-\frac{1}{4}}). \end{aligned}$$

This is the basis of the Riemann-Siegel formula. All that is required to apply this formula to compute $Z(t)$ is a more precise formula for the error term. The formula is derived at length in Chapter 7 of [48].

The Riemann-Siegel Formula. *For all $t \in \mathbb{R}$,*

$$Z(t) = 2 \sum_{n=1}^{N} \left(\frac{\cos\left(\theta(t) - t \log n\right)}{n^{\frac{1}{2}}} \right) + \frac{e^{-i\theta(t)} e^{-\frac{t\pi}{2}}}{(2\pi)^{\frac{1}{2}+it} e^{-\frac{i\pi}{4}} (1 - ie^{-t\pi})} \int_{C_N} \frac{(-x)^{-\frac{1}{2}+it} e^{-Nx} dx}{e^x - 1},$$

where C_N is a positively oriented closed contour containing all of the points $\pm 2\pi i N$, $\pm 2\pi i (N-1)$, $\ldots$, $\pm 2\pi i$ and 0.

In practice, algorithms use the truncated series, and a numerical approximation to the integral, in order to realize a useful approximation to $Z(t)$ and thus to $\zeta(\frac{1}{2} + it)$. Note that the term "Riemann-Siegel Formula" can refer to the above formula, or to the Approximate Functional Equation given at the start of the section, depending on the source.

3.5 Gram's Law

We have that $Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right)$, from which it easily follows that

$$\begin{aligned} \zeta\left(\frac{1}{2} + it\right) &= e^{-i\theta(t)} Z(t) \\ &= Z(t) \cos\theta(t) - iZ(t) \sin\theta(t). \end{aligned}$$

Thus,

$$\Im\,\zeta\left(\frac{1}{2}+it\right)=-Z(t)\sin\theta(t).$$

Now the sign changes of $\zeta(\frac{1}{2}+it)$ depend on the sign changes of $Z(t)$ and $\sin\theta(t)$. Gram showed that we can find the zeros of $\sin\theta(t)$ relatively easily; we call these points "Gram points". We define the n^{th} Gram point, g_n, to be the unique real number satisfying $\theta(g_n)=n\pi$. This definition leads us to the formulation of Gram's Law.

Gram's Law. *Hardy's function, $Z(t)$, satisfies $(-1)^n Z(g_n)>0$ at the Gram points g_n.*

Although named Gram's Law, this statement was originally formulated by Hutchinson [70], and is frequently broken (although up to large values of t exceptions are surprisingly few).

Gram points which conform to Gram's Law are called "good" and those which do not are called "bad". We give the definition of Gram blocks first formulated by Rosser et al. [128]. A Gram block of length k is an interval $B_j=[g_j,g_{j+k})$ such that g_j and g_{j+k} are good Gram points and $g_{j+1},\ldots,g_{j+k-1}$ are bad Gram points. These blocks were introduced to deal with exceptions to Gram's Law [25]. This motivates the formulation of Rosser's Rule,

Rosser's Rule. *The Gram block B_m satisfies Rosser's Rule if $Z(t)$ has at least k zeros in $B_m=[g_m,g_{m+k})$.*

Rosser's Rule was proven to fail infinitely often by Lehman in 1970 [86], however it still provides a useful tool for counting zeros of $\zeta(s)$ on the critical line. Brent, in [25], gives the following result.

Theorem 3.3. *If K consecutive Gram blocks with union $[g_n,g_p)$ satisfy Rosser's Rule, where*

$$K\geq 0.0061\ln^2(g_p)+0.08\ln(g_p),$$

then

$$N(g_n)\leq n+1\quad \textit{and}\quad p+1\leq N(g_p).$$

These rules lead to computationally feasible methods of verifying the Riemann Hypothesis up to a desired height.

3.6 Turing

Alan Turing [149] presented an algorithm that gave higher precision calculations of the zeta function. However, this development was made redundant with better estimates for the error terms in the Riemann-Siegel formula [118].

Turing also developed a method for calculating $N(T)$ that is much better than Backlund's method outlined in section 3.2. Turing's method only requires information about the behaviour of $\zeta(s)$ on the critical line.

Let g_n denote the n^{th} Gram point, and suppose g_n is a Gram point at which Gram's law fails. So $(-1)^n Z(g_n) > 0$ fails. We want to find a point close to g_n, $g_n + h_n$, such that $(-1)^n Z(g_n + h_n) > 0$. Turing showed that if $h_m = 0$ and if the values of h_n for n near m are not too large, then $N(g_m)$ must take the value $N(g_m) = m + 1$ [48].

Let $S(N)$ denote the error in the approximation $N(T) \sim \pi^{-1}\theta(T) + 1$, so $S(N) = N(T) - \pi^{-1}\theta(T) - 1$. Then from Turing's results, in order to prove that $S(g_m) = 0$ one only need prove that $-2 < S(g_m) < 2$ as $S(g_m)$ is an even integer [48, p. 173].

3.7 The Odlyzko-Schönhage Algorithm

The Odlyzko-Schönhage algorithm is currently the most efficient algorithm for determining values $t \in \mathbb{R}$ for which $\zeta(\frac{1}{2} + it) = 0$. It uses the fact that the most computationally complex part of the evaluation using the Riemann-Siegel formula are computations of the form,

$$g(t) = \sum_{k=1}^{M} k^{-it}.$$

The algorithm employs the Fast Fourier Transform (FFT) to convert sums of this type into rational functions. The authors also give a fast algorithm for computing rational function values. The details are found in [118]. The algorithm presented computes the first n zeros of $\zeta(\frac{1}{2} + it)$ in $O(n^{1+\varepsilon})$ (as opposed to $O(n^{\frac{3}{2}})$ using previous methods).

3.8 A Simple Algorithm for the Zeta Function

This method uses tools for the acceleration of series convergence to evaluate

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s).$$

Algorithm 1. *Let*

$$d_k := n \sum_{i=0}^{k} \frac{(n+i-1)!4^i}{(n-i)!(2i)!}$$

then

$$\zeta(s) = \frac{-1}{d_n(1-2^{1-s})} \sum_{k=0}^{n-1} \frac{(-1)^k(d_k - d_n)}{(k+1)^s} + \gamma_n(s)$$

where for $s = \sigma + it$ *with* $\sigma \geq \frac{1}{2}$

$$\begin{aligned} |\gamma_n(s)| &\leq \frac{2}{(3+\sqrt{8})^n} \frac{1}{|\Gamma(s)|} \frac{1}{|(1-2^{1-s})|} \\ &\leq \frac{3}{(3+\sqrt{8})^n} \frac{(1+2|t|)e^{\frac{|t|\pi}{2}}}{|(1-2^{1-s})|} \end{aligned}$$

The proof relies on the properties of Chebyshev polynomials and can be found in [23].

3.9 Further Reading

A detailed exposition on strategies for evaluating $\zeta(s)$ can be found in [22]. The authors consider the advantages of several algorithms, and several variations of those described above. The material includes time and space complexity benchmarks for specific algorithms, and the more general ideas/context from which these algorithms arise.

4

Empirical Evidence

It would be very discouraging if somewhere down the line you could ask a computer if the Riemann Hypothesis is correct and it said, 'Yes, it is true, but you won't be able to understand the proof.' [69]

Ron Graham

The Riemann Hypothesis has endured for more than a century as a widely believed conjecture. There are many reasons why it has endured, and captured the imagination of mathematicians worldwide. In this chapter we will explore the most direct form of evidence for the Riemann Hypothesis, empirical evidence. Arguments for the Riemann Hypothesis often include its widespread ramifications and appeals to mathematical beauty; however, we also have a large corpus of hard facts. With the advent of powerful computational tools over the last century, mathematicians have increasingly turned to computational evidence to support conjectures, and the Riemann Hypothesis is no exception. To date ten trillion zeros of the Riemann zeta function have been calculated, and all conform to the Riemann Hypothesis. It is, of course, impossible to verify the Hypothesis through direct computation; however, computational evidence fulfills several objectives. First, and most obviously, it helps to convince mathematicians that the Hypothesis is worth attempting to prove. It allows us to prove tangential results that are intricately connected to the Riemann Hypothesis (for example disproving the Mertens Conjecture). Finally, numerical data allows us the opportunity to recognize other patterns, and develop new ideas with which to attack the Hypothesis (for example the Hilbert-Pólya conjecture, discussed in Section 4.3).

4.1 Verification in an Interval

In Chapter 2 we presented a thorough definition of the Riemann zeta function using analytic continuation. From that analysis it is clear that calculating zeros of the zeta function is not a simple undertaking. The tools needed to compute zeros are discussed in Chapter 3. However, in order to verify the Riemann Hypothesis it is not sufficient to merely compute zeros. A rigourous verification of the Hypothesis in a given interval can be effected in the following way [18].

First, we state Cauchy's Residue Theorem.

Theorem 4.1. *Let C be a positively oriented simple closed contour. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \ldots, n)$ inside C, then*

$$\sum_{k=1}^{n} Res_{z=z_k} f(z) = \frac{1}{2\pi i} \int_C f(z)dz,$$

where $Res_{z=z_k} f(z)$ is the residue of $f(z)$ at the pole $z = z_k$.

This is a fundamental result in complex analysis and can be found in any text on the subject. For a proof of this theorem see Section 63 of [31].

Now, let $s = \frac{1}{2} + it$ for $t \in \mathbb{C}$, and let $N_1(T)$ be the number of zeros of $\zeta(s)$ in the rectangle R with vertices at $-1 - iT, 2 - iT, 2 + iT, -1 + iT$ (note that $N_1(T) = 2N(T)$ where $N(T)$ is as defined in the first section of Chapter 2). Then by Theorem 4.1 we have

$$N_1(T) - 1 = \frac{1}{2\pi i} \int_{\partial R} -\frac{\zeta'(s)}{\zeta(s)} ds,$$

as long as T is not the imaginary part of a zero of $\zeta(s)$. The -1 term, which appears on the left hand side of the equation, corresponds to the pole of $\zeta(s)$ located at $s = 1$.

Both $\zeta(s)$ and $\zeta'(s)$ can be computed to arbitrarily high precision, for example using the Riemann-Siegel or Euler-Maclaurin summation formula (see Section 3.4). The exact value for $N_1(T) - 1$ is obtained by dividing the integral by $2\pi i$ and rounding off to the nearest integer.

In his 1859 paper [124], Riemann introduced the function

$$\xi(t) = \frac{1}{2} s(s-1)\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

which is analytic everywhere in $\mathbb{C}$ and has zeros with imaginary part between $-\frac{1}{2}$ and $\frac{1}{2}$ [124]. Riemann "conjectured" that all zeros α of $\xi(t)$ are real, that is, all nontrivial zeros of the Riemann zeta function are of the form $\frac{1}{2} + i\alpha$.

Let $\hat{t}$ be a real variable. Then, in view of the intermediate value theorem in calculus, $\xi(\hat{t})$ is continuous and real, so there will be a zero of odd order between any two points where the sign of $\xi(\hat{t})$ changes. If the number of sign changes of $\xi(\hat{t})$ in $[-T, T]$ equals $N_1(T)$, then all zeros of $\zeta(s)$ in $\mathcal{R}$ are simple and lie on the critical line. It has been verified that the first ten trillion zeros are simple and satisfy the Riemann Hypothesis [54].

Conjecture 4.2. *All non-trivial zeros of $\zeta(s)$ are simple.*

All zeros of the Riemann zeta function are believed to be simple [36], though the falsity of this conjecture would not disprove the Riemann Hypothesis. However, zeros of higher order would be problematic for many current computational techniques. For further details regarding Riemann Hypothesis verification see Sections 3.2, 3.5 and 3.6.

4.2 A Brief History of Computational Evidence

The first computations of zeros were performed by Riemann himself, prior to the presentation of his paper 12.2. These computations were never published, but they formed the basis of his famous conjecture. Siegel, in his study of Riemann's notes, uncovered both the fact that Riemann had performed some calculations of the values of $\zeta(s)$, and his brilliant method. The formula Riemann devised was published by Siegel in the 1930s, and it subsequently became known as the Riemann-Siegel formula (see 3.4). It has formed the basis of all large-scale computations of the Riemann zeta function [34, pp 343–344].

The following table outlines the history of computational evidence supporting the Riemann Hypothesis.

Year	Number of computed zeros above the real axis	Computed by
1859 (approx.)	1	B. Riemann [48, p. 159]
1903	15	J. P. Gram [55]
1914	79	R. J. Backlund [10]
1925	138	J. I. Hutchinson [70]
1935	1041	E. C. Titchmarsh [145]
1953	1,104	A. M. Turing [150]
1956	15,000	D. H. Lehmer [88]
1956	25,000	D. H. Lehmer [87]
1958	35,337	N. A. Meller [96]
1966	250,000	R. S. Lehman [85]
1968	3,500,000	J. B. Rosser, et al [128]
1977	40,000,000	R. P. Brent [160]
1979	81,000,001	R. P. Brent [25]
1982	200,000,001	R. P. Brent, et al [26]
1983	300,000,001	J. van de Lune, H. J. J. te Riele [151]
1986	1,500,000,001	J. van de Lune, et al [152]
2001	10,000,000,000	J. van de Lune (unpublished)
2004	900,000,000,000	S. Wedeniwski [160]
2004	10,000,000,000,000	X. Gourdon [54]

Even a single exception to Riemann's conjecture would have enormously strange consequences for the distribution of prime numbers... If the Riemann Hypothesis turns out to be false, there will be huge oscillations in the distribution of primes. In an orchestra, that would be like one loud instrument that drowns out the others—an aesthetically distasteful situation [19].

Enrico Bombieri

This body of evidence seems impressive; however, there are reasons why one should be wary of computational evidence. The Riemann Hypothesis is equivalent to the assertion that, for real t, all local maxima of $\xi(t)$ are positive and all local minima of $\xi(t)$ are negative [18]. It has been suggested that a counterexample, if one exists, would be found in the neighborhood of unusually large peaks of $|\zeta(\frac{1}{2}+it)|$. However, such peaks do occur, but only at very large heights [18]. This fact, among others, should discourage the reader from being persuaded solely by the numbers in the table above.

4.3 The Riemann Hypothesis and Random Matrices

How could it be that the Riemann zeta function so convincingly mimics a quantum system without being one? [32]

M. Berry

One of the more popular ideas regarding the Riemann Hypothesis is that the zeros of the zeta function can be interpreted as eigenvalues of certain matrices. This line of thinking is attractive and is potentially a good way to attack the Hypothesis, as it gives a connection to physical phenomena. This connection to the natural world grounds the abstract statement of the Hypothesis to the real world in a way that many mathematicians find compelling.

The connection between the Riemann Hypothesis and random matrices has its roots in the work of Hilbert and Pólya. Both, independently, enquired whether for zeros $\frac{1}{2}+i\gamma_j$ of $\zeta(s)$, the numbers γ_j belong to a set of eigenvalues of a Hermitian operator. Not only would the truth of the conjecture indicate that all the numbers γ_j are real, but it would link the zeros to an important tool of physics. As such, the conjecture might give a natural reason for all the nontrivial zeros to be on the critical line [17].

