16

Multipole Fields

In Chapters 3 and 4 on electrostatics the spherical harmonic expansion of the scalar potential was used extensively for problems possessing some symmetry property with respect to an origin of coordinates. Not only was it useful in handling boundary-value problems in spherical coordinates, but with a source present it provided a systematic way of expanding the potential in terms of multipole moments of the charge density. For time-varying electromagnetic fields the scalar spherical harmonic expansion can be generalized to an expansion in vector spherical waves. These vector spherical waves are convenient for electromagnetic boundary-value problems possessing spherical symmetry properties and for the discussion of multipole radiation from a localized source distribution. In Chapter 9 we have already considered the simplest radiating multipole systems. In the present chapter we present a systematic development.

16.1 Basic Spherical Wave Solutions of the Scalar Wave Equation

As a prelude to the vector spherical wave problem, we consider the scalar wave equation. A scalar field $\psi(x, t)$ satisfying the source-free wave equation,

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (16.1)$$

can be Fourier-analyzed in time as

$$\psi(x, t) = \int_{-\infty}^{\infty} \psi(x, \omega)e^{-i\omega t} d\omega \quad (16.2)$$

with each Fourier component satisfying the Helmholtz wave equation,

$$(\nabla^2 + k^2)\psi(x, \omega) = 0 \quad (16.3)$$
with \( k^2 = \omega^2/c^2 \). For problems possessing symmetry properties about some origin it is convenient to have fundamental solutions appropriate to spherical coordinates. The representation of the Laplacian operator in spherical coordinates is given in equation (3.1). The separation of the angular and radial variables follows the well-known expansion,

\[
\psi(x, \omega) = \sum_{l,m} f_l(r) Y_{lm}(\theta, \phi)
\]

(16.4)

where the spherical harmonics \( Y_{lm} \) are defined by (3.53). The radial functions \( f_l(r) \) satisfy the radial equation,

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{k^2}{r^2} - \frac{l(l + 1)}{r^2} \right] f_l(r) = 0
\]

(16.5)

With the substitution,

\[
f_l(r) = r^{\frac{l}{2}} u_l(r)
\]

(16.6)

equation (16.5) is transformed into

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{k^2}{r^2} - \frac{l(l + \frac{3}{2})^2}{r^2} \right] u_l(r) = 0
\]

(16.7)

This equation is just Bessel's equation (3.75) with \( \nu = l + \frac{1}{2} \). Thus the solutions for \( f_l(r) \) are

\[
f_l(r) \sim \frac{1}{r^{\frac{l}{2}}} J_{l+\frac{1}{2}}(kr), \quad \frac{1}{r^{\frac{l}{2}}} N_{l+\frac{1}{2}}(kr)
\]

(16.8)

It is customary to define spherical Bessel and Hankel functions, denoted by \( j_l(x), n_l(x), h_l^{(1,2)}(x) \), as follows:

\[
\begin{align*}
  j_l(x) &= \left( \frac{\pi}{2x} \right)^{\frac{l}{2}} J_{l+\frac{1}{2}}(x) \\
  n_l(x) &= \left( \frac{\pi}{2x} \right)^{\frac{l}{2}} N_{l+\frac{1}{2}}(x) \\
  h_l^{(1,2)}(x) &= \left( \frac{\pi}{2x} \right)^{\frac{l}{2}} \left[ J_{l+\frac{1}{2}}(x) \pm i N_{l+\frac{1}{2}}(x) \right]
\end{align*}
\]

(16.9)

For real \( x \), \( h_l^{(2)}(x) \) is the complex conjugate of \( h_l^{(1)}(x) \). From the series expansions (3.82) and (3.83) one can show that

\[
\begin{align*}
  j_l(x) &= (-x)^l \left( \frac{1}{x} \frac{d}{dx} \right)^l \left( \frac{\sin x}{x} \right) \\
  n_l(x) &= -(-x)^l \left( \frac{1}{x} \frac{d}{dx} \right)^l \left( \frac{\cos x}{x} \right)
\end{align*}
\]

(16.10)
For the first few values of $l$ the explicit forms are:

\[
\begin{align*}
  j_0(x) &= \frac{\sin x}{x}, \quad n_0(x) = -\frac{\cos x}{x}, \quad h_0^{(1)}(x) = \frac{e^{ix}}{ix} \\
  j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \\
  h_1^{(1)}(x) &= -\frac{e^{ix}}{x} \left(1 + \frac{i}{x}\right) \\
  j_2(x) &= \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin x - \frac{3 \cos x}{x^2}, \quad n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right) \cos x - \frac{3 \sin x}{x^2} \\
  h_2^{(1)}(x) &= \frac{ie^{ix}}{x} \left(1 + \frac{3i}{x} - \frac{3}{x^2}\right)
\end{align*}
\]
(16.11)

From the asymptotic forms (3.89)-(3.91) it is evident that the small argument limits are

\[
\begin{align*}
  x \ll l \\
  j_l(x) &\to \frac{x^l}{(2l + 1)!} \\
  n_l(x) &\to -\frac{(2l - 1)!}{x^{l+1}}
\end{align*}
\]
(16.12)

where $(2l + 1)! = (2l + 1)(2l - 1)(2l - 3) \cdots (5)(3)(1)$. Similarly the large argument limits are

\[
\begin{align*}
  x \gg l \\
  j_l(x) &\to \frac{1}{x} \sin \left(x - \frac{l\pi}{2}\right) \\
  n_l(x) &\to -\frac{1}{x} \cos \left(x - \frac{l\pi}{2}\right) \\
  h_l^{(1)}(x) &\to (-i)^{l+1} \frac{e^{ix}}{x}
\end{align*}
\]
(16.13)

The spherical Bessel functions satisfy the recursion formulas,

\[
\begin{align*}
  \frac{2l + 1}{x} z_l(x) &= z_{l-1}(x) + z_{l+1}(x) \\
  z_l'(x) &= \frac{1}{2l + 1} \left[ lz_{l-1}(x) - (l + 1)z_{l+1}(x) \right]
\end{align*}
\]
(16.14)
where \( z_i(x) \) is any one of the functions \( j_l(x), n_l(x), h_l^{(1)}(x), h_l^{(2)}(x) \). The Wronskians of the various pairs are

\[
W(j_l, n_l) = \frac{1}{i} W(j_l, h_l^{(1)}) = -W(n_l, h_l^{(1)}) = \frac{1}{x^2} \quad (16.15)
\]

The general solution of (16.3) in spherical coordinates can be written

\[
\psi(x) = \sum_{l,m} [A_{lm}^{(1)} h_l^{(1)}(kr) + A_{lm}^{(2)} h_l^{(2)}(kr)] Y_{lm}(\theta, \phi) \quad (16.16)
\]

where the coefficients \( A_{lm}^{(1)} \) and \( A_{lm}^{(2)} \) will be determined by the boundary conditions.

For reference purposes we present the spherical wave expansion for the outgoing wave Green’s function \( G(x, x') \), which is appropriate to the equation,

\[
(V^2 + k^2)G(x, x') = -\delta(x - x') \quad (16.17)
\]

in the infinite domain. The closed form for this Green’s function, as was shown in Chapter 9, is

\[
G(x, x') = \frac{e^{ik|x-x'|}}{4\pi|x-x'|} \quad (16.18)
\]

The spherical wave expansion for \( G(x, x') \) can be obtained in exactly the same way as was done in Sections 3.8 and 3.10 for Poisson’s equation [see especially equation (3.117) and below, and (3.138) and below]. An expansion of the form,

\[
G(x, x') = \sum_{l,m} g_l(r, r') Y_l^*(\theta', \phi') Y_{lm}(\theta, \phi) \quad (16.19)
\]

substituted into (16.17) leads to an equation for \( g_l(r, r') \):

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} \right] g_l = -\frac{1}{r^2} \delta(r - r') \quad (16.20)
\]

The solution which satisfies the boundary conditions of finiteness at the origin and outgoing waves at infinity is

\[
g_l(r, r') = Aj_l(kr_<)h_l^{(1)}(kr_>) \quad (16.21)
\]

The correct discontinuity in slope is assured if \( A = ik \). Thus the expansion of the Green’s function is

\[
\frac{e^{ik|x-x'|}}{4\pi|x-x'|} = ik \sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \quad (16.22)
\]
Our emphasis so far has been on the radial functions appropriate to the scalar wave equation. We now re-examine the angular functions in order to introduce some concepts of use in considering the vector wave equation. The basic angular functions are the spherical harmonics \( Y_{lm}(\theta, \phi) \) (3.53), which are solutions of the equation,

\[
- \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_{lm} = l(l + 1) Y_{lm} \quad (16.23)
\]

As is well known in quantum mechanics, this equation can be written in the form:

\[
L^2 Y_{lm} = l(l + 1) Y_{lm} \quad (16.24)
\]

The differential operator \( L^2 = L_x^2 + L_y^2 + L_z^2 \), where

\[
L = \frac{1}{i} (\mathbf{r} \times \nabla) \quad (16.25)
\]

is the orbital angular-momentum operator of wave mechanics.

The components of \( L \) can be written conveniently in the combinations,

\[
\begin{align*}
L_+ &= L_x + iL_y = e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) \\
L_- &= L_x - iL_y = e^{-i\phi} \left( - \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) \\
L_z &= -i \frac{\partial}{\partial \phi}
\end{align*} \quad (16.26)
\]

We note that \( L \) operates only on angular variables and is independent of \( r \). From definition (16.25) it is evident that

\[
r \cdot L = 0 \quad (16.27)
\]

holds as an operator equation. From the explicit forms (16.26) it is easy to verify that \( L^2 \) is equal to the operator on the left side of (16.23).