Empirical results indicate that the zeros of the Riemann zeta function are indeed distributed like the eigenvalues of certain matrix ensembles, in particular the Gaussian unitary ensemble. This suggests that random matrix theory might provide an avenue for the proof of the Riemann Hypothesis.

From Hilbert and Pólya the idea of connecting the zeros of the zeta function to eigenvalues progressed in the work of Montgomery and Dyson. Recall from Section 2.3 that the number of zeros of $\zeta(s)$ in the critical strip with $0 \leq \Re(s) < T$ is given by

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{4.3.1}$$

Denote the imaginary parts of the first n zeros above the real axis by

$$\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots \leq \gamma_n,$$

in order of increasing height (if two zeros are equal in their imaginary parts, then we order them according to their real parts in increasing order). It follows from (4.3.1) that

$$\gamma_n \sim \frac{2\pi n}{\log n}.$$

Under the Riemann Hypothesis, Montgomery studied the gaps $\delta_j = \gamma_{j+1} - \gamma_j$ between consecutive zeros, first normalizing them to have mean length 1, asymptotically. The normalized gaps, denoted $\hat{\delta}_j$, are given by

$$\hat{\delta}_j = \hat{\gamma}_{j+1} - \hat{\gamma}_j \, ,$$

where

$$\hat{\gamma}_n = N(\gamma_n)\, .$$

The difference

$$\hat{\gamma}_{j+k} - \hat{\gamma}_j$$

is known as the kth consecutive spacing [138].

Let α and β be positive real numbers, with $\alpha < \beta$. Let $\hat{\gamma}_1 < \cdots < \hat{\gamma}_n$ be the non-trivial zeros of the Riemann zeta function along $\sigma = \frac{1}{2}$, normalized as above. Now let the number of k^{th} consecutive spacings in $[\alpha, \beta]$ be given by

$$\frac{|\{(j,k)\colon 1 \leq j \leq n, k \geq 0, (\hat{\gamma}_{j+k} - \hat{\gamma}_j) \in [\alpha, \beta]\}|}{n}.$$

Montgomery conjectured in the early 1970's that the number of k^{th} consecutive spacings in $[\alpha, \beta]$ behaves asymptotically as follows.

Conjecture 4.3.

$$\frac{|\{(j,k)\colon 1 \leq j \leq n, k \geq 0, (\hat{\gamma}_{j+k} - \hat{\gamma}_j) \in [\alpha, \beta]\}|}{n} \sim \int_\alpha^\beta \left(1 - \frac{sin^2(\pi x)}{(\pi x)^2}\right) dx,$$

as $n \to \infty$ [138, 99].

This conjecture is based on results that are provable assuming the Riemann Hypothesis, as well as on conjectures for the distribution of twin primes and other prime pairs [34].

The function

$$1 - \frac{sin^2(\pi x)}{(\pi x)^2}$$

is called the *pair correlation function* for zeros of the Riemann zeta function [138]. Part of the folklore surrounding the Riemann Hypothesis and random matrices is a meeting between the physicist Freeman Dyson and the mathematician Hugh Montgomery at Princeton [129]. Montgomery showed Dyson the pair correlation function for the zeta function, and the latter recognized that it is also the pair correlation function for suitably normalized eigenvalues in a Gaussian Unitary Ensemble. For a discussion of the theory of random matrices, including the theory of Gaussian Unitary Ensembles, see Mehta [95].

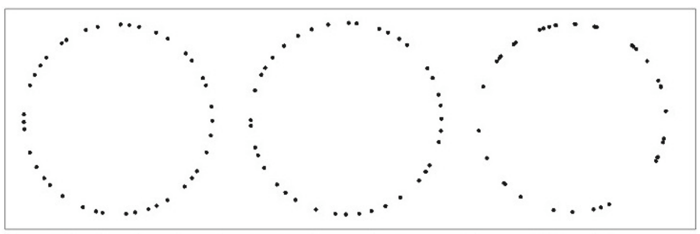

Fig. 4.1. The eigenvalues of a random 40×40 unitary matrix, 40 consecutive zeros of $\zeta(s)$ scaled to wrap once around the circle, and 40 randomly chosen points on the unit circle. Figure from [34] (see Section 11.3).

Dyson's insight prompted Montgomery to reformulate his conjecture as follows.

Conjecture 4.4 (Montgomery-Odlyzko Law). *The distribution of spacings between nontrivial zeros of the Riemann zeta function is statistically identical to the distribution of eigenvalue spacings in a Gaussian Unitary Ensemble [77].*

In the 1980s, Odlyzko verified the Riemann Hypothesis in large intervals around the 10^{20}th zero of the zeta function [34, 118]. Using the normalization

$$\hat{\delta}_n = (\gamma_{n+1} - \gamma_n)\frac{\log(\gamma_n/2\pi)}{2\pi},$$

Odlyzko calculated the pair correlation for zeros in these intervals (see [114] for Odlyzko's results and other computations).

The following figure is a plot of the correlation function (solid line) with a superimposed scatterplot of empirical data.

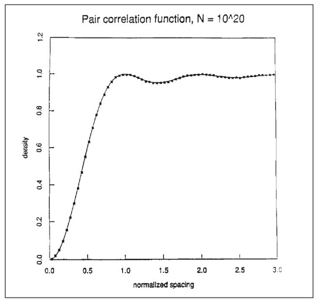

Fig. 4.2. The pair-correlation function for GUE (solid) and for 8×10^6 zeros near the 10^{20}th zero of $\zeta(s)$ above the real axis. Figure from [34] (see Section 11.3).

Repulsion between successive zeros is predicted by Montgomery's conjecture. However, some pairs of zeros occur very close together. The zeros at $\frac{1}{2} + (7005 + t)i$ for $t \approx 0.06286617$ and $t \approx 0.1005646$ form one such pair. The appearance of closely-spaced nontrivial zeros is known as Lehmer's phenomenon.

4.4 Skewes Number

In Section 4.2, we mentioned that there are reasons to be wary of computational evidence. Skewes number is a perfect example of the potential misleading nature of numerical data, and serves as another warning to those who would be convinced by numbers alone. Skewes number is the smallest n such that $\pi(n) < \mathrm{Li}(n)$ fails.

The Prime Number Theorem gives

$$\pi(n) \sim \mathrm{Li}(n),$$

as $n \to \infty$. Preliminary calculations suggested that $\pi(n) < \mathrm{Li}(n)$ for all n, and was conjectured by many mathematicians including Riemann and Gauss. This conjecture was refuted in 1912 by Littlewood, who proved that the inequality fails for some n [58]. Two years later, Littlewood proved that in fact the inequality fails for infinitely many n.

We have seen the close connection of the Riemann Hypothesis and the Prime Number Theorem. As a consequence of this connection, bounds on Skewes number depend on the truth or falsity of the Riemann Hypothesis.

The following bounds all require the assumption of the Riemann Hypothesis. In 1933, Skewes gave the upper bound $n \le 10^{10^{10^{34}}}$ [136]. In 1966, Lehman showed that there are more than 10^{500} successive integers between 1.53×10^{1165} and 1.65×10^{1165} for which the inequality fails [84]. Using better approximations to the zeros of $\zeta(s)$, te Riele in 1987 showed that there are at least 10^{180} successive integers in $[6.627 \times 10^{370}, 6.687 \times 10^{370}]$ for which the inequality fails [142]. This was improved to 1.39×10^{316} by Bays and Hudson in 2000 [16].

Skewes also found a bound that does not require the Riemann Hypothesis. First Skewes defines Hypothesis H to be

Hypothesis H. *Every complex zero $s = \sigma + it$ of the Riemann zeta function satisfies*

$$\sigma - \frac{1}{2} \le X^{-3} \log^{-2}(X)$$

provided that $|t| < X^3$.

Note that Hypothesis H is weaker than the Riemann Hypothesis, so the negation of Hypothesis H is stronger than the negation of the Riemann Hypothesis. In 1955, assuming the negation of Hypothesis H, Skewes computed the upper bound $n \le 10^{10^{10^{10^{3}}}}$ [137].

The best current bounds on Skewes numbers lie well beyond the limits of our computing power. It is entirely possible that a counter-example to the Riemann Hypothesis lies beyond our computational limits, and hence that our efforts will inevitably and unfailingly generate positive evidence for a false conjecture.

> *So for all practical purposes, the Riemann zeta function does not show its true colours in the range available by numerical investigations. You should go up to the height 10^{10000} then I would be much more convinced if things were still pointing strongly in the direction of the Riemann Hypothesis. So numerical calculations are certainly very impressive, and they are a triumph of computers and numerical analysis, but they are of limited capacity. The Riemann Hypothesis is a very delicate mechanism. It works as far as we know for all existing zeros, but we cannot, of course, verify numerically an infinity of zeros, so other theoretical ways of approach must be found, and for the time being they are insufficient to yield any positive conclusion [129].*
>
> **Aleksandar Ivić**

5

Equivalent Statements

Riemann's insight was that the frequencies of the basic waveforms that approximate the psi function are determined by the places where the zeta function is equal to zero ... To me, that the distribution of prime numbers can be so accurately represented in a harmonic analysis is absolutely amazing and incredibly beautiful. It tells of an arcane music and a secret harmony composed by the prime numbers [19].

Enrico Bombieri

... there is a sense in which we can give a one-line non technical statement of the Riemann Hypothesis: 'The primes have music in them' [14].

M. Berry and J. P. Keating

In this chapter we discuss several statements that are equivalent to the Riemann Hypothesis. By restating the Riemann Hypothesis in different language, and in entirely different disciplines, we gain more possible avenues of attack. We group these equivalences into three categories: equivalences that are entirely number theoretic; equivalences that are closely related to the analytic properties of the zeta function and other functions; and equivalences which are truly cross-disciplinary. These equivalences range from old to relatively new, from central to arcane and from deceptively simple to staggeringly complex.

5.1 Number Theoretic Equivalences

Number theoretic equivalences of the Riemann Hypothesis provide a natural method of explaining the Hypothesis to non-mathematicians without appealing to complex analysis. While it is unlikely that any of these equivalences will lead directly to a solution, they provide a sense of how intricately the Riemann zeta function is tied to the primes. We begin by repeating our definition of the Liouville function.

Definition 5.1. *The Liouville Function is defined by*

$$\lambda(n) := (-1)^{\omega(n)}$$

where $\omega(n)$ is the number of, not necessarily distinct, prime factors in n, with multiple factors counted multiply.

This leads to an equivalence that can be roughly stated in terms of probability. Namely, the Riemann Hypothesis is equivalent to the statement that an integer has an equal probability of having an odd number or an even number of distinct prime factors. More formally we have:

Equivalence 5.2. *The Riemann Hypothesis is equivalent to*

$$\lambda(1) + \lambda(2) + \cdots + \lambda(n) \ll n^{1/2+\varepsilon},$$

for every positive ε.

See Section 1.2 for further discussion of this equivalence.

The next equivalence we discuss has a long history. Recall that the Prime Number Theorem states that $\pi(x) \sim \mathrm{Li}(x)$, where $\mathrm{Li}(x)$ is the logarithmic integral, defined as follows.

Definition 5.3. *The logarithmic integral,* Li*, of x is defined as*

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

We also define $\pi(x)$, the prime counting function.

Definition 5.4. *The prime counting function, denoted $\pi(x)$, is the number of primes less than or equal to a real number x.*

The Prime Number Theorem, in the form $\pi(x) \sim \mathrm{Li}(x)$, was conjectured by Gauss in 1849, in a letter to Enke [52]. Gauss made his conjecture on the basis of his calculations (done by hand) of $\pi(x)$ and contemporary tables of values of $\mathrm{Li}(x)$. A short table of his findings appears in his letter to Enke.

Below	Here are Prime	Integral $\int \frac{dn}{\log n}$	Error
500,000	41,556	41,606.4	+50.4
1,000,000	78,501	79,627.5	+126.5
1,500,000	114,112	114,263.1	+151.1
2,000,000	148,883	149,054.8	+171.8
2,500,000	183,016	183,245.0	+229.0
3,000,000	216,745	216,970.6	+225.6

Table 5.1. Table of Gauss' calculations of $\pi(x)$ from his letter to Enke [52].

The Prime Number Theorem, proved independently by Hadamard (see Section 12.3) and de la Valleé Poussin (see Section 12.4), requires showing that $\zeta(s) \neq 0$ when $\Re(s) = 1$. However, the Riemann Hypothesis gives us a more accurate asymptotic estimation to the error in the Prime Number Theorem.

Equivalence 5.5. *The assertion that*

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x} \log x)$$

is equivalent to the RH [18].

The next equivalence involves the Mertens function, for which we will need the Möbius function.

Definition 5.6. *The Möbius function, $\mu(n)$, is defined in the following way:*

$$\mu(n) := \begin{cases} 0 & \textit{if } n \textit{ has a square factor} \\ 1 & \textit{if } n = 1 \\ (-1)^k & \textit{if } n \textit{ is a product of } k \textit{ distinct primes.} \end{cases}$$

We now define the Mertens function as follows.

Definition 5.7. *The Mertens function, denoted $M(x)$, is defined by*

$$M(x) := \sum_{n \leq x} \mu(n)\,,$$

where x is real [147, p. 370].

In terms of the Mertens function we have the following equivalence.

Equivalence 5.8. *The Riemann Hypothesis is equivalent to*

$$M(x) = O(x^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$ [147].

Proving that $M(x) = O(x^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon < \frac{1}{2}$ is open and would be an impressive result.

We can also reformulate the Riemann Hypothesis in terms of the sum of divisors function. The sum of divisors function is defined as follows.

Definition 5.9. *For $n \in \mathbb{N}$,*

$$\sigma(n) := \sum_{d|n} d.$$

This function is a special case of the divisor function $\sigma_k(n)$ (i.e., for $k = 1$), which is defined as the sum of the k^{th} powers of the divisors of n [82]. The following equivalence is due to Robin.

Equivalence 5.10. *The Riemann Hypothesis is equivalent to the statement that, for all* $n > 5040$,

$$\sigma(n) < e^{\gamma} n \log \log n$$

where γ *is Euler's constant [127].*

Robin also showed, unconditionally, that

$$\sigma(n) < e^{\gamma} n \log \log n + 0.6482 \frac{n}{\log \log n}$$

for all $n \geq 3$.

Building on Robin's work, Lagarias proved another equivalence to the Riemann Hypothesis which involves the sum of divisors function.

Equivalence 5.11. *The following statement is equivalent to the RH,*

$$\sigma(n) \leq H_n + \exp(H_n) \log(H_n),$$

for all $n \geq 1$, *with equality only for* $n = 1$ *[82].*

Here H_n is the n^{th} harmonic number defined as follows.

Definition 5.12. *The* n^{th} *harmonic number is given by*

$$H_n := \sum_{j=1}^{n} \frac{1}{j} .$$

Of particular interest is the fact that Lagarias' equivalence is a precise inequality. Eric Rains verified the inequality for $1 \leq n \leq 5040$.