From the explicit forms (16.26) and recursion relations for \( Y_{lm} \) the following useful relations can be established:

\[
\begin{align*}
L_+ Y_{lm} &= \sqrt{(l - m)(l + m + 1)} Y_{l,m+1} \\
L_- Y_{lm} &= \sqrt{(l + m)(l - m + 1)} Y_{l,m-1} \\
L_z Y_{lm} &= m Y_{lm}
\end{align*} \quad (16.28)
\]
Finally we note the following operator equations concerning the commutation properties of \( L \), \( L^2 \), and \( \nabla^2 \):

\[
\begin{align*}
L^2 L &= LL^2 \\
L \times L &= iL \\
L_j \nabla^2 &= \nabla^2 L_j
\end{align*}
\tag{16.29}
\]

where

\[
\nabla^2 = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r) - \frac{L^2}{r^2}
\tag{16.30}
\]

### 16.2 Multipole Expansion of the Electromagnetic Fields

In a source-free region Maxwell's equations are

\[
\begin{align*}
\nabla \times E &= -\frac{1}{c} \frac{\partial B}{\partial t}, \\
\nabla \times B &= \frac{1}{c} \frac{\partial E}{\partial t} \\
\nabla \cdot E &= 0, \\
\nabla \cdot B &= 0
\end{align*}
\tag{16.31}
\]

With the assumption of a time dependence, \( e^{-i\omega t} \), these equations become

\[
\begin{align*}
\nabla \times E &= ikB, \\
\nabla \times B &= -ikE \\
\nabla \cdot E &= 0, \\
\nabla \cdot B &= 0
\end{align*}
\tag{16.32}
\]

If \( E \) is eliminated between the two curl equations, we obtain the following equations,

\[
(\nabla^2 + k^2)B = 0, \\
\nabla \cdot B = 0
\]

and the defining relation,

\[
E = \frac{i}{k} \nabla \times B
\tag{16.33}
\]

Alternatively \( B \) can be eliminated to yield

\[
(\nabla^2 + k^2)E = 0, \\
\nabla \cdot E = 0
\]

plus

\[
B = \frac{-i}{k} \nabla \times E
\tag{16.34}
\]

Either (16.33) or (16.34) is a set of three equations which is equivalent to Maxwell's equations (16.32).

We now wish to determine multipole solutions for \( E \) and \( B \). From (16.33) it is evident that each rectangular component of \( B \) satisfies the Helmholtz wave equation (16.3). Hence each component of \( B \) can be
represented by the general solution (16.16). These can be combined to yield the vectorial result:

$$\mathbf{B} = \sum_{l,m} \left[ A_{lm}^{(1)} h_{l}^{(1)}(kr) + A_{lm}^{(2)} h_{l}^{(2)}(kr) \right] Y_{lm}(\theta, \phi)$$  \hspace{1cm} (16.35)

where \( A_{lm} \) are arbitrary constant vectors.

The coefficients \( A_{lm} \) in (16.35) are not completely arbitrary. The divergence condition \( \nabla \cdot \mathbf{B} = 0 \) must be satisfied. Since the radial functions are linearly independent, the condition \( \nabla \cdot \mathbf{B} = 0 \) must hold for the two sets of terms in (16.35) separately. Thus we require the coefficients \( A_{lm} \) to be so chosen that

$$\nabla \cdot \sum_{l,m} h_{l}(kr) A_{lm} Y_{lm}(\theta, \phi) = 0$$  \hspace{1cm} (16.36)

The gradient operator can be written in the form:

$$\nabla = \frac{\mathbf{r}}{r} \frac{\partial}{\partial r} - \frac{i}{r^{2}} \mathbf{r} \times \mathbf{L}$$  \hspace{1cm} (16.37)

where \( \mathbf{L} \) is the operator (16.25). When this is applied in (16.36), we obtain the requirement,

$$\mathbf{r} \cdot \sum_{l} \left[ \frac{\partial h_{l}}{\partial r} \sum_{m} A_{lm} Y_{lm} - \frac{i h_{l}}{r} \mathbf{L} \times \sum_{m} A_{lm} Y_{lm} \right] = 0$$  \hspace{1cm} (16.38)

From recursion formulas (16.14) it is evident that in general the coefficients \( A_{lm} \) for a given \( l \) will be coupled with those for \( l' = l \pm 1 \). This will happen unless the \((2l + 1)\) vector coefficients for each \( l \) value are such that

$$\mathbf{r} \cdot \sum_{m} A_{lm} Y_{lm} = 0$$  \hspace{1cm} (16.39)

For this special circumstance, the second term in (16.38) shows that the final condition on the coefficients is

$$\mathbf{r} \cdot (\mathbf{L} \times \sum_{m} A_{lm} Y_{lm}) = 0$$  \hspace{1cm} (16.40)

The assumption (16.39) that the field is transverse to the radius vector, together with (16.40), is sufficient to determine a unique set of vector angular functions of order \( l \), one for each \( m \) value. These can be found in a straightforward manner from (16.39) and (16.40), and the properties of the \( Y_{lm} \)'s. But it is expedient, and not too damaging at this point, to observe that the appropriate angular solution is

$$\sum_{m} A_{lm} Y_{lm'} = \sum_{m} a_{lm} Y_{lm}$$  \hspace{1cm} (16.41)

From (16.27) it is clear that the transversality condition (16.39) is satisfied. Similarly, from the second commutation relation in (16.29) and (16.27),
the final condition (16.40) is obeyed. That the functions \( f_i(r)Y_{lm} \) satisfy the wave equation (16.3) follows from the last commutation relation in (16.29).