By using the Mertens function we can re-cast an earlier equivalence in terms of Farey series. The term Farey series is entirely misleading, as they were not discovered by Farey, nor are they series. Farey series were in fact discovered by Haros, and are defined as follows.

Definition 5.13. *The Farey series of order* n *is the set of rationals* $\frac{a}{b}$ *with* $0 \leq a \leq b \leq n$ *and* $(a, b) = 1$, *arranged in increasing order [48].*

For example, the Farey series of order 3 is given by

$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}.$$

We denote the j^{th} term of F_n by $F_n(j)$. It is easy to see that the number, m, of terms in the Farey series of order n is $m = 1 + \sum_{j=1}^{n} \phi(j)$ where $\phi(j)$ is the Euler totient function. Now we can formulate the following equivalence.

Equivalence 5.14. *The RH is equivalent to*

$$\sum_{j=1}^{m} \left| F_n(j) - \frac{j}{m} \right| = O(n^{\frac{1}{2}+\varepsilon})$$

where $\varepsilon > 0$ *[48].*

5.2 Analytic Equivalences

In this section we explore various equivalences that are connected to the Riemann zeta function analytically. These problems vary from direct consequences of the definition of the zeta function to more abstruse re-formulations. Generally these problems are classical, and have arisen in the course of research into the properties of $\zeta(s)$. They offer more promising avenues of attack on the Hypothesis.

We begin by re-stating the Riemann Hypothesis in terms of the Dirichlet eta function, or the alternating zeta function.

Equivalence 5.15. *The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function*

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} = (1 - 2^{1-s})\zeta(s)$$

which fall in the critical strip $0 < \Re(s) < 1$ *lie on the critical line* $\Re(s) = 1/2$.

We can also consider the convergence of $1/\zeta(s)$ and the values of the derivative, $\zeta'(s)$.

Equivalence 5.16. *The convergence of*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\sigma > 1/2$ is necessary and sufficient for the RH [147, pp 369–370]. Note that for $\sigma > 1$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} .$$

Equivalence 5.17. *The RH is equivalent to the non-vanishing of $\zeta'(s)$ in the region $0 < \sigma < 1/2$ [140].*

Recall from Chapter 2 that in deriving the functional equation for $\zeta(s)$ we defined the ξ function,

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Lagarias derived the following criterion on $\xi(s)$ [81].

Equivalence 5.18. *The RH is equivalent to*

$$\Re\left(\frac{\xi'(s)}{\xi(s)}\right) > 0.$$

Continuing to work with $\xi(s)$, we define λ_n as

$$\lambda_n := \frac{1}{(n-1)!}\frac{d^n}{ds^n}(s^{n-1}\log\xi(s)).$$

Now we can state an equivalence relating the Riemann Hypothesis to the value of λ_n.

Equivalence 5.19. *The Riemann Hypothesis is equivalent to the non-negativity of λ_n for all $n \geq 1$ [90].*

Since the Riemann Hypothesis is a statement regarding zeros of the zeta function, it is no surprise that we can re-formulate the Hypothesis into other statements involving the zeros of $\zeta(s)$. The following two equivalences give examples of non-trivial reformulations that use these zeros.

Equivalence 5.20. *The RH is equivalent to the following statement:*

$$\sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^n\right) > 0$$

for each $n = 1, 2, 3, \ldots$, where ρ runs over the complex zeros of $\zeta(s)$ [20].

Note that

$$\sum_{\rho}\left(1-\left(1-\frac{1}{\rho}\right)^n\right) = \lambda_n|_{s=1},$$

where λ_n is defined as in Equivalence 5.19.

Equivalence 5.21. *The RH is true if and only if the integral*

$$I = \int_{\sigma=\frac{1}{2}} \frac{\log(|\zeta(s)|)}{|s|^2} dt = 0$$

where $s = \sigma + it$ [14]. Moreover, this integral can be exactly expressed as the sum

$$I = 2\pi \sum_{\Re(\rho)>\frac{1}{2}} \log \left| \frac{\rho}{1-\rho} \right|,$$

where ρ is a zero of $\zeta(s)$ in the region indicated.

Hardy and Littlewood derived another equivalence to the Riemann Hypothesis that sums values of $\zeta(s)$, evaluated at odd integers.

Equivalence 5.22. *The RH holds if and only if*

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k!\zeta(2k+1)} = O(x^{-\frac{1}{4}}),$$

as $x \to \infty$ [60].

For the next equivalence we will need some definitions. We first define the Xi function as,

$$\Xi(iz) := \frac{1}{2}\left(z^2 - \frac{1}{4}\right) \pi^{-z/2-\frac{1}{4}} \Gamma\left(\frac{1}{2}z + \frac{1}{4}\right) \zeta\left(z + \frac{1}{2}\right).$$

We note that $\Xi(\frac{z}{2})/8$ is the Fourier transform of the signal $\Phi(t)$, given by,

$$\Phi(t) := \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}) e^{-\pi n^2 e^{4t}},$$

for $t \in \mathbb{R}$ and $t \geq 0$. Now we consider the Fourier transform of $\Phi(t)e^{\lambda t^2}$, which we will denote as $H(\lambda, z)$,

$$H(\lambda, z) := \mathcal{F}_t[\Phi(t)e^{\lambda t^2}](z),$$

where $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$. We can see that $H(0, z) = \Xi(\frac{z}{2})/8$. It was proven in 1950 by de Bruijn that H has only real zeros for $\lambda \geq \frac{1}{2}$. Furthermore, it is known that there exists a constant Λ such that if $H(\lambda, z) = 0$, then z is real if and only if $\lambda \geq \Lambda$. The value Λ is called the de Bruijn-Newman constant. Now we can state the Riemann Hypothesis in terms of the de Bruijn-Newman constant.

Equivalence 5.23. *The Riemann Hypothesis is equivalent to the conjecture that $\Lambda \leq 0$ [39, 38].*

There are several lower bounds on Λ, the best at this time being $-2.7 \cdot 10^{-9} < \Lambda$, proven by Odlyzko in 2000 [117]. It had been conjectured by C. M. Newman that Λ satisfies $\Lambda \geq 0$ [104].

Salem, in 1953, gave the following criterion for the truth of the Riemann Hypothesis.

Equivalence 5.24. *The Riemann Hypothesis holds if and only if the integral equation*

$$\int_{-\infty}^{\infty} \frac{e^{-\sigma y}\varphi(y)}{e^{e^{x-y}}+1} dy = 0$$

has no bounded solution, $\varphi(y)$, other than the trivial solution $\varphi(y) = 0$, for $\frac{1}{2} < \sigma < 1$ [131].

The following elegant equivalence, due to Volchkov, connects the zeros of the Riemann zeta function to Euler's constant γ.

Equivalence 5.25. *The Riemann Hypothesis is equivalent to*

$$\int_0^{\infty} (1 - 12t^2)(1 + 4t^2)^{-3} \int_{1/2}^{\infty} \log |\zeta(\sigma + it)| \, d\sigma dt = \pi(3 - \gamma)/32,$$

where γ is Euler's constant [157].

Julio Alacántara-Bode, building on the work of Beurling, re-fomulated the Riemann Hypothesis using the Hilbert-Schmidt integral operator. The Hilbert-Schmidt integral operator A on $L^2(0,1)$ is given by

$$[Af](\theta) := \int_0^1 f(x) \left\{ \frac{\theta}{x} \right\} dx,$$

where $\{x\}$ is the fractional part of x. Now the Riemann Hypothesis can be stated as follows.

Equivalence 5.26. *The Riemann hypothesis is true if and only if the Hilbert-Schmidt operator A is injective [5].*

We can also re-state the Riemann Hypothesis in terms of ergodic theory. A theorem of Dani gives that, for each $y > 0$, there exist ergodic measures, $m(y)$, on the space $M = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$, supported on closed orbits of period $1/y$ of the horocyclic flow.

Equivalence 5.27. *The Riemann Hypothesis is equivalent to the statement that for any smooth function f on M,*

$$\int_M f dm(y) = o(y^{\frac{3}{4}-\varepsilon})$$

for any $\varepsilon > 0$ as $y \to 0$ [154].

5.3 Other Equivalences

At first glance the remaining equivalences seem to have little to do with the zeros of a complex-valued function. They are interesting as examples of the intricate connections between various mathematical disciplines, but they have not indicated novel approaches to tackling the Riemann Hypothesis.

We begin by defining the Redheffer Matrix of order n. The $n \times n$ Redheffer matrix, $R_n := [R_n(i,j)]$, is defined by

$$R_n(i,j) = \begin{cases} 1 & \text{if } j = 1 \text{ or if } i \mid j \\ 0 & \text{otherwise.} \end{cases}$$

It can easily be shown that $\det R_n = \sum_{k=1}^{n} \mu(k)$. We can simply use this to state the Riemann Hypothesis by employing a previous equivalence.

Equivalence 5.28. *The Riemann Hypothesis is true if and only if*

$$\det(R_n) = O(n^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$*.*

This was shown by Redheffer in [123]. Now we can use the Redheffer matrices to translate the Riemann Hypothesis into the language of graph theory. We let R_n be the Redheffer matrix as defined above, and set $B_n := R_n - I_n$ (where I_n is the $n \times n$ identity matrix). Now we let G_n be the directed graph whose adjacency matrix is B_n. Finally we let the graph $\overline{G}_n$ be the graph obtained from G_n by adding a loop at node 1 of G_n. Now we can re-state the Riemann Hypothesis in terms of the cycles of $\overline{G}_n$.

Equivalence 5.29. *The following statement is equivalent to the Riemann Hypothesis:*

$$|\#(\textit{odd cycles in } \overline{G}_n) - \#(\textit{even cycles in } \overline{G}_n)| = O(n^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$ *[15].*

The Nyman-Beurling equivalent form translates the Riemann Hypothesis into a statement about the span of a set of functions.

Equivalence 5.30. *The closed linear span of* $\{\rho_\alpha(t) : 0 < \alpha < 1\}$ *is* $L^2(0,1)$ *if and only if the Riemann Hypothesis is true, where*

$$\rho_\alpha(t) := \left\{\frac{\alpha}{t}\right\} - \alpha\left\{\frac{1}{t}\right\},$$

$\{x\}$ *is the fractional part of* x*, and* $L^2(0,1)$ *is the space of square integrable functions on* $(0,1)$ *[13].*

We can also re-formulate the Riemann Hypothesis in terms of a problem regarding the order of group elements.

Equivalence 5.31. *The Riemann Hypothesis is equivalent to the statement that for sufficiently large n,*

$$\log g(n) < \frac{1}{\sqrt{\mathrm{Li}(n)}}$$

where $g(n)$ is the maximal order of elements of the symmetric group S_n of degree n [94].

Finally, a problem about the rate of convergence of certain discrete measures is equivalent to the Riemann Hypothesis. Denote the nonzero reals by $\mathbb{R}^*$, and let $C_c^r(\mathbb{R}^*)$ be the set of all functions $f\colon \mathbb{R}^* \to \mathbb{C}$ that are r times differentiable and have compact support. For $y \in \mathbb{R}^*$ and for $f \in C_c^2(R^*)$, define

$$m_y(f) := \sum_{n \in \mathbb{N}} y\phi(n) f(y^{1/2} n)\,,$$

with ϕ the Euler totient function. Let

$$m_0(f) := \int_0^\infty \left(\frac{6}{\pi^2}\right) u f(u)\, du\,.$$

Equivalence 5.32. *The RH is true if and only if $m_y(f) = m_0(f) + o(y^{\frac{3}{4}-\varepsilon})$ as $y \to 0$, for every function $f \in C_c^2(\mathbb{R}^*)$ and every $\varepsilon > 0$ [154].*

6

Extensions of the Riemann Hypothesis

It is common in mathematics to generalize hard problems to even more difficult ones. Sometimes this is productive and sometimes not. However, in the Riemann Hypothesis case this is a valuable enterprise that allows us to see a larger picture. These stronger forms of the RH allow the proof of many conditional results that appear to be plausible (see papers 11.1 and 11.2) and add to the heuristic evidence for the RH.

We begin by once again stating the RH in its classical form. We then give a variety of extensions to other related problems. There is some variation in the literature on these problems as to names and abbreviations. We attempt to use standard notations as reflected in the source material; however, this is not always possible.

6.1 Riemann Hypothesis

A usual formulation of the problem is as follows (see [18]),

Conjecture 6.1 (The RH). *The nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.*

The fundamental object under consideration is $\zeta(s)$. As before, the Riemann zeta function, $\zeta(s)$, is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and in the whole complex plane by analytic continuation (see Chapter 2).

This statement of the problem can be simplified by introducing the Dirichlet eta function, also known as the alternating zeta function.

Definition 6.2. *The Dirichlet eta function is defined as*

$$\eta(s) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} = (1 - 2^{1-s})\zeta(s).$$

Since $\eta(s)$ converges for all $s \in \mathbb{C}$, with $\Re(s) > 0$, one needs not consider analytic continuation. The Riemann Hypothesis is true if and only if the zeros of $\eta(s)$ in the strip $0 < \Re(s) < 1$ all lie on the line $\Re(s) = \frac{1}{2}$. This equivalence, and other elementary equivalences, are discussed further in Chapter 5.

6.2 Generalized Riemann Hypothesis

The Riemann zeta function is the object considered by the Riemann Hypothesis. However, $\zeta(s)$ is a special case within a larger, more general class of functions. The Generalized Riemann Hypothesis (gRH) considers this larger class of functions, the class of Dirichlet L-functions [18]. We will build these functions in the same way we built the Riemann zeta function in Chapter 2. We define a Dirichlet L-series to be,

$$L(s, \chi_k) := \sum_{n=1}^{\infty} \chi_k(n) n^{-s}$$

where $\chi_k(n)$ is a number theoretic character modulo k, defined by the properties,

$$\chi_k(1) = 1,$$
$$\chi_k(n) = \chi_k(n + k),$$

for all m and n

$$\chi_k(m)\chi_k(n) = \chi_k(mn),$$

and for $(k, n) \neq 1$,

$$\chi_k(n) = 0.$$

A character modulo k is said to be primitive if no $\chi_d(n)$ exists such that $\chi_k(n) = \chi_d(n)$ and $d|k$, $d \neq k$. The unique $\chi_k(n)$ such that $\chi_k(n) = 1$ for all n with $(k, n) = 1$ is called the principal character modulo k.

A Dirichlet L-function, $L(s, \chi_k)$, is the analytic continuation of the associated Dirichlet L-series. We can see that $\zeta(s)$ belongs in this class of functions as

$$L(s, \chi_1) = \sum_{n=1}^{\infty} \chi_1(n) n^{-s} = \sum_{n=1}^{\infty} n^{-s} = \zeta(s).$$

We now state the gRH as follows.

Extension 6.3 (The gRH). *All non-trivial zeros of $L(s, \chi_k)$ have real part equal to $\frac{1}{2}$ [30, p. 3].*

Non-trivial in this case means $L(s, \chi_k) = 0$ for $s \in \mathbb{C}$, such that $0 < \Re(s) < 1$.