By assumption (16.39) we have found a special set of electromagnetic multipole fields,

\[
\begin{align*}
B_{lm} &= f_i(kr)LY_{lm}(\theta, \phi) \\
E_{lm} &= \frac{i}{k} \nabla \times B_{lm}
\end{align*}
\]  

(16.42)

where

\[
f_i(kr) = A_i^{(1)}h_i^{(1)}(kr) + A_i^{(2)}h_i^{(2)}(kr)
\]  

(16.43)

Any linear combination of these fields, summed over \( l \) and \( m \), satisfies the set of equations (16.33). They have the characteristic that the magnetic induction is perpendicular to the radius vector \( \mathbf{r} \cdot B_{lm} = 0 \). They therefore do not represent a general solution to equations (16.33). They are, in fact, the spherical equivalent of the *transverse magnetic* (TM), or electric, cylindrical fields of Chapter 8.

If we had started with the set of equations (16.34) instead of (16.33), we would have obtained an alternative set of multipole fields in which \( E \) is transverse to the radius vector:

\[
\begin{align*}
E_{lm} &= f_j(kr)LY_{lm}(\theta, \phi) \\
B_{lm} &= -\frac{i}{k} \nabla \times E_{lm}
\end{align*}
\]  

(16.44)

These are the spherical wave analogs of the *transverse electric* (TE), or magnetic, cylindrical fields of Chapter 8.

Just as for the cylindrical wave-guide case, the two sets of multipole fields (16.42) and (16.44) can be shown to form a complete set of vector solutions to Maxwell's equations. The terminology electric and magnetic multipole fields will be used, rather than TM and TE, since the sources of each type of field will be seen to be the electric-charge density and the magnetic-moment density, respectively. Since the vector spherical harmonic, \( LY_{lm} \), plays an important role, it is convenient to introduce the normalized form,*

\[
X_{lm}(\theta, \phi) = \frac{1}{\sqrt{l(l+1)}} LY_{lm}(\theta, \phi)
\]  

(16.45)

with the orthogonality property,

\[
\int X_{l'm'}^* \cdot X_{lm} \, d\Omega = \delta_{ll'}\delta_{mm'}
\]  

(16.46)

* \( X_{lm} \) is defined to be identically zero for \( l = 0 \). Spherically symmetric solutions to the source-free Maxwell's equations exist only in the static limit \( k \to 0 \).
By combining the two types of fields we can write the general solution to Maxwell's equations (16.32):

\begin{align*}
\mathbf{B} &= \sum_{l,m} \left[ a_E(l,m)f_l(kr)\mathbf{X}_{lm} - \frac{i}{k} a_M(l,m)\nabla \times g_l(kr)\mathbf{X}_{lm} \right] \\
\mathbf{E} &= \sum_{l,m} \left[ \frac{i}{k} a_E(l,m)\nabla \times f_l(kr)\mathbf{X}_{lm} + a_M(l,m)g_l(kr)\mathbf{X}_{lm} \right]
\end{align*}  \quad (16.47)

where the coefficients \( a_E(l,m) \) and \( a_M(l,m) \) specify the amounts of electric \((l,m)\) multipole and magnetic \((l,m)\) multipole fields. The radial functions \( f_l(kr) \) and \( g_l(kr) \) are of form (16.43). The coefficients \( a_E(l,m) \) and \( a_M(l,m) \), as well as the relative proportions in (16.43), will be determined by the sources and boundary conditions.

16.3 Properties of Multipole Fields; Energy and Angular Momentum of Multipole Radiation

Before considering the connection between the general solution (16.47) and a localized source distribution, we examine the properties of the individual multipole fields (16.42) and (16.44). In the near zone \((kr \ll 1)\) the radial function \( f_l(kr) \) is proportional to \( n_l \), given by (16.12), unless its coefficient vanishes identically. Excluding this possibility, the limiting behavior of the magnetic induction for an electric \((l,m)\) multipole is

\[ \mathbf{B}_{lm} \to \frac{k}{l} \mathbf{L} \frac{Y_{lm}}{r^l+1} \]  \quad (16.48)

where the proportionality coefficient is chosen for later convenience. To find the electric field we must take the curl of the right-hand side. A useful operator identity is

\[ i\nabla \times \mathbf{L} = r\nabla^2 - \nabla \left( 1 + r \frac{\partial}{\partial r} \right) \frac{Y_{lm}}{r^l+1} \]  \quad (16.49)

The electric field (16.42) is

\[ \mathbf{E}_{lm} \to \frac{-i}{l} \nabla \times \mathbf{L} \frac{Y_{lm}}{r^l+1} \]  \quad (16.50)

Since \((Y_{lm}/r^{l+1})\) is a solution of Laplace's equation, the first term in (16.49) vanishes. The second term merely gives a factor \( l \). Consequently the electric field at close distances for an electric \((l,m)\) multipole is

\[ \mathbf{E}_{lm} \to -\nabla \left( \frac{Y_{lm}}{r^{l+1}} \right) \]  \quad (16.51)
This is exactly the electrostatic multipole field of Section 4.1. We note that the magnetic induction \( \mathbf{B}_{im} \) is smaller in magnitude than \( \mathbf{E}_{im} \) by a factor \( kr \). Hence, in the near zone, the magnetic induction of an electric multipole is always much smaller than the electric field. For the magnetic multipole fields (16.44) evidently the roles of \( \mathbf{E} \) and \( \mathbf{B} \) are interchanged according to the transformation,

\[
\mathbf{E}_E \rightarrow -\mathbf{B}_M, \quad \mathbf{B}_E \rightarrow \mathbf{E}_M
\]  

(16.52)

In the far or radiation zone \( (kr \gg 1) \) the multipole fields depend on the boundary conditions imposed. For definiteness we consider the example of outgoing waves, appropriate to radiation by a localized source. Then the radial function \( f_i(kr) \) is proportional to the spherical Hankel function \( h_l^{(1)}(kr) \). From the asymptotic form (16.13) we see that in the radiation zone the magnetic induction for an electric \( (1, m) \) multipole goes as

\[
\mathbf{B}_{lm} \rightarrow (-i)^l+1 \frac{e^{ikr}}{kr} Y_{lm}
\]

(16.53)

Then the electric field can be written

\[
\mathbf{E}_{lm} = \frac{(-i)^l}{k^2} \left[ \nabla \left( \frac{e^{ikr}}{r} \right) \times Y_{lm} + \frac{e^{ikr}}{r} \nabla \times Y_{lm} \right]
\]

(16.54)

Since we have already used the asymptotic form of the spherical Hankel function, we are not justified in keeping higher powers in \( (1/r) \) than the first. With this restriction and use of the identity (16.49) we find

\[
\mathbf{E}_{lm} = -(-i)^l+1 \frac{e^{ikr}}{kr} \left[ \mathbf{n} \times Y_{lm} - \frac{1}{k} (\mathbf{r} \nabla^2 - \nabla) Y_{lm} \right]
\]

(16.55)

where \( \mathbf{n} = (\mathbf{r}/r) \) is a unit vector in the radial direction. The second term is evidently \( 1/kr \) times some dimensionless function of angles and can be omitted in the limit \( kr \gg 1 \). Then we find that the electric field in the radiation zone is

\[
\mathbf{E}_{lm} = \mathbf{B}_{lm} \times \mathbf{n}
\]

(16.56)

where \( \mathbf{B}_{lm} \) is given by (16.53). These fields are typical radiation fields, transverse to the radius vector and falling off as \( r^{-1} \). For magnetic multipoles we merely make the interchanges (16.52).

The multipole fields of a radiating source can be used to calculate the energy and angular momentum carried off by the radiation. For definiteness we consider an electric \( (l, m) \) multipole and, following (16.47), write the fields as

\[
\begin{align*}
\mathbf{B}_{lm} &= a_p(l, m) h_l^{(1)}(kr) X_{lm} e^{-i\omega t} \\
\mathbf{E}_{lm} &= \frac{i}{k} \nabla \times \mathbf{B}_{lm}
\end{align*}
\]

(16.57)
For harmonically varying fields the time-averaged energy density is

\[ u = \frac{1}{16\pi} (E \cdot E^* + B \cdot B^*) \]  \hspace{1cm} (16.58)

In the radiation zone the two terms are equal. Consequently the energy in a spherical shell between \( r \) and \( (r + dr) \) (for \( kr \gg 1 \)) is

\[ dU = \frac{|a_E(l, m)|^2}{8\pi} |h^{(1)}_l(kr)|^2 r^2 dr \int X^*_l m \cdot X_{l m} d\Omega \]  \hspace{1cm} (16.59)

With the orthogonality integral (16.46) and the asymptotic form (16.13) of the spherical Hankel function, this becomes

\[ \frac{dU}{dr} = \frac{|a_E(l, m)|^2}{8\pi k^2} \]  \hspace{1cm} (16.60)

independent of the radius. For a magnetic \((l, m)\) multipole we merely replace \(a_E(l, m)\) by \(a_M(l, m)\).

The time-averaged angular-momentum density is

\[ m = \frac{1}{8\pi c} \text{Re} [r \times (E \times B^*)] \]  \hspace{1cm} (16.61)

The triple cross product can be expanded and the electric field (16.57) substituted to yield, for electric multipoles,

\[ m = \frac{1}{8\pi \omega} \text{Re} [B^* (L \cdot B)] \]  \hspace{1cm} (16.62)

Then the angular momentum in a spherical shell between \( r \) and \( (r + dr) \) is

\[ dM = \frac{|a_E(l, m)|^2}{8\pi \omega} |h^{(1)}_l(kr)|^2 r^2 \int \text{Re} \left[ X^*_l m \cdot X_{l m} \right] d\Omega \]  \hspace{1cm} (16.63)

With the explicit form (16.45) for \( X_{l m} \), (16.63) becomes in the radiation zone

\[ \frac{dM}{dr} = \frac{|a_E(l, m)|^2}{8\pi \omega k^2} \int \text{Re} \left[ Y^*_l m \right] Y_{l m} d\Omega \]  \hspace{1cm} (16.64)

From the properties of \( L Y_{l m} \) listed in (16.28) and the orthogonality of the spherical harmonics we see that only the \( z \)-component of \( dM \) exists. It has the value,

\[ \frac{dM_z}{dr} = \frac{m}{\omega} \frac{|a_E(l, m)|^2}{8\pi k^2} \]  \hspace{1cm} (16.65)