We see that gRH implies RH, as $\zeta(s)$ is a member of the class of Dirichlet L-functions. We also note that, unlike $\zeta(s)$, $L(s, \chi_k)$ may have zeros on the line $\Im(s) = 0$; however, all of these zeros are known and are considered to be trivial zeros.

Note that there is variation in the literature regarding generalized vs. extended Riemann Hypthesis. Also, the theory of global L-functions is rich and extends far beyond the points considered here (see [75] for a more thorough discussion).

6.3 Extended Riemann Hypothesis

We define the Legendre symbol, $\left(\frac{n}{p}\right)$, as

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & \text{if the congruence } x^2 \equiv n \pmod{p} \text{ has a solution} \\ 0, & \text{if } p|n \\ -1, & \text{if the congruence } x^2 \equiv n \pmod{p} \text{ has no solution.} \end{cases}$$

Consider the series,

$$L_p(s) := \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) \frac{1}{n^s}.$$

The Extended Riemann Hypothesis considers the extension of $L_p(s)$ to the entire complex plane through analytic continuation.

Extension 6.4 (The eRH). *The zeros of $L_p(s)$ with $0 < \Re(s) < 1$ all lie on the line $\Re(s) = \frac{1}{2}$ [30, p. viii].*

Since $\left(\frac{a}{b}\right)$ is a specific example of a number theoretic character, the Extended Riemann Hypothesis is an instance of the Generalized Riemann Hypothesis, presented in Section 6.2.

6.4 Equivalent Extended Riemann Hypothesis

If we define $\pi(x;k,l)$ to be

$$\pi(x;k,l) := |\{p \ : \ p \leq x, \ p \text{ is prime, and } p \equiv l \pmod{k}\}|$$

then an equivalent eRH is as follows.

Extension 6.5 (An Equivalent eRH). *For* $(k,l)=1$ *and* $\varepsilon > 0$,

$$\pi(x;k,l) = \frac{\mathrm{Li}(x)}{\phi(k)} + O(x^{\frac{1}{2}+\varepsilon})$$

where $\phi(k)$ *is Euler's totient function, and* $\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}$.

This statement can be found in [9]. The authors of [9] credit Titchmarsh (circa 1930).

6.5 Another Extended Riemann Hypothesis

Let K be a number field with ring of integers $\mathcal{O}_K$. The Dedekind zeta function of K is given by,

$$\zeta_K(s) := \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$$

for $\Re(s) > 1$, where the sum is over all integral ideals of $\mathcal{O}_K$, and $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$. Again we consider the extension of $\zeta_K(s)$ to the entire complex plane through analytic continuation.

Extension 6.6 (Another eRH). *All zeros of the Dedekind zeta function of any algebraic number field, with* $0 < \Re(s) < 1$, *lie on the line* $\Re(s) = \frac{1}{2}$.

This statement includes the Riemann Hypothesis as the Riemann zeta function is the Dedekind zeta function over the field of rational numbers [3].

6.6 Grand Riemann Hypothesis

In order to state the Grand Riemann Hypothesis we need to define automorphic L-functions. This requires sophisticated tools and we refer the interested reader to another source, such as [163], in order to develop the necessary background.

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and let π be an automorphic cuspidal representation of the general linear group $GL_m(\mathbb{A})$ with central character χ. The representation π is equivalent to $\bigotimes_v \pi_v$ with $v = \infty$ or $v = p$ and π_v an irreducible unitary representation of $GL_m(\mathbb{Q}_v)$. For each prime and local representation π_p we define

$$L(s, \pi_p) := \prod_{j=1}^{m} \left(1 - \alpha_{j,\pi}(p)p^{-s}\right)^{-1}$$

where the m complex parameters $\alpha_{j,\pi}(p)$ are determined by π_p. For $v = \infty$, π_∞ determines parameters $\mu_{j,\pi}(\infty)$ such that

$$L(s, \pi_\infty) := \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s - \mu_{j,\pi}(\infty))$$

where

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

The standard automorphic L-function associated with π is defined by

$$L(s, \pi) := \prod_{p} L(s, \pi_p)$$

Now if we define

$$\Lambda(s, \pi) := L(s, \pi_\infty)L(s, \pi),$$

then $\Lambda(s, \pi)$ is entire and satisfies the functional equation

$$\Lambda(s, \pi) = \epsilon_\pi N_\pi^{\frac{1}{2}-s}\Lambda(1 - s, \tilde{\pi})$$

where $N_\pi \geq 1$ is an integer, ϵ_π is of modulo 1 and $\tilde{\pi}$ is the contragredient representation $\tilde{\pi}(g) = \pi({}^t g^{-1})$. General conjectures of Langlands assert these standard L-functions multiplicatively generate all L-functions.

As a simple and explicit example of automorphic L-function, consider the discriminant

$$\Delta(z) := e^{2\pi iz} \prod_{n=1}^{\infty} \left(1 - e^{2\pi inz}\right)^{24} := \sum_{n=1}^{\infty} \tau(n)e^{2\pi inz}.$$

Then $\Delta(z)$ is a meromophic function in the upper half plane satisfying the transformation rule

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

for $a, b, c, d \in \mathbb{Z}$ satisfying $ad - bc = 1$. Associated to Δ, we let

$$L(s,\mathit{\Delta}) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{11/2}} n^{-s} = \prod_p \left(1 - \frac{\tau(p)}{p^{11/2}} p^{-s} + p^{-2s}\right)^{-1}.$$

Now $L(s,\mathit{\Delta})$ is entire and satisfies the functional equation

$$\mathit{\Lambda}(s,\mathit{\Delta}) := \mathit{\Gamma}_{\mathbb{R}}\left(s+\frac{11}{2}\right)\mathit{\Gamma}_{\mathbb{R}}\left(s+\frac{13}{2}\right)L(s,\mathit{\Delta}) = \mathit{\Lambda}(1-s,\mathit{\Delta}).$$

The Grand Riemann Hypothesis asserts that the zeros of $\mathit{\Lambda}(s,\pi)$ (and in particular, those of $\mathit{\Lambda}(s,\mathit{\Delta})$) all lie on $\Re(s) = \frac{1}{2}$.

Extension 6.7. *All of the zeros, of any automorphic L-function, with* $0 < \Re(s) < 1$, *lie on the line* $\Re(s) = \frac{1}{2}$.

Sarnak discusses the Grand Riemann Hypothesis in more detail in 11.2.

The objects considered by extensions to the Riemann Hypothesis become more specialized and complicated. We refer the interested reader to more comprehensive resources for further material on these problems [75, 76].

7

Assuming the RH and its Extensions ...

Right now, when we tackle problems without knowing the truth of the Riemann Hypothesis, it's as if we have a screwdriver. But when we have it, it'll be more like a bulldozer [79].

Peter Sarnak

The consequences of a proof of the Riemann Hypothesis to prime numbers and elementary number theory are apparent, as are their connection to practical applications such as cryptography. Aside from these considerations a large body of theory has been built on the Riemann Hypothesis. This body of mathematics is further heuristic evidence in support of the Riemann Hypothesis, as a positive result would confirm many other plausible conjectures in number theory. These results also provide further motivation for mathematicians to prove the Riemann Hypothesis and add to its importance to mathematics as a whole. In this chapter we consider some of the more important statements which would become true should a proof of the Riemann Hypothesis be found.

7.1 The Prime Number Theorem

The Prime Number Theorem was proved independently in 1896 and 1899 by Hadamard and de la Vallée Poussin respectively.

Theorem 7.1 (The Prime Number Theorem). $\pi(n) \sim \mathrm{Li}(n)$ *as* $n \to \infty$.

The proof relies on showing that $\zeta(s)$ has no zeros of the form $1+it$ for $t \in \mathbb{R}$. Thus it follows from the truth of the Riemann Hypothesis.

It is interesting to note that Erdős and Selberg both found an "elementary" proof of the Prime Number Theorem. Here elementary means that the proofs do not use any advanced theory in complex analysis, in particular they use no advanced theory of $\zeta(s)$. These proofs appear in Sections 12.11 and 12.10 respectively. The simplest proof we know of is due to Newman (included in Section 12.15) and is elegantly presented by Korevaar in Section 12.16. The fact that the Prime Number Theorem can be proven independently of the Riemann zeta function adds some plausibility to the Riemann Hypothesis.

7.2 Goldbach's Conjecture

In 1742, Goldbach conjectured, in a letter to Euler, that every natural number, $n \geq 5$, is a sum of three prime numbers (Euler gave the equivalent reformulation; every even number greater than 3 is a sum of two primes) [75]. This conjecture is referred to as Goldbach's strong conjecture and is one of the oldest unsolved problems in number theory. A related problem is Goldbach's weak conjecture.

Conjecture 7.2 (Goldbach's Weak Conjecture). *Every odd number, $n > 7$, can be expressed as a sum of three odd primes.*

Hardy and Littlewood proved in [61] that the gRH implies Goldbach's weak conjecture for sufficiently large n. In 1937, Vinogradov gave the following result without assuming any variant of the RH.

Theorem 7.3. *Every sufficiently large odd number, $N \geq N_0$, is the sum of 3 prime numbers [155].*

The number N_0 is effectively computable. However, the best known N_0 is still too large to be verified, even with the aid of a computer. In 1997 Deshouillers, Effinger, te Riele, and Zinoviev proved the following result.

Theorem 7.4. *Assuming the gRH, every odd number, $n > 5$, can be expressed as a sum of three prime numbers. [47]*

7.3 More Goldbach

Hardy and Littlewood [62] proved that if the gRH (see Section 6.2) is true, then almost all even numbers are a sum of two primes. Specifically, if $E(N)$ denotes the number of even integers, $n < N$, that are not a sum of two primes, then $E(N) = O(N^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$.

Another important result along these lines was proven by J.J. Chen in 1973.

Theorem 7.5 (Chen's Theorem). *Every sufficiently large even integer is a sum of a prime and a product of at most two primes [29].*

7.4 Primes in a Given Interval

A question of interest in the study of prime numbers concerns the existence of a prime p such that $a < p < b$ for given a and b. Bertrand's Postulate of 1845 states that between a and $2a$ there is always a prime number (for $a > 1$). It is not difficult to show the following under the assumption of the Riemann Hypothesis.

Theorem 7.6. *If the Riemann Hypothesis is true, then for sufficiently large x and for any $\alpha > \frac{1}{2}$, there exists a prime, p, in $(x, x + x^{\alpha})$ [122].*

7.5 The Least Prime in Arithmetic Progressions

The extended Riemann Hypothesis can be applied to the problem of computing (or estimating) $\pi(x; k, l)$, where $\pi(x; k, l)$ is defined as,

$$\pi(x; k, l) := |\{p \ : \ p \text{ is prime}, \ p \leq x, \text{ and } p \equiv l \pmod{k}\}|.$$

Titchmarsh proved the following result given the eRH (see Section 6.4).

Theorem 7.7. *If the eRH is true, then the least prime $p \equiv l \pmod{k}$ is less than $k^{2+\varepsilon}$, where $\varepsilon > 0$ is arbitrary and $k > k_0(\varepsilon)$ [30, p. xiv].*

Unconditionally, the best known result is due to Heath-Brown that the least prime p satisfies $p \ll k^{5.5}$ [65].

7.6 Primality Testing

The performance of a variety of algorithms for primality testing rely on the gRH. The probabilistic Miller-Rabin primality test runs in deterministic polynomial-time assuming the gRH [98]. Also, the probabilistic Solovay-Strassen algorithm is provably deterministic under the gRH [2]. However both of these results are superseded by the results of Agrawal, Kayal and Saxena (presented in Section 12.20). Their paper contains an unconditional deterministic polynomial-time algorithm for primality testing. Again the removal of the gRH as an assumption adds to the plausibility of the RH.

7.7 Artin's Conjecture

There is no statement in the literature that is uniquely identified as Artin's Conjecture. We present one such conjecture related to primitive roots in $\mathbb{Z}/p\mathbb{Z}$.

Definition 7.8. *Given a prime, p, an integer a is a primitive root modulo p if $a^\alpha \not\equiv 1 \pmod{p}$ for any $0 < \alpha \leq p-2$.*

Artin's Conjecture is as follows.

Conjecture 7.9. *Every $a \in \mathbb{Z}$, where a is not square and $a \neq -1$, is a primitive root modulo p for infinitely many primes p.*

The Conjecture was proven by Hooley in 1967 assuming the gRH [68]. Artin's Conjecture was proven unconditionally for infinitely many a by Ram Murty and Gupta using sieve methods [101, 102]. This result was improved in 1986 by Heath-Brown to the following.

Theorem 7.10. *If q, r, s are three nonzero multiplicatively independent integers such that none of $q, r, s, -3qr, -3qs, -3rs, qrs$ is a square, then the number $n(x)$ of primes $p \leq x$ for which at least one of q, r, s is a primitive root modulo p satisfies $n(x) \gg \frac{x}{\log^2 x}$ [64].*

From this it follows that there are at most two positive primes k for which Artin's conjecture fails. These results also add to the plausibility of the RH as they are unconditional.

7.8 Bounds on Dirichlet L-series

Let $L(1, \chi_D)$ be the Dirichlet L-series,

$$L(1, \chi_D) = \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n},$$

at 1, where χ_D is a non-principal Dirichlet character with modulus D (see Section 6.2). It is well known [56] that $|L_D(1, \chi)|$ is bounded by

$$D^{-\varepsilon} \ll_\epsilon |L_D(1, \chi)| \ll \log D.$$

Littlewood [92], assuming the gRH, gave the following bounds on $|L_D(1, \chi)|$:

$$\frac{1}{\log \log D} \ll |L_D(1, \chi)| \ll \log \log D.$$

7.9 The Lindelöf Hypothesis

The Lindelöf Hypothesis (LH) states that $\zeta(\frac{1}{2} + it) = O(t^{\varepsilon})$ for any $\varepsilon > 0$, $t \geq 0$. We can make the statement that $\zeta(\sigma + it) = O(|t|^{\max\{1-2\sigma,0\}+\varepsilon})$ for any $\varepsilon > 0$, $0 \leq \sigma \leq 1$. If the Riemann Hypothesis holds, then the Lindelöf Hypothesis follows; however, it is not known whether the converse is true [147, p. 328]. We define the function $N(\sigma, T)$ as follows,

Definition 7.11. *$N(\sigma, T)$ is the number of zeros of $\zeta(\beta+it)$ such that $\beta > \sigma$, for $0 < t \leq T$.*

We can see that if the RH is true, then $N(\sigma, T) = 0$ if $\sigma \neq \frac{1}{2}$. Also, if we take $N(T)$ to be the number of zeros of $\zeta(\sigma + it)$ in the rectangle, $0 \leq \sigma < 1$, $0 \leq t \leq T$, then $N(\sigma, T) \leq N(T)$ [147]. The Lindelöf Hypothesis is equivalent to the following statement.

Equivalence 7.12 (An Equivalent LH). *For every $\sigma > \frac{1}{2}$,*

$$N(\sigma, T+1) - N(\sigma, T) = o(\log T) \quad [11].$$

This statement shows the relevance of the Lindelöf Hypothesis in terms of the distribution of the zeros of $\zeta(s)$.

7.10 Titchmarsh's $\mathcal{S}(T)$ Function

In [147], Titchmarsh considers at length some consequences of a proof of the Riemann Hypothesis. He introduces the functions $\mathcal{S}(T)$ and $\mathcal{S}_1(T)$, defined as follows.

Definition 7.13. *For a real variable $T \geq 0$, set*

$$\mathcal{S}(T) := \pi^{-1} \arg\zeta\left(\frac{1}{2} + iT\right),$$

and

$$\mathcal{S}_1(T) := \int_0^T \mathcal{S}(t)dt.$$

The connection between $\zeta(s)$ and $\mathcal{S}_1(T)$ is given by the following theorem [147, p. 221].

Theorem 7.14. *We have that,*

$$\mathcal{S}_1(T) = \frac{1}{\pi}\int_{\frac{1}{2}}^{2} \log|\zeta(\sigma + iT)|d\sigma + O(1).$$

The following results concerning $\mathcal{S}(T)$ and $\mathcal{S}_1(T)$ follow under the assumption of the RH.

Theorem 7.15. *Under the RH, for any $\varepsilon > 0$, each of the inequalities*

$$\mathcal{S}(T) > (\log T)^{\frac{1}{2}-\varepsilon},$$

and

$$\mathcal{S}_1(T) < -(\log T)^{\frac{1}{2}-\varepsilon}$$

have solutions for arbitrarily large values of T.

Recall that $f(x) = \Omega(g(x))$ if $|f(x)| \geq A|g(x)|$ for some constant A, and all values of $x > x_0$ for some x_0.

Theorem 7.16. *Under the RH we have for any $\varepsilon > 0$ that*

$$\mathcal{S}_1(T) = \Omega\left((\log T)^{\frac{1}{2}-\varepsilon}\right).$$

Titchmarsh proves that $\mathcal{S}(T) = O(\log T)$ and $\mathcal{S}_1(T) = O(\log T)$ without the RH. If we assume the RH then we can make the following improvements.

Theorem 7.17. *Under the RH we have*

$$\mathcal{S}(T) = O\left(\frac{\log T}{\log\log T}\right),$$

and

$$\mathcal{S}_1(T) = O\left(\frac{\log T}{(\log\log T)^2}\right).$$

Proofs of these theorems and their connection to the behaviour of $\zeta(s)$ are given in detail in [147, § 14.10].

7.11 Mean Values of $\zeta(s)$

An ongoing theme in the study of the Riemann zeta function, $\zeta(s)$, has been the estimation of the mean values (or moments) of $\zeta(s)$,

$$I_k(T) := \frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt,$$

for integer $k \geq 1$. Upper bounds for $I_k(T)$ have numerous applications, for example, in zero-density theorems for $\zeta(s)$ and various divisor problems.

In 1918, Hardy and Littlewood [60] proved that

$$I_1(T) \sim \log T$$

as $T \to \infty$. In 1926, Ingham [71] showed that

$$I_2(T) \sim \frac{1}{2\pi^2}(\log T)^4.$$

Although it has been conjectured that

$$I_k(T) \sim c_k(\log T)^{k^2}$$

for some positive constant c_k, no asymptotic estimate for I_k when $k \geq 3$ is known. Conrey and Ghosh (in an unpublished work) conjectured a more precise form of the constant c_k, namely

$$I_k(T) \sim \frac{a(k)g(k)}{\Gamma(k^2+1)}(\log T)^{k^2},$$

where

$$a(k) := \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)} \right)^2 p^{-m} \right),$$

and $g(k)$ is an integer whenever k is an integer. The results of Hardy-Littlewood and Ingham give $g(1) = 1$ and $g(2) = 2$ respectively. Keating and Snaith [78] suggested that the characteristic polynomial of a large random unitary matrix can be used to model the value distribution of the Riemann zeta function near a large height T. Based on this, Keating and Snaith made an explicit conjecture on the integer $g(k)$ in terms of Barnes' G-function.

Conrey and Ghosh showed in [35] that the RH implies

$$I_k(T) \geq (a(k) + o(1))(\log T)^{k^2}$$

as $T \to \infty$. Later, Balasubramanian and Ramachandra [12] removed the assumption of the RH. Their result was subsequently improved by Soundararajan in [139] to

$$I_k(T) \geq 2(a_k + o(1))(\log T)^{k^2}$$

for $k \geq 2$.

8

Failed Attempts at Proof

George Pólya once had a young mathematician confide to him that he was working on the great Riemann Hypothesis. " I think about it every day when I wake up in the morning," he said. Pólya sent him a reprint of a faulty proof that had been once been submitted by a mathematician who was convinced he'd solved it, together with a note: "If you want to climb the Matterhorn you might first wish to go to Zermatt where those who have tried are buried." [67]

In this chapter we discuss four famous failed attempts at the Riemann Hypothesis. Though flawed, all of these attempts spurred further research into the behaviour of the Riemann zeta function.

8.1 Stieltjes and The Mertens Conjecture

Recall that the Mertens' function is defined as

$$M(n) := \sum_{k=1}^{n} \mu(k)$$

where $\mu(k)$ is the Möbius function. The statement generally referred to as the Mertens Conjecture is that for any $n \geq 1$, $|M(n)| < n^{\frac{1}{2}}$. The truth of the Mertens conjecture implies the truth of the Riemann Hypothesis. In 1885, Stieltjes published a note in the *Comptes Rendus of the Paris Academy of Sciences* in which he claimed to have proved that $M(n) = O(n^{\frac{1}{2}})$. However he died without publishing his result, and no proof was found in his papers posthumously [46]. Odlyzko and te Riele prove in [119] that the Mertens conjecture is false. They show that,

$$\limsup_{n\to\infty} M(n)n^{-\frac{1}{2}} > 1.06 \quad \text{and,}$$
$$\liminf_{n\to\infty} M(n)n^{-\frac{1}{2}} < -1.009.$$

Their proof relies on accurate computation of the first 2000 zeros of $\zeta(s)$. While it is not impossible that $M(n) = O(n^{\frac{1}{2}})$ they feel that it is improbable. The Riemann Hypothesis is in fact equivalent to the conjecture that $M(n) = O(n^{\frac{1}{2}+\varepsilon})$, for any $\varepsilon > 0$ (see chapter 5 for equivalences).

8.2 Hans Rademacher and False Hopes

In 1945, Time Magazine reported that Hans Rademacher had submitted a flawed proof of the Riemann Hypothesis to the journal Transactions of the American Mathematical Society. The text of the article follows:

> A sure way for any mathematician to achieve immortal fame would be to prove or disprove the Riemann hypothesis. This baffling theory, which deals with prime numbers, is usually stated in Riemann's symbolism as follows: "All the nontrivial zeros of the zeta function of s, a complex variable, lie on the line where sigma is 1/2 (sigma being the real part of s)." The theory was propounded in 1859 by Georg Friedrich Bernhard Riemann (who revolutionized geometry and laid the foundations for Einstein's theory of relativity). No layman has ever been able to understand it and no mathematician has ever proved it.
>
> One day last month electrifying news arrived at the University of Chicago office of Dr. Adrian A. Albert, editor of the Transactions of the American Mathematical Society. A wire from the society's secretary, University of Pennsylvania Professor John R. Kline, asked Editor Albert to stop the presses: a paper disproving the Riemann hypothesis was on the way. Its author: Professor Hans Adolf Rademacher, a refugee German mathematician now at Penn.
>
> On the heels of the telegram came a letter from Professor Rademacher himself, reporting that his calculations had been checked and confirmed by famed Mathematician Carl Siegel of Princeton's Institute for Advanced Study. Editor Albert got ready to publish the historic paper in the May issue. U.S. mathematicians, hearing the wildfire rumor, held their breath. Alas for drama, last week the issue went to press without the Rademacher article. At the last moment the professor wired meekly that it was all a mistake; on rechecking. Mathematician Siegel had discovered a flaw (undisclosed) in the Rademacher reasoning. U.S. mathematicians felt much like the morning after a phony armistice celebration. Sighed Editor Albert: "The whole thing certainly raised a lot of false hopes." [143]

8.3 Turán's Condition

Turán showed that if for all N sufficiently large, the N^{th} partial sum of $\zeta(s)$ does not vanish for $\sigma > 1$, then the Riemann Hypothesis follows [148]. However, H. Montgomery proved that this approach will not work as for any positive $c < \frac{4}{\pi} - 1$, the N^{th} partial sum of $\zeta(s)$ has zeros in the half-plane $\sigma > 1 + c\frac{\log\log N}{\log N}$. [100]

8.4 Louis de Branges' Approach

Louis de Branges is currently Edward C. Elliott Distinguished Professor of Mathematics at Purdue University. In 1985, de Branges published a proof of another famous open problem, the Bieberbach Conjecture. The conjecture considers the class, S, of all $f(s) = s + a_2 s^2 + a_3 s^3 + \ldots$ which are analytic and satisfy $|f(s)| = 1$, for any $|s| = 1$.

Theorem 8.1. *For all $n > 1$, and all $f \in S$, $|a_n| \leq n$.*

The proof followed several incomplete attempts by de Branges. Since proving the Bieberbach Conjecture, he has claimed several times to have a proof of the Riemann Hypothesis. However, none of these are accepted by the mathematical community. In the words of Karl Sabbagh,

> ...another mathematician was quietly working away, ignored or actively shunned by his professional colleagues, on his proof of the Riemann Hypothesis, a proof to which he was putting the final touches early in 2002. He was Louis de Branges, a man to be taken seriously, one would think, because he had previously proved another famous theorem – the Bieberbach Conjecture. [130, p. 115]

An extensive profile of de Branges and one of his earlier proofs can be found in [130]. A press release from Purdue regarding de Branges' latest claim can be found in [24].

The approach used by de Branges has been disputed by Conrey and Li in [37]. The relevant papers by de Branges are [42, 43, 44, 45]. Also papers can be found on de Branges' website, [41].

8.5 No Really Good Idea

> *There have probably been very few attempts at proving the Riemann hypothesis, because, simply, no one has ever had any really good idea for how to go about it [32]!*

Atle Selberg

Many proofs have been offered by less illustrious mathematicians. Several popular anecdotes to this effect involve the editor of a mathematical journal staving off enthusiastic amateur mathematicians who are ignorant of the subtleties of complex analysis. Roger Heath-Brown relates his experience with these attempts,

> I receive unsolicited manuscripts quite frequently – one finds a particular person who has an idea, and no matter how many times you point out a mistake they correct that and produce something else that is also a mistake. [130, p. 112]

Any basic web-search will reveal several short and easy 'proofs' of the Hypothesis. Given the million dollar award for a proof, we suspect many more will be offered.

9

Formulae

The following is a reference list of useful formulae involving $\zeta(s)$, and its relatives. Throughout the variable n is assumed to lie in $\mathbb{N}$ [1].

The Euler Product Formula (pg 11):

$$\boxed{\begin{gathered}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \\ \text{for } \Re(s) > 1, \text{ and } p \text{ prime.}\end{gathered}}$$

Formula for $\zeta(s)$ at an even integer:

$$\boxed{\begin{gathered}\zeta(2n) = \frac{2^{2n-1}|B_{2n}|\pi^{2n}}{(2n)!}, \\ \text{where } B_n \text{ is the } n^{th} \text{ Bernoulli number.}\end{gathered}}$$

Another formula for $\zeta(s)$ at an even integer:

$$\zeta(2n) = \frac{(-1)^{n+1}2^{2n-3}\pi^{2n}}{(2^{2n}-1)(2n-2)!}\int_0^1 E_{2(n-1)}(x)dx,$$

where E_n is the n^{th} Euler polynomial.

Formula for $\zeta(s)$ at odd integer:

$$\zeta(2n+1) = \frac{(-1)^{n}2^{2n-1}\pi^{2n+1}}{(2^{2n+1}-1)(2n)!}\int_0^1 E_{2n}(x)\tan\left(\frac{\pi}{2}x\right)dx,$$

where E_n is the n^{th} Euler polynomial.

Analytic Continuation for $\zeta(s)$ (pg 14):

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left(\frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)\left(\frac{\vartheta(x)-1}{2}\right)dx\right),$$

where $\vartheta(x)$ is the Jacobi ϑ function.

Functional Equation for $\zeta(s)$ (pg 14):

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

Definition of $\xi(s)$ (pg 14):

$$\boxed{\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)}$$

Functional Equation for $\xi(s)$ (pg 15):

$$\boxed{\xi(s) = \xi(1-s)}$$

Definition of $\eta(s)$ (pg 58):

$$\boxed{\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} = (1-2^{1-s})\zeta(s)}$$

Definition of $\Xi(iz)$ (pg 53):

$$\boxed{\Xi(iz) = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right)}$$

Definition of $\mathcal{S}(T)$ (pg 67):

$$\boxed{\mathcal{S}(T) = \pi^{-1} \arg \zeta \left(\frac{1}{2} + iT \right)}$$

Definition of $\mathcal{S}_1(T)$ (pg 67):

$$\boxed{\mathcal{S}_1(T) = \int_0^T \mathcal{S}(t) dt}$$

Definition of $Z(t)$ (pg 34):

$$\boxed{Z(t) = e^{i\left(\Im \log \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2} \log \pi\right)} \zeta \left(\frac{1}{2} + it \right)}$$

The Approximate Functional Equation (pg 34):

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} + \chi(s) \sum_{n \le y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1}),$$

where $s = \sigma + it$, $2\pi xy = |t|$ for $x, y \in \mathbb{R}^+$, $0 \le \sigma \le 1$ and,

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s).$$

The Riemann-Siegel Formula (pg 35):

$$Z(t) = 2\sum_{n=1}^{N}\left(\frac{\cos\left(\theta(t) - t\log n\right)}{n^{\frac{1}{2}}}\right) + O(t^{-\frac{1}{4}}),$$
$$\text{where } N = \lfloor\sqrt{|t|/2\pi}\rfloor$$

Power series expansion of $\ln\Gamma(1+z)$:

$$\ln\Gamma(1+z) = -\ln(1+z) - z(\gamma-1) + \sum_{n=2}^{\infty}(-1)^n(\zeta(n)-1)\frac{z^n}{n}$$

An integral representation of $\zeta(s)$:

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx$$

Another integral representation of $\zeta(s)$:

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x + 1}dx$$

Laurent Series for $\zeta(s)$ at $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n,$$

where,

$$\gamma_n = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1} \right).$$

10

Timeline

There is over three hundred years of history surrounding the Riemann Hypothesis and the Prime Number Theorem. While the authoritative history of these ideas has yet to appear; this timeline briefly summarizes the high and low points.