Comparison with the energy radiated (16.60) shows that the ratio of \( z \)-component of angular momentum to energy is

\[ \frac{M_z}{U} = \frac{m}{\omega} = \frac{m\hbar}{\hbar\omega} \]  \hspace{1cm} (16.66)
This has the obvious quantum interpretation that the radiation from a multipole of order \((l, m)\) carries off \(mh\) units of \(z\) component of angular momentum per photon of energy \(\hbar \omega\). In further analogy with quantum mechanics we would expect the ratio of the magnitude of the angular momentum to the energy to have the value,

\[
\frac{M^{(q)}}{U} = \frac{(M^2_x + M^2_y + M^2_z)^{1/2}}{U} = \sqrt{\frac{l(l+1)}{\omega}} \tag{16.67}
\]

But from (16.64) and (16.65) the classical result is

\[
\frac{M^{(c)}}{U} = \frac{|M_z|}{U} = \frac{|m|}{\omega} \tag{16.68}
\]

The reason for this difference lies in the quantum nature of the electromagnetic fields for a single photon. If the \(z\) component of angular momentum of a single photon is known precisely, the uncertainty principle requires that the other components be uncertain, with mean square values such that (16.67) holds. On the other hand, for a state of the radiation field containing many photons (the classical limit) the mean square values of the transverse components of angular momentum can be made negligible compared to the square of the \(z\) component. Then the classical limit (16.68) applies.*

The quantum-mechanical interpretation of the radiated angular momentum per photon for multipole fields contains the selection rules for multipole transitions between quantum states. A multipole transition of order \((l, m)\) will connect an initial quantum state specified by total angular momentum \(J\) and \(z\) component \(M\) to a final quantum state with \(J'\) in the range \(|J - l| \leq J' \leq J + l\) and \(M' = M - m\). Or, alternatively, with two states \((J, M)\) and \((J', M')\), possible multipole transitions have \((l, m)\) such that \(|J - J'| \leq l \leq J + J'\) and \(m = M - M'\).

To complete the quantum-mechanical specification of a multipole transition it is necessary to state whether the parities of the initial and final states are the same or different. The parity of the initial state is equal to the product of the parities of the final state and the multipole field. To determine the parity of a multipole field we merely examine the behavior of the magnetic induction \(B_{lm}\) under the parity transformation of inversion through the origin \((r \rightarrow -r)\). One way of seeing that \(B_{lm}\) specifies the parity of a multipole field is to recall that the interaction of a charged particle and the electromagnetic field is proportional to \((\mathbf{v} \cdot \mathbf{A})\). If \(B_{lm}\) has

* For a detailed discussion of this point, see C. Morette De Witt and J. H. D. Jensen, Z. Naturforsch., 8a, 267 (1953). They show that for a multipole field containing \(N\) photons the square of the angular momentum is equal to \([N^2m^2 + Nl(l + 1) - m^2]\hbar^2\).
a certain parity (even or odd) for a multipole transition, then the corresponding $A_{lm}$ will have the opposite parity, since the curl operation changes parity. Then, because $\mathbf{v}$ is a polar vector with odd parity, the states connected by the interaction operator $(\mathbf{v} \cdot \mathbf{A})$ will differ in parity by the parity of the magnetic induction $B_{lm}$.

For electric multipoles the magnetic induction is given by (16.57). The parity transformation $(\mathbf{r} \rightarrow -\mathbf{r})$ is equivalent to $(r \rightarrow r, \theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi)$ in spherical coordinates. The operator $\mathbf{L}$ is invariant under inversion. Consequently the parity properties of $B_{lm}$ for electric multipoles are specified by the transformation of $Y_{lm}(\theta, \phi)$. From (3.53) and (3.50) it is evident that the parity of $Y_{lm}$ is $(-1)^l$. Thus we see that the parity of fields of an electric multipole of order $(l, m)$ is $(-1)^l$. Specifically, the magnetic induction $B_{lm}$ has parity $(-1)^l$, while the electric field $E_{lm}$ has parity $(-1)^{l+1}$, since $E_{lm} \sim \nabla \times B_{lm}$.

For a magnetic multipole of order $(l, m)$ the parity is $(-1)^{l+1}$. In this case the electric field $E_{lm}$ is of the same form as $B_{lm}$ for electric multipoles. Hence the parities of the fields are just opposite to those of an electric multipole of the same order.

Correlating the parity changes and angular-momentum changes in quantum transitions, we see that only certain combinations of multipole transitions can occur. For example, if the states have $J = \frac{1}{2}$ and $J' = \frac{3}{2}$, the allowed multipole orders are $l = 1, 2$. If the parities of the two states are the same, we see that parity conservation restricts the possibilities, so that only magnetic dipole and electric quadrupole transitions occur. If the states differ in parity, then electric dipole and magnetic quadrupole radiation can be emitted or absorbed.

### 16.4 Angular Distribution of Multipole Radiation

For a general localized source distribution the fields in the radiation zone are given by the superposition,

$$
B \rightarrow \frac{e^{ikr - i\omega t}}{kr} \sum_{l, m} (-i)^{l+1} \left[ a_B(l, m)X_{lm} + a_M(l, m)n \times X_{lm} \right]
$$

$$
E \rightarrow B \times n.
$$

The coefficients $a_B(l, m)$ and $a_M(l, m)$ will be related to the properties of the source in the next section. The time-averaged power radiated per unit solid angle is

$$
\frac{dP}{d\Omega} = \frac{c}{8\pi k^2} \left| \sum_{l, m} (-i)^{l+1} \left[ a_B(l, m)X_{lm} \times n + a_M(l, m)X_{lm} \right] \right|^2
$$

(16.70)
Within the absolute value signs the polarization of the radiation is specified by the directions of the vectors. We note that electric and magnetic multipoles of a given \((l, m)\) have the same angular dependence, but have polarizations at right angles to one another. Thus the multipole order can be determined by measurement of the angular distribution of radiated power, but the character of the radiation (electric or magnetic) can be determined only by a polarization measurement.

For pure multipole of order \((l, m)\) the angular distribution (16.70) reduces to a single term,

\[
\frac{dP(l, m)}{d\Omega} = \frac{c}{8\pi k^2} |a(l, m)|^2 |X_{lm}|^2
\]  

(16.71)

From definition (16.45) of \(X_{lm}\) and properties (16.28), this can be transformed into the explicit form:

\[
\frac{dP(l, m)}{d\Omega} = \frac{c |a(l, m)|^2}{8\pi k^2 l(l+1)} \left( \frac{1}{2}(l - m)(l + m + 1) |Y_{l,m+1}|^2 + \frac{1}{2}(l + m)(l - m + 1) |Y_{l,m-1}|^2 + m^2 |Y_{lm}|^2 \right)
\]  

(16.72)

The table lists some of the simpler angular distributions.

<table>
<thead>
<tr>
<th>(l)</th>
<th>(m)</th>
<th>(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>±1</td>
</tr>
<tr>
<td>1</td>
<td>3 (\frac{\sin^2 \theta}{8\pi})</td>
<td>(3 \frac{1 + \cos^2 \theta}{16\pi})</td>
</tr>
<tr>
<td>Dipole</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(15 \frac{\sin^2 \theta \cos^2 \theta}{8\pi})</td>
<td>(5 \frac{1 - 3 \cos^2 \theta}{16\pi})</td>
</tr>
<tr>
<td>Quadrupole</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The dipole distributions are seen to be those of a dipole oscillating parallel to the \(z\) axis \((m = 0)\) and of two dipoles, one along the \(x\) axis and one along the \(y\) axis, 90° out of phase \((m = \pm 1)\). The dipole and quadrupole angular distributions are plotted as polar intensity diagrams in Fig. 16.1. These are representative of \(l = 1\) and \(l = 2\) multipole angular distributions, although a general multipole distribution of order \(l\) will involve a coherent superposition of the \((2l + 1)\) amplitudes for different \(m\), as shown in (16.70).
Fig. 16.1 Dipole and quadrupole radiation patterns for pure $(l, m)$ multipoles.
It can be shown by means of (3.69) that the absolute squares of the vector spherical harmonics obey the sum rule,

$$\sum_{m=-l}^{l} |X_{lm}(\theta, \phi)|^2 = \frac{2l + 1}{4\pi}$$  \hspace{1cm} (16.73)

Hence the radiation distribution will be isotropic from a source which consists of a set of multipoles of order $l$, with coefficients $a(l, m)$ independent of $m$, superposed incoherently. This situation usually prevails in atomic and nuclear radiative transitions unless the initial state has been prepared in a special way.

The total power radiated by a pure multipole of order $(l, m)$ is given by the integral of (16.71) over all angles. Since the $X_{lm}$ are normalized to unity, the power radiated is

$$P(l, m) = \frac{c}{8\pi k^2} |a(l, m)|^2$$  \hspace{1cm} (16.74)

For a general source the angular distribution is given by the coherent sum (16.70). On integration over angles it is easy to show that the interference terms do not contribute. Hence the total power radiated is just an incoherent sum of contributions from the different multipoles:

$$P = \frac{c}{8\pi k^2} \sum_{l,m} [|a_k(l, m)|^2 + |a_M(l, m)|^2]$$  \hspace{1cm} (16.75)

### 16.5 Sources of Multipole Radiation; Multipole Moments

Having discussed the properties of multipole fields, the radiation patterns, and the angular momentum and energy carried off, we now turn to the connection of the fields with the sources which generate them. We assume that there exists a localized distribution of charge $\rho(x, t)$, current $J(x, t)$, and intrinsic magnetization $H(x, t)$. Furthermore, we assume that the time dependence can be analyzed into its Fourier components, and we consider only harmonically varying sources,

$$\rho(x)e^{-i\omega t}, \quad J(x)e^{-i\omega t}, \quad H(x)e^{-i\omega t}$$  \hspace{1cm} (16.76)

where it is understood that we take the real part of such complex quantities. A more general time dependence can be obtained by linear superposition.

Since we are considering a magnetization density, we must preserve the
Evidently, outside the source these sets of equations reduce to (16.33) and (16.34). Consequently the general solution for \( \mathbf{B} \) and \( \mathbf{E}' \) outside the source is given by (16.47). Furthermore, even inside the source both fields still have vanishing divergence. Thus the general structure of (16.47) will be preserved, with modifications arising only in the form of the radial functions \( f_i(r) \) and \( g_i(r) \), in a way that is familiar in scalar field problems such as electrostatics or wave mechanics.