1737 Euler proves the Euler Product Formula [46]; namely, for real s

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

1738 Euler invents the "Euler-Maclaurin summation method."

1742 Maclaurin invents the "Euler-Maclaurin summation method."

1742 Goldbach proposes the Goldbach Conjecture to Euler in a letter.

1792 Gauss proposes what would later become the Prime Number Theorem.

1802 Haros discovers and proves results concerning the general properties of Farey Series.

1816 Farey re-discovers Farey series and is given credit for their invention. In the somewhat unfair words of Hardy,
"*Farey is immortal because he failed to understand a theorem which Haros had proved perfectly fourteen years before* [59]."

1845 Bertrand postulates that for $a > 1$ there is always a prime that lies between a and $2a$.

1850 Chebyshev proves Bertrand's Postulate using elementary methods.

1859 Riemann publishes his *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* in which he proposes the RH. The original statement is,

"*One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation* [124]."

Riemann's paper contains the functional equation,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

and the formula,

$$\xi(t) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

for $s = \frac{1}{2} + it$. His work also contains analysis of the Riemann zeta function. Some of Riemann's analysis is over-looked until it is re-invigourated by Siegel (see 1932).

1885 Stieltjes claims to have a proof of what would later be called the Mertens Conjecture. His proof is never published, nor found amongst his papers posthumously. The Mertens Conjecture implies the Riemann Hypothesis. This is perhaps the first significant failed attempt at a proof.

1890 Some time after this date Lindelöf proposed the Lindelöf Hypothesis. This conjecture concerns the distribution of the zeros of Riemann's zeta [147, p. 328]. It is still unproven.

1896 Hadamard and de la Vallée Poussin independently prove the Prime Number Theorem. The proof relies on showing that $\zeta(s)$ has no zeros of the form $1 + it$ for $t \in \mathbb{R}$.

1897 Mertens publishes the Mertens Conjecture. (This conjecture is proved incorrect by Odlyzko and te Riele, see 1985).

1901 Von Koch proves that the Riemann Hypothesis is equivalent to the statement that,

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x}\log x).$$

1903 Gram calculates the first 15 zeros of $\zeta(s)$ on the critical line.

1903 In the decade following 1903, Gram, Backlund and Hutchinson independently apply Euler-Maclaurin summation to calculate $\zeta(s)$ and to verify the RH for $t \leq 300$ (where $s = \frac{1}{2} + it$).

1912 Littlewood proves that the Mertens Conjecture implies the RH.

1912 Littlewood proves that $\pi(n) < \mathrm{Li}(n)$ fails for some n.

1912 Backlund develops a method of determining the number of zeros of $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$ up to a given height [48, p. 128]. This method is used through 1932.

1914 Hardy proves that infinitely many zeros of $\zeta(s)$ lie on the critical line.

1914 Backlund calculates the first 79 zeros of $\zeta(s)$ on the critical line.

1914 Littlewood proves that $\pi(n) < \mathrm{Li}(n)$ fails for infinitely many n.

1914 Bohr and Landau prove that if $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ in the rectangle $0 \leq \Im(s) \leq T$, $\sigma < \Re(s) \leq 1$ then $N(\sigma, T) = O(T)$, for any fixed $\sigma \geq \frac{1}{2}$.

1919 Pólya conjectures that the summatory Liouville function,

$$L(x) := \sum_{n=1}^{x} \lambda(n),$$

where $\lambda(n)$ is the Liouville function, satisfies $L(x) \leq 0$ for $x \geq 2$. (This conjecture is proved incorrect by Haselgrove, see 1958).

1920 Carlson proves the density theorem.

Theorem 10.1 (The Density Theorem). *For any* $\varepsilon > 0$ *and* $\frac{1}{2} \leq \sigma \leq 1$, *we have* $N(\sigma, T) = O(T^{4\sigma(1-\sigma)+\varepsilon})$.

1922 Hardy and Littlewood show that the gRH implies Goldbach's Weak Conjecture.

1923 Hardy and Littlewood [62] prove that if the gRH is true, then almost all even numbers are the sum of two primes. Specifically, if $E(N)$ denotes the number of even integers, less than N, that are not a sum of two primes, then $E(N) \ll N^{\frac{1}{2}+\varepsilon}$.

1924 Franel and Landau discover an equivalence to the RH involving Farey series. The details are not complicated, but are rather lengthy.

1925 Hutchinson calculates the first 138 zeros of $\zeta(s)$ on the critical line.

1928 Littlewood [92] shows that the gRH bounds $L_D(1, \chi)$ as,

$$\frac{1}{\log \log D} \ll |L_D(1, \chi)| \ll \log \log D.$$

1932 Siegel analyzes Riemanns private (and public) papers. He finds (amongst other things) a formula for calculating values of $\zeta(s)$ that is more efficient than Euler-Maclaurin summation. The method is referred to as the Riemann-Siegel Formula and is used in some form up to the present.

Siegel is credited with re-invigourating Riemann's most important results regarding $\zeta(s)$. In the words of Edwards, "It is indeed fortunate that Siegel's concept of scholarship derived from the older tradition of respect for the past rather than the contemporary style of novelty [48, p. 136]."

1934 Speiser shows that the Riemann Hypothesis is equivalent to the non-vanishing of $\zeta'(s)$ in $0 < \sigma < \frac{1}{2}$.

1935 Titchmarsh calculates the first 1,041 zeros of $\zeta(s)$ on the critical line.

1937 Vinogradov proves the following result related to Goldbach's Conjecture without assuming any variant of the RH.

Theorem 10.2. *Every sufficiently large odd number is a sum of 3 prime numbers.*

1940 Ingham shows that $N(\sigma, T) = O(T^{3\left(\frac{1-\sigma}{2-\sigma}\right)} \log^5 T)$. This is still the best known result for $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$.

1941 Weil proves that the Riemann Hypothesis is true for function fields [162].

1942 Ingham publishes a paper building on the conjectures of Mertens and Pólya. He proves that not only do both conjectures imply the truth of the Hypothesis, and the simplicity of the zeros, but they also imply a linear dependence between the imaginary parts of the zeros.

1942 Selberg proves that a positive proportion of the zeros of $\zeta(s)$ lie on the critical line.

1943 Alan Turing publishes two important developments. The first is an algorithm for computing $\zeta(s)$ (made obsolete by better estimates to the error terms in the Riemann-Siegel Formula). The second is a method for calculating $N(T)$, and gives a powerful tool for verifying the RH up to a given height.

1945 Time magazine publishes a short article detailing a recent failed attempt at a proof of the RH. The proof was submitted for review and publication to Transactions of the American Mathematical Society by Hans Rademacher and subsequently withdrawn.

1948 Turán shows that if for all N sufficiently large, the N^{th} partial sum of $\zeta(s)$ does not vanish for $\sigma > 1$ then the Riemann Hypothesis follows [148].

1949 Selberg and Erdős, building on the work of Selberg, both find 'elementary' proofs of the Prime Number Theorem.

1951 In [147], Titchmarsh considers at length some consequences of a proof of the Riemann Hypothesis. He considers sharper bounds for $\zeta(s)$, as well as the functions $\mathcal{S}(T)$ and $\mathcal{S}_1(T)$.

1951 Sometime before 1951, Titchmarsh found that the eRH can be applied in considering the problem of computing (or estimating) $\pi(x;k,l)$. He showed that if the eRH is true, then the least $p \equiv l \pmod{k}$ is less than $k^{2+\varepsilon}$, where $\varepsilon > 0$ is arbitrary and $k > k_0(\varepsilon)$. [30, p. xiv]

1953 Turing calculates the first 1,104 zeros of $\zeta(s)$ on the critical line.

1955 Skewes bounds the first n such that $\pi(n) < \mathrm{Li}(n)$ fails. This bound is improved in the future, but retains the name 'Skewes number'.

1955 Beurling finds the Nyman-Beurling equivalent form.

1956 Lehmer calculates the first 15,000 zeros of $\zeta(s)$ on the critical line, and later in the same year the first 25,000 zeros.

1958 Meller calculates the first 35,337 zeros of $\zeta(s)$ on the critical line.

1958 Haselgrove disproves Pólya's conjecture.

1966 Lehman improves Skewes' bound.

1966 Lehman calculates the first 250,000 zeros of $\zeta(s)$ on the critical line.

1967 Hooley proves that Artin's Conjecture holds under the assumption of the eRH. Artin's Conjecture is,

Conjecture 10.3. *Every $a \in \mathbb{Z}$, where a is not square and $a \neq -1$, is a primitive root modulo p for infinitely many primes p.*

Hooley's proof appears in [68].

1968 Rosser, Yohe and Schoenfeld calculate the first 3,500,000 zeros of $\zeta(s)$ on the critical line.

1968 Louis de Branges makes the first of his several attempts to prove the RH. Other proofs were offered in 1986, 1992 and 1994.

1973 Montgomery conjectures that the correlation for the zeros of the zeta function is,

$$1 - \frac{\sin^2(\pi x)}{(\pi x)^2}.$$

1973 Chen proves that every sufficiently large even integer is a sum of a prime and a product of at most two primes.

1974 The probabilistic Solovay-Strassen algorithm for primality testing is published. It can be made deterministic under the gRH [2].

1975 The probabilistic Miller-Rabin algorithm for primality testing is published. It runs in polynomial time under the gRH [2].

1977 Redheffer shows that the Riemann Hypothesis is equivalent to the statement that,

$$\det(R_n) = O(n^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$, where $R_n := [R_n(i,j)]$ is the $n \times n$ matrix with entries,

$$R_n(i,j) = \begin{cases} 1 & \text{if } j = 1 \text{ or if } i|j \\ 0 & \text{otherwise.} \end{cases}$$

1977 Brent calculates the first 40,000,000 zeros of $\zeta(s)$ on the critical line.

1979 Brent calculates the first 81,000,001 zeros of $\zeta(s)$ on the critical line.

1982 Brent, van de Lune, te Riel and Winter calculate the first 200,000,001 zeros of $\zeta(s)$ on the critical line.

1983 van de Lune and te Riele calculate the first 300,000,001 zeros of $\zeta(s)$ on the critical line.

1983 Montgomery proves that the 1948 approach of Turán will not lead to a proof of the RH. This is because for any positive $c < \frac{4}{\pi} - 1$ the N^{th} partial sum of $\zeta(s)$ has zeros in the half-plane $\sigma > 1 + c\frac{\log\log N}{\log N}$. [100]

1984 Ram Murty and Gupta prove that Artin's Conjecture holds for infinitely many a without assuming any variant of the Riemann Hypothesis.

1985 Odlyzko and te Riele prove in [119] that the Mertens conjecture is false. They speculate that, while not impossible, it is improbable that $M(n) = O(n^{\frac{1}{2}})$. The Riemann Hypothesis is in fact equivalent to the conjecture, $M(n) = O(n^{\frac{1}{2}+\varepsilon})$.

1986 van de Lune, te Riele and Winter calculate the first 1,500,000,001 zeros of $\zeta(s)$ on the critical line.

1986 Heath-Brown proves that Artin's Conjecture fails for at most two primes.

1987 te Riele lowers the Skewes number.

1988 Odlyzko and Schönhage publish an algorithm for calculating values of $\zeta(s)$. The Odlyzko-Schönhage algorithm is currently the most efficient algorithm for determining values $t \in \mathbb{R}$ for which $\zeta(\frac{1}{2}+it) = 0$. The algorithm (found in [118]) computes the first n zeros of $\zeta(\frac{1}{2}+it)$ in $O(n^{1+\varepsilon})$ (as opposed to $O(n^{\frac{3}{2}})$ using previous methods).

1988 Barratt, Forcade and Pollington formulate a graph theoretic equivalent to the Riemann Hypothesis through Redheffer matrices.

1989 Odylzko computes 175 million consecutive zeros around $t = 10^{20}$.

1989 Conrey proves that more than 40% of the nontrivial zeros of $\zeta(s)$ lie on the critical line.

1993 Alcántara-Bode shows that the Riemann Hypothesis is true if and only if the operator A is injective, where A is the Hilbert-Schmidt integral operator on $L^2(0,1)$.

1994 Verjovsky proves that the Riemannn Hypothesis is equivalent to a problem about the rate of convergence of certain discrete measures.

1995 Volchkov proves that the statement,

$$\int_0^\infty (1-12t^2)(1+4t^2)^{-3} \int_{\frac{1}{2}}^\infty \log|\zeta(\sigma+it)|d\sigma dt = \frac{\pi(3-\gamma)}{32}$$

is equivalent to the Riemann Hypothesis, where γ is Euler's constant.

1995 Amoroso proves that the statement that $\zeta(s)$ does not vanish for $\Re(z) \geq \lambda + \varepsilon$ is equivalent to,

$$\tilde{h}(F_N) \ll N^{\lambda+\varepsilon}$$

where,

$$\tilde{h}(F_N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |F_N(e^{i\theta})|d\theta$$

and $F_N(z) = \prod_{n\leq N} \Phi_n(z)$ where $\Phi_n(z)$ denotes the n^{th} cyclotomic polynomial, and $\log^+(x) = \max(0, \log x)$.

1997 Hardy and Littlewood's 1922 result concerning Goldbach's Conjecture is improved by Deshouillers, Effinger, te Riele and Zinoviev. They prove,

Theorem 10.4. *Assuming the gRH, every odd number greater than 5 can be expressed as a sum of 3 prime numbers. [47]*

2000 Conrey and Li argue that the approach used by de Branges cannot lead to proof of the RH [37].

2000 Bays and Hudson lower the Skewes number.

2000 The Riemann Hypothesis is named by the Clay Mathematics Institute as one of seven "Millennium Prize Problems." The solution to each problem is worth one million US dollars.

2001 van de Lune calculates the first 10,000,000,000 zeros of $\zeta(s)$ on the critical line.

2004 Wedeniwski calculates the first 900,000,000,000 zeros of $\zeta(s)$ on the critical line.

2004 Gourdon calculates the first 10,000,000,000,000 zeros of $\zeta(s)$ on the critical line.

Part II

Original Papers

11

Expert Witnesses

> *Hilbert included the problem of proving the Riemann hypothesis in his list of the most important unsolved problems which confronted mathematics in 1900, and the attempt to solve this problem has occupied the best efforts of many of the best mathematicians of the twentieth century. It is now unquestionably the most celebrated problem in mathematics and it continues to attract the attention of the best mathematicians, not only because it has gone unsolved for so long but also because it appears tantalizingly vulnerable and because its solution would probably bring to light new techniques of far-reaching importance [48].*
>
> **H. M. Edwards**

This chapter contains four expository papers on the Riemann Hypothesis. These are our 'expert witnesses', and they provide the perspective of specialists in the fields of analytic number theory and complex analysis. The first two papers were commissioned by the Clay Mathematics Institute to serve as official prize descriptions. They give a thorough description of the problem, the surrounding theory, and probable avenues of attack. The last paper outlines reasons why mathematicians should remain skeptical of the Hypothesis, and possible sources of disproof.