Consider, for example, the magnetic induction,

\[
\mathbf{B} = \sum_{l,m} \left[ f_{lm}(r) \mathbf{X}_{lm} - \frac{i}{k} \nabla \times g_{lm}(r) \mathbf{X}_{lm} \right]
\]

where, outside the source, to conform to the notation of (16.57) and (16.69),

\[
f_{lm}(r) \rightarrow a_E(l, m)h_i^{(1)}(kr) \quad \text{and} \quad g_{lm}(r) \rightarrow a_M(l, m)h_i^{(1)}(kr)
\]
To determine the equation satisfied by the electric multipole function \( f_{lm}(r) \) inside the source, we substitute (16.82) into the first equation of (16.80), take the scalar product of both sides with a typical \( \mathbf{X}_{lm}^* \), and integrate over all angles. All the terms on the left-hand side of the equation involving \( g_{lm}(r) \) vanish because of orthogonality, and only one term involving an \( f_{lm}(r) \) survives:

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + k^2 - \frac{l(l + 1)}{r^2} \right] f_{lm}(r) =
- \frac{4\pi}{c} \int \mathbf{X}_{lm}^* \cdot \left[ \nabla \times \mathbf{J} + c \nabla \times (\nabla \times \mathbf{M}) \right] d\Omega \quad (16.84)
\]

By substituting the equivalent expansion for \( \mathbf{E}' \) into the first equation of (16.81) and carrying out the same manipulations, we obtain an equation for \( g_{lm}(r) \):

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + k^2 - \frac{l(l + 1)}{r^2} \right] g_{lm}(r) =
-4\pi ik \int \mathbf{X}_{lm}^* \cdot \left[ \nabla \times \mathbf{M} + \frac{1}{ck^2} \nabla \times (\nabla \times \mathbf{J}) \right] d\Omega \quad (16.85)
\]

These inhomogeneous equations for \( f_{lm}(r) \) and \( g_{lm}(r) \) can be solved by means of the Green's function technique. The appropriate Green's function is (16.21), which satisfies (16.20). Denoting the right-hand side of (16.84) as \( -K_E(r) \), we can write the solution for \( f_{lm}(r) \) in the form:

\[
f_{lm}(r) = ik \int_0^\infty r'^2 j_l(kr') h_l^{(1)}(kr') K_E(r') \, dr' \quad (16.86)
\]

Outside the source \( r_\leq = r' \) and \( r_\geq = r \). Then

\[
f_{lm}(r) \to ik h_l^{(1)}(kr) \int_0^\infty r'^2 j_l(kr') K_E(r') \, dr' \quad (16.87)
\]

Comparison with (16.83) allows us to identify the electric multipole coefficient \( a_E(l, m) \). With the explicit form of \( K_E(r) \) from the right side of (16.84), we have

\[
a_E(l, m) = \frac{4\pi ik}{c} \int j_l(kr) \mathbf{X}_{lm}^* \cdot \left[ \nabla \times \mathbf{J} + c \nabla \times (\nabla \times \mathbf{M}) \right] d^3x \quad (16.88)
\]

Similarly the magnetic multipole coefficient \( a_M(l, m) \) is

\[
a_M(l, m) = -4\pi k^2 \int j_l(kr) \mathbf{X}_{lm}^* \cdot \left[ \nabla \times \mathbf{M} + \frac{1}{ck^2} \nabla \times (\nabla \times \mathbf{J}) \right] d^3x \quad (16.89)
\]
Results (16.88) and (16.89) can be transformed into more useful forms by means of the following identity:

\[
\int j_i(kr)X_{lm}^* \cdot (\nabla \times A) \, d^3x
= \frac{i}{\sqrt{l(l+1)}} \int Y_{lm}^* \left[ (\nabla \cdot A) \frac{\partial}{\partial r} [rj_i(kr)] - k^2r \cdot A j_i(kr) \right] \, d^3x \quad (16.90)
\]

\(A\) is any well-behaved vector field which vanishes at infinity faster than \(r^{-2}\).

The proof of (16.90) involves integration by parts to cast the curl operator over on \(X_{lm}\), then use of the operator relation (16.49), and another integration by parts. With \(A\) equal to \(J, \mathcal{M}, \nabla \times J, \nabla \times \mathcal{M}\), the various terms in (16.88) and (16.89) can be transformed to yield the expressions:

\[
a_E(l, m) = \frac{4\pi k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left[ \rho \frac{\partial}{\partial r} [rj_i(kr)] + \frac{ik}{c} (r \cdot J)j_i(kr) \right] \, d^3x
\]

\[
-ik\nabla \cdot (r \times \mathcal{M})j_i(kr)
\]

\[
(16.91)
\]

and

\[
a_M(l, m) = \frac{4\pi k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left[ \nabla \cdot \left( \frac{r \times J}{c} \right)j_i(kr) + \nabla \cdot \mathcal{M} \frac{\partial}{\partial r} [rj_i(kr)] \right] \