11.1 E. Bombieri, *Problems of the millennium: The Riemann hypothesis*, 2000–2001

> *The failure of the Riemann Hypothesis would create havoc in the distribution of prime numbers. This fact alone singles out the Riemann Hypothesis as the main open question of prime number theory [63].*
>
> **Enrico Bombieri**

This paper is the official problem description for the Millenium Prize offered by the Clay Mathematics Institute. Bombieri gives the classical formulation of the Riemann Hypothesis. He also writes about the history and significance of the Riemann Hypothesis to the mathematical community. The paper also includes some justification for the prize, by way of heuristic argument for the truth of the Hypothesis. The paper includes an extensive bibliography of standard sources on the Riemann zeta function, as well as sources which consider broader extensions of the Riemann Hypothesis.

11.2 P. Sarnak, *Problems of the millennium: The Riemann hypothesis* 2004

> *The Riemann Hypothesis is the central problem and it implies many, many things. One thing that makes it rather unusual in mathematics today is that there must be over five hundred - somebody should go and count - which start "Assume the Riemann Hypothesis, " and the conclusion is fantastic. And those [conclusions] would then become theorems . . . With this one solution you would have proven five hundred theorems or more at once [129].*
>
> **Peter Sarnak**

This paper was written for the 2004 annual report of the Clay Mathematics Institute. In the paper, Peter Sarnak elaborates on Bombieri's official problem description. Sarnak focuses his attention on the Grand Riemann Hypothesis and presents some connected problems. He suggests that a proof of the Grand Riemann Hypothesis would follow analogously to the proof of the Weil conjectures, and states his skepticism towards the random matrix approach.

11.3 J. B. Conrey
The Riemann hypothesis
2003

> *It's a whole beautiful subject and the Riemann zeta function is just the first one of these, but it's just the tip of the iceberg. They are just the most amazing objects, these L-functions - the fact that they exist, and have these incredible properties are tied up with all these arithmetical things - and it's just a beautiful subject. Discovering these things is like discovering a gemstone or something. You're amazed that this thing exists, has these properties and can do this. [129]*
>
> **J. Brian Conrey**

J. Brian Conrey presents a variety of theory on the Riemann Hypothesis motivated by three workshops sponsored by the American Institute of Mathematics. This paper gives a detailed account of the various approaches that mathematicians have undertaken to attack the Riemann Hypothesis. Conrey reports on the most active areas of current research into the Hypothesis, and focuses on the areas that currently enjoy the most interest. Particularly, he highlights recent work on the connections between random matrix theory, the Riemann zeta function and L-functions. He also discusses the Landau-Siegel zero and presents the recent innovative approach of Iwaniec towards its elimination.

11.4 A. Ivić *On some reasons for doubting the Riemann hypothesis*, 2003

> *...I don't believe or disbelieve the Riemann Hypothesis. I have a certain amount of data and a certain amount of facts. These facts tell me definitely that the thing has not been settled. Until it's been settled it's a hypothesis, that's all. I would like the Riemann Hypothesis to be true, like any decent mathematician, because it's a thing of beauty, a thing of elegance, a thing that would simplify many proofs and so forth, but that's all [129].*
>
> **Aleksandar Ivić**

In this article Ivić discusses several arguments against the truth of the Riemann Hypothesis. Ivić points to heuristics on the moments of zeta functions as evidence that the Hypothesis is false. He discusses this and other conditional disproofs of the Riemann Hypothesis. This paper questions widely-held belief in the Riemann Hypothesis and cautions mathematicians against accepting the Hypothesis as fact, even in light of the empirical data.

12

The Experts Speak For Themselves

> *To appreciate the living spirit rather than the dry bones of mathematics, it is necessary to inspect the work of a master at first hand. Textbooks and treatises are an unavoidable evil ... The very crudities of the first attack on a significant problem by a master are more illuminating than all the pretty elegance of the standard texts which has been won at the cost of perhaps centuries of finicky polishing.*
>
> **Eric Temple Bell**

This chapter contains several original papers. These give the most essential sampling of the enormous body of material on the Riemann zeta function, the Riemann Hypothesis and related theory. They give a chronology of milestones in the development of the theory contained in previous chapters. We begin with Chebyshev's ground-breaking work on $\pi(x)$, continue through Riemann's proposition of the Riemann Hypothesis, and end with an ingenious algorithm for primality testing. These papers place the material into historical context and illustrate the motivations for research on and around the Riemann Hypothesis. Each paper is preceeded by a short biographical note on the author(s) and a short review of the material they present.

12.1 P. L. Chebyshev, *Sur la fonction qui détermine la totalité des nombres premiers inférieurs á une limite donnée*, 1852

Pafutney Lvovich Chebyshev was born in Okatovo, Russia, on May 16^{th}, 1821. Chebyshev's mother, Agrafena Ivanova Pozniakova, acted as his teacher and taught him basic language skills. His cousin tutored him in French and arithmetic. Chebyshev was schooled at home by various tutors until he entered Moscow University in 1837. In 1847 he was appointed to the University of St. Petersburg. During his career he contributed to several fields of mathematics. Specifically, in the theory of prime numbers he proved Bertrand's postulate in 1850, and came very close to proving the Prime Number Theorem. He was recognized internationally as a pre-eminent mathematician. He was elected to academies in Russia, France, Germany, Italy, England, Belgium and Sweden. He was also given honourary positions at every Russian university. Chebyshev died in St. Petersburg in 1894. [111]

This paper is considered the first substantial result towards the Prime Number Theorem. The Prime Number Theorem is the statement that $\pi(x) \sim \frac{x}{\log x}$. Legendre had worked on the Prime Number Theorem and had attempted to give accurate approximations to $\pi(x)$. Chebyshev shows that if any approximation to $\pi(x)$ is within order $\frac{x}{\log^N x}$, for any fixed large $N \in \mathbb{Z}$, then the approximation is $\mathrm{Li}(x)$. Chebyshev considers the expression $\frac{\pi(x)\log x}{x}$, and proves that if $\lim_{x\to\infty} \frac{\pi(x)\log x}{x}$ exists, then the limit is 1.

12.2 B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, 1859

Bernhard Riemann was born in 1826 in what is now Germany. The son of a Lutheran minister, Riemann excelled in Hebrew and theology, but had a keen interest in mathematics. He entered the Univeristy of Göttingen in 1846 as a student of theology. However, with the permission of his father, he began to study mathematics under Stern and Gauss. In 1847 Riemann moved to Berlin to study under Steiner, Jacobi, Dirichlet and Eisenstein. In 1851 he submitted his celebrated doctoral thesis on Riemann surfaces to Gauss. In turn Gauss recommended Riemann for a post at Göttingen. The name Riemann is ubiquitous in modern mathematics. Riemann's work is full of brilliant insight, and mathematicians struggle to this day with the questions he raised. Riemann died in Italy in 1866. [106]

In 1859 Riemann was appointed to the Berlin Academy of Sciences. New members of the academy were required to report on their research, and in complying with this convention Riemann presented his report *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (*On the number of primes less than a given magnitude*). In this paper, Riemann considers the properties of the Riemann zeta function. His results are stated concisely, and motivated future researchers to supply rigourous proofs. This paper contains the original statement of the Riemann Hypothesis.

We have included a digitization of Riemann's original hand-written memoir. It is followed by a translation into English, due to David R. Wilkins.

12.3 J. Hadamard, *Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques*, 1896

Jacques Hadamard was born in France in 1865. Both of his parents were teachers, and Hadamard excelled in all subjects. He entered the École Normale Supérieure in 1884. Under his professors, which included Hermite and Goursat, he began to consider research problems in mathematics. In 1892 Hadamard received his doctorate for his thesis on functions defined by Taylor series. He won several prizes for research in the fields of physics and mathematics. Hadamard was elected the president of the French Mathematical Society in 1906. His *Leçons sur le calcul des variations* of 1910 laid the foundations for functional analysis, and introduced the term 'functional'. In 1912 he was elected to the Academy of Sciences. Throughout his life Hadamard was active politically, and campaigned for peace following the second world war. His output as a researcher encompasses over 300 papers and books. Hadamard died in Paris in 1963. [109]

This paper gives Hadamard's proof of the Prime Number Theorem. The statement $\pi(x) \sim \mathrm{Li}(x)$ is proven indirectly, by considering the zeros of $\zeta(s)$. Hadamard increases the zero-free region of the Riemann zeta function by proving that $\zeta(\sigma + it) = 0$ implies that $\sigma \neq 1$. This is sometimes referred to as the Hadamard-Vallée Poussin Theorem, and from it the desired result follows. An exposition of Hadamard's method in English can be found in Chapter 3 of [147]. This paper, in conjuction with 12.4, gives the most important result in the theory of prime numbers since Chebyshev's 1852 result (see 12.1).

12.4 C. de la Vallée Poussin, *Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs a une limite donnée*, 1899

Charles de la Vallée Poussin was born in 1866 in Belgium. Though he came from a family of diverse intellectual background, he was originally interested in becoming a Jesuit priest. He attended the Jesuit College at Mons, but eventually found the experience unappealing. He changed the course of his studies and obtained a diploma in engineering. While at the University of Louvain, de la Vallée Poussin became inspired to study mathematics under Louis-Philippe Gilbert. In 1891 he was appointed Gilbert's assistant. Following Gilbert's death one year later, de la Vallée Poussin was elected to his chair at Louvain. De la Vallée Poussin's early work focused on analysis; however, his most famous result was in the theory of prime numbers. In 1896 he proved the long-standing Prime Number Theorem independently of Jacques Hadamard. His only other works in number theory are two papers on the Riemann zeta function, published in 1916. His major work was his *Cours d'Analyse* . Written in two volumes it provides a comprehensive introduction to analysis both for the beginner and the specialist. De la Vallée Poussin was elected to academies in Belgium, Spain, Italy, America and France. In 1928 the King of Belgium conferred on de la Vallée Poussin the title of Baron. He died in Louvain, Belgium, in 1962. [107]

This paper details de la Vallée Poussin's historic proof of the Prime Number Theorem. The author proves that $\pi(x) \sim \mathrm{Li}(x)$ by considering the zeros of $\zeta(\sigma + it)$. The paper contains a proof of the fact that $\zeta(\sigma + it) = 0$ implies $\sigma \neq 1$. From this fact, sometimes referred to as the Hadamard-Vallée Poussin Theorem, the desired result is deduced. The basics of de la Vallée Poussin's approach are explained in English in Chapter 3 of [147].

12.5 G. H. Hardy, *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, 1914

Godfrey Harold Hardy was born in England in 1877 to Isaac and Sophia. He performed exceptionally in school but did not develop a passion for mathematics during his youth. Instead he was drawn to the subject as a way to assert his intellectual superiority over his peers. He won scholarships to Winchester College in 1889 and to Trinity College, Cambridge, in 1896. He was elected fellow of Trinity in 1900. Hardy was an excellent collaborator and much of his best work was done with Littlewood and Ramanujan. He also worked with Titchmarsh, Ingham, Landau and Pólya. He died in Cambridgeshire, England in 1947 [108].

In this paper Hardy proves that there are infinitely many zeros of $\zeta(s)$ on the critical line. This is the first appearance of such a result. It gives a valuable piece of heuristic evidence and is requisite to the truth of the Hypothesis.

12.6 G. H. Hardy, *Prime numbers*, 1915

The Theory of Numbers has always been regarded as one of the most obviously useless branches of Pure Mathematics. The accusation is one against which there is no valid defence; and it is never more just than when directed against the parts of the theory which are more particularly concerned with primes. A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life. The theory of prime numbers satisfies no such criteria. Those who pursue it will, if they are wise, make no attempt to justify their interest in a subject so trivial and so remote, and will console themselves with the thought that the greatest mathematicians of all ages have found in it a mysterious attraction impossible to resist.

G. H. Hardy

Hardy's abilities as a mathematician were complemented by his literary abilities. He is respected not only for his research, but for his ability to communicate mathematics clearly and beautifully. His "A Mathematician's Apology" is considered one of the best accounts of the life of a creative artist.

In this paper, Hardy gives a brief exposition on the Prime Number Theorem. He begins by presenting a brief history of results in the field of prime number theory. The bulk of the paper discusses the application of analysis to number theory and in particular to the asymptotic distribution of the prime numbers. Hardy gives Chebychev's results which appear in section 12.1. He then proceeds chronologically to give Riemann's innovations, which appear in section 12.2. Finally Hardy gives an outline of the proofs given by Hadamard and de la Vallée Poussin, which appear in sections 12.3 and 12.4 respectively.

12.7 G. H. Hardy and J. E. Littlewood, *New proofs of the prime-number theorem and similar theorems*, 1915

> *I believe this to be false. There is no evidence whatever for it (unless one counts that it is always nice when any function has only real roots). One should not believe things for which there is no evidence. In the spirit of this anthology I should also record my feeling that there is no imaginable reason why it should be true. Titchmarsh devised a method, of considerable theoretical interest, for calculating the zeros. The method reveals that for a zero to be off the critical line a remarkable number of 'coincidences' have to happen. I have discussed the matter with several people who know the problem in relation to electronic calculation; they are all agreed that the chance of finding a zero off the line in a lifetime's calculation is millions to one against. It looks then as if we may never know.*
> *It is true that the existence of an infinity of L-functions raising the same problems creates a remarkable situation. Nonetheless life would be more comfortable if one could believe firmly that the hypothesis is false [93].*
>
> **John E. Littlewood**

John Edensor Littlewood was born in England in 1885 to Edward and Sylvia. When Littlewood was seven he and his family moved to South Africa so his father could take a position as the headmaster of a new school. His father, a mathematician by trade, recognized his son's talents and their failure to be realized in Africa. Littlewood was sent back to England at the age of fifteen to study at St. Paul's school in London. In 1902 he won a scholarship to Cambridge and entered Trinity College in 1903. In 1906 he began to research under the direction of E. W. Barnes. After solving the first problem assigned him by Barnes, Littlewood was assigned the Riemann Hypothesis. His legendary 35 year collaboration with G. H. Hardy started in 1911 and its products include the following paper [110]. For biographical information on G. H. Hardy see 12.6 or 12.5.

In this paper Hardy and Littlewood develop new analytic methods to prove the Prime Number Theorem. The tools they develop are general and concern Dirichlet series. To apply these methods to the Prime Number Theorem one can consider any of

$$M(x) = o(x), \quad \sum_{n \le x} \Lambda(n) \sim x, \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0.$$

Each of these statements can be shown to be equivalent to the Prime Number Theorem without employing the theory of functions of a complex variable.

12.8 A. Weil, *On the Riemann Hypothesis in function fields*, 1941

André Weil was born in France in 1906. He became interested in mathematics at a very young age and studied at the École Normale Supérieure in Paris. Weil received his doctorate degree from the University of Paris under the supervision of Hadamard. His thesis developed ideas on the theory of algebraic curves. Life during the second world war was difficult for Weil, who was a conscientious objector. After the war he moved to the United States where he became a member of the Institute for Advanced Study at Princeton. he retired from the Institute in 1976 and became a Professor Emeritus.

Weil's research focused on the areas of number theory, algebraic geometry and group theory. He laid the foundations for abstract algebraic geometry as well as the theory of abelian varieties. His work led to two Field's Medals (Deligne in 1978 for solving the Weil conjectures, and Yau in 1982 for work in three dimensional algebraic geometry). Weil was an honourary member of the London Mathematical Society, a fellow of the Royal Society of London, and a member of both the Academy of Sciences in Paris and the National Academy of Sciences in the United States [105]. In this paper Weil gives a proof of the Riemann Hypothesis for function fields. This proof builds on Weil's earlier work, presented in [161].

12.9 P. Turán, *On some approximative Dirichlet-polynomials in the theory of the zeta-function of Riemann*, 1948

Paul Turán was born in 1910 in Hungary. He demonstrated a talent for mathematics early in his schooling. Turán attended Pázmány Péter University in Budapest where he met Paul Erdős, with whom he would collaborate extensively. He completed his Ph.D. under Fejér in 1935 and published his thesis *On the number of prime divisors of integers*. Turán found it extremely difficult to find employment owing to the widespread anti-semitism of the period. He spent the war in forced labour camps which not only saved his life, but according to Turán, gave him time to develop his mathematical ideas. During his lifetime he published approximately 150 papers in several mathematical disciplines. Among his several honours are the Kossuth Prize from the Hungarian government (which he won on two occasions), the Szele Prize from the János Bolyai Mathematical Society and membership in the Hungarian Academy of Sciences [113].

In this paper Turán considers partial sums of the Riemann zeta function, $\sum_{k=1}^{n} k^{-s}$. Turán proves the condition that if there is an n_0 such that for $n > n_0$ the partial sums do not vanish in the half plane $\Re(s) > 1$ then the Riemann Hypothesis follows. Hugh Montgomery later proved that this condition cannot lead to a proof of the Hypothesis as the partial sums of the zeta function have zeros in the region indicated.

12.10 A. Selberg, *An elementary proof of the prime number theorem*, 1949

In this paper Selberg presents an elementary proof of the Prime Number Theorem. Together with the proof of Erdős (see Section 12.11) this is the first elementary proof of the PNT. Selberg proves that

$$\sum_{p\le x}\log^2 p+\sum_{pq\le x}\log p\log q=2x\log x+O(x)$$

where p and q are primes. He then uses the statement (due to Erdős) that for any $\lambda > 1$ the number of primes in $(x, \lambda x)$ is at least $Kx/\log x$ for $x > x_0$ (here K and x_0 are constants dependent on λ) to derive the Prime Number Theorem without appealing to the theory of functions of a complex variable.

12.11 P. Erdős, *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, 1949

Paul Erdős is one of the most famous and brilliant mathematicians of the 20^{th} century. He was born in Hungary in 1913. His early years were chaotic owing to the political situation at the time, as Erdős was of Jewish descent. After completing his doctoral studies at Pázmány Péter University in Budapest in 1934 Erdős moved to Manchester. At the start of the second world war he moved to the United States and took up a fellowship at Princeton. Erdős' tenure at Princeton was short lived due to his mannerisms and work habits; throughout his life he would move frequently from university to university. Erdős spent his career posing and solving difficult mathematical problems in several disciplines. He believed strongly in collaboration and few of his publications bear his name alone. His many accomplishments and eccentricities are compiled in several biographies [112].

In this paper Erdős presents an elementary proof of the Prime Number Theorem. Together with the proof of Selberg (see Section 12.10) this is the first elementary proof of the PNT. Erdős states that,

$$\sum_{p\leq x}\log^2 p+\sum_{pq\leq x}\log p\log q=2x\log x+O(x)$$

citing the paper of Selberg as proof. He then goes on to prove that for any $\lambda>1$ the number of primes in $(x,\lambda x)$ is at least $Kx/\log x$ for $x>x_0$ (here K and x_0 are constants dependent on λ). From there he gives an account of Selberg's proof of the PNT and a simplification deduced by both himself and Selberg.

12.12 S. Skewes, *On the difference* $\pi(x) - \mathrm{Li}(x)$ *(II)*, 1955

Skewes completed his Ph.D. at Cambridge in 1938 under the supervision of J. E. Littlewood. His dissertation was titled *On the difference* $\pi(x) - \mathrm{Li}(x)$. This paper is a continuation of his paper *On the difference* $\pi(x) - \mathrm{Li}(x)$ *(I)*, published in 1933. In that paper Skewes proved that $\pi(x) - \mathrm{Li}(x) > 0$ for some x satisfying $2 \leq x < X$ where an explicit bound X is given. In this paper Skewes gives a better bound X under the assumption of Hypothesis H.

Hypothesis H. *Every complex zero* $s = \sigma + it$ *of the Riemann zeta function satisfies*

$$\sigma - \frac{1}{2} \leq X^{-3} \log^{-2}(X)$$

provided that $|t| < X^3$.

Skewes shows that under the assumption of Hypothesis H, the inequality $\pi(x) - \mathrm{Li}(x) > 0$ holds for some $2 \leq x < X = \exp\exp\exp(7.703)$. He also shows that under the negation of Hypothesis H, the inequality holds for some $2 \leq x < X = 10^{10^{10^{10^3}}}$.

12.13 C. B. Haselgrove, *A disproof of a conjecture of Pólya*, 1958

In this paper Haselgrove disproves a conjecture of Pólya intricately connected to the Riemann Hypothesis. The conjecture is as follows.

Conjecture 12.1 (Pólya's Conjecture). *The quantity, $L(x) := \sum_{n \le x} \lambda(n)$ satisfies $L(x) \le 0$ for $x \ge 2$, where $\lambda(n)$ is Liouville's function.*

This conjecture had been verified for all $x \le 800000$. The conjecture implies the Riemann Hypothesis, so in his disproof Haselgrove assumes the Hypothesis. The proof is based on the result of Ingham that if the Riemann Hypothesis is true, then

$$A_T^*(u) \le \limsup A_T^*(u) \le \limsup A(u).$$

Here the functions $A(u)$ and $A_T^*(u)$ are defined as

$$\begin{aligned} A(u) &:= e^{-\frac{u}{2}} L(e^u), \\ A_T^*(u) &:= \alpha_0 + 2\Re\left\{\sum \left(1 - \frac{\gamma_n}{T}\right) \alpha_n e^{iu\gamma_n}\right\} \end{aligned}$$

where the sum ranges over the values of n for which $0 < \gamma_n < T$ and γ_n is the imaginary part of the n^{th} non-trivial zero, ρ_n, of the Riemann zeta function. Furthermore,

$$\alpha_0 := \frac{1}{\zeta\left(\frac{1}{2}\right)}, \quad \text{and} \quad \alpha_n := \frac{\zeta(2\rho_n)}{\rho_n \zeta'(\rho_n)}.$$

So, if there are T and u that give $A_T^*(u) > 0$ then the conjecture of Pólya is false. Haselgrove finds $T = 1000$ and $u = 831.847$ for which $A_T^*(u) = 0.00495$ and so disproves the conjecture. This finding is based on the calculation of the first 1500 values of ρ_n.

12.14 H. Montgomery, *The pair correlation of zeros of the zeta function*, 1973

In a certain standard terminology the Conjecture may be formulated as the assertion that $1-((\sin \pi u)/\pi u)^2$ is the pair correlation function of the zeros of the zeta function. F. J. Dyson has drawn my attention to the fact that the eigenvalues of a random complex Hermitian or unitary matrix of large order have precisely the same pair correlation function. This means that the Conjecture fits well with the view that there is a linear operator (not yet discovered) whose eigenvalues characterize the zeros of the zeta function. The eigenvalues of a random real symmetric matrix of large order have a different pair correlation, and the eigenvalues of a random symplectic matrix of large order have yet another pair correlation. In fact the "form factors" $F_r(\alpha)$, $F_s(\alpha)$ of these latter pair correlations are nonlinear for $0 < \alpha < 1$, so our Theorem enables us to distinguish the behaviour of the zeros of $\zeta(s)$ from the eigenvalues of such matrices. Hence, if there is a linear operator whose eigenvalues characterize the zeros of the zeta function, we might expect that it is complex Hermitian or unitary.

H. Montgomery

In this paper Montgomery assumes the Riemann Hypothesis and considers the differences between the imaginary parts of the non-trivial zeros. To this end Montgomery defines,

$$F(\alpha) = F(\alpha, T) := \left(\left(\frac{T}{2\pi} \right) \log T \right)^{-1} \sum_{0<\gamma\leq T;\, 0<\gamma'\leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'),$$

where α and $T \geq 2$ are real, $w(u) := \frac{4}{4+u^2}$, and γ and γ' are the imaginary parts of non-trivial zeros of the zeta function. Montgomery proves that $F(\alpha) = F(-\alpha)$ is real, $F(\alpha) \geq -\varepsilon$ for all α whenever $T > T_0(\varepsilon)$ and that for fixed $\alpha \in [0,1)$, $F(\alpha) = (1+o(1))T^{-2\alpha} \log T + \alpha + o(1)$ uniformly for $0 \leq \alpha \leq 1-\varepsilon$ as $T \to \infty$. Montgomery also makes some further conjectures on the differences $\gamma - \gamma'$ and suggests a connection with the eigenvalues of a random complex Hermitian or unitary matrix of large order.

12.15 D. J. Newman, *Simple analytic proof of the prime number theorem*, 1980

This paper endeavours to give a simple, but not elementary, proof of the Prime Number Theorem. Newman offers two proofs of the Prime Number Theorem. In the first he proves the following result, due to Ingham, in a novel way.

Theorem 12.2. *Suppose* $|a_n| \leq 1$ *and form the series* $\sum a_n n^{-z}$ *which clearly converges to an analytic function* $F(z)$ *for* $\Re z > 1$. *If, in fact,* $F(z)$ *is analytic throughout* $\Re z \geq 1$, *then* $\sum a_n n^{-z}$ *converges throughout* $\Re z \geq 1$.

The novelty of the proof lies in the cleverly chosen contour integral,

$$\int_{\Gamma} f(z) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz,$$

where Γ is a specific finite contour. From this result the convergence of $\sum \mu(n)/n$ follows directly (here $\mu(n)$ is the Möbius function). This result is equivalent to the Prime Number Theorem as shown by Landau. In his second proof of the Prime Number Theorem, Newman applies the theorem to show that

$$\sum_{p<n} \frac{\log p}{p} - \log n$$

converges to a limit. This result is also equivalent to the Prime Number Theorem.

12.16 J. Korevaar, *On Newman's quick way to the prime number theorem*, 1982

In this paper Korevaar presents Newman's simple proof of the Prime Number Theorem (see Section 12.15). Korevaar's paper is expository in nature and he presents Newman's method in greater detail. He starts with a brief historical note on the Prime Number Theorem and presents some elementary properties of the zeta function that Newman takes as given. Korevaar proceeds to prove the Prime Number Theorem applying Newman's method to Laplace integrals as opposed to Dirichlet series, and replacing the Ikehara-Wiener Tauberian theorem by a "poor man's" version. Korevaar's presentation is clear and detailed, and is an excellent starting point for studying proofs of the Prime Number Theorem.

12.17 H. Daboussi, *Sur le théorème des nombres premiers*, 1984

In this paper Daboussi gives an elementary proof of the Prime Number Theorem (for other elementary proofs see Sections 12.10, 12.11 and 12.18). This elementary proof differs from those of Selberg and Erdős in that it makes no use of the formula

$$\sum_{p \le x} \log^2 p + \sum_{pq \le x} \log p \log q = 2x \log x + O(x)$$

where p and q are primes. Daboussi uses the identity $\log n = \sum_{m|n} \Lambda(m)$ of Chebyshev and a sieving technique. He does not prove the classical form of the Prime Number Theorem directly, however he proves an equivalent statement, $M(x)/x \to 0$ as $x \to \infty$ where $M(x) = \sum_{n \le x} \mu(n)$ and $\mu(n)$ is the Möbius function.

12.18 A. Hildebrand, *The prime number theorem via the large sieve*, 1986

In this paper Hildebrand gives an elementary proof of the Prime Number Thoerem (for other elementary proofs see Sections 12.10, 12.11 and 12.17). Hildebrand proves the following statement which is equivalent to the Prime Number Theorem, $M(x) = o(1)$ where $M(x) = \sum_{n \leq x} \mu(n)$ and $\mu(n)$ is the Möbius function. His proof, in common with the proof of Daboussi, does not use Selberg's formula

$$\sum_{p \leq x} \log^2 p + \sum_{pq \leq x} \log p \log q = 2x \log x + O(x)$$

where p and q are primes. Hildebrand's proof makes use of a large sieve inequality to show that $M(x) - M(x') \to 0$ as $x \to \infty$, uniformly for $x \leq x' \leq x^{1+\eta(x)}$ where $\eta(x) \to 0$. From this statement he deduces the Prime Number Thoerem in the abovementioned form.

12.19 D. Goldston and H. Montgomery, *Pair correlation of zeros and primes in short intervals*, 1987

In this paper the authors prove two main results. First they prove that, if the Riemann Hypothesis is true, then,

$$\int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^2 x^{-2} dx \ll \delta(\log X)\left(\log\left(\frac{2}{\delta}\right)\right)$$

is valid for $0 < \delta \leq 1$, $X \geq 2$. Here the function $\psi(x)$ is defined as

$$\psi(x) := \sum_{p^m \leq x} \log p.$$

This extends a result of Selberg from 1943. The second result builds on the work of Montgomery presented in Section 12.14. On the Riemann Hypothesis and the hypothesis that

$$F(X,T) = \sum_{0<\gamma,\gamma'\leq T} X^{i(\gamma-\gamma')} w(\gamma-\gamma') \sim \frac{T}{2\pi}\log T$$

holds uniformly for $X^{-B_1}(\log X)^{-3} \leq T \leq X^{B_2}(\log X)^3$, where $w(u) = 4/(4+u^2)$ and γ, γ' are the imaginary parts of non-trivial zeros of $\zeta(s)$, the authors show that

$$\int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^2 dx \sim \frac{1}{2}\delta X^2 \log\left(\frac{1}{\delta}\right)$$

holds uniformly for $X^{-B_2} \leq \delta \leq X^{-B_1}$.

12.20 M. Agrawal, N. Kayal and N. Saxena, *PRIMES is in P*, 2004

The majority of the papers presented so far are focused on the Prime Number Theorem. This theorem gives an approximation to the distribution of prime numbers. However, it does not give any information as to whether or not a specific number is prime. This question is practical, as prime numbers (or determining which numbers are prime) are essential in terms of gathering computational evidence for some of the conjectures we have presented, and more modernly, in terms of cryptography and internet security. More specifically, the algorithm presented in this paper removes the Extended Riemann Hypothesis as a condition to performing primality testing in deterministic polynomial time.

In this paper, the authors present an algorithm for primality testing that runs in deterministic polynomial time without assuming the Riemann Hypothesis. The algorithm returns "PRIME" or "COMPOSITE" on input n in $O(\log^{15/2} n)$ operations. The authors present a rigourous proof, and discuss further improvements to the running time of the algorithm. Particularly striking is the simplicity of both the algorithm and its proof.

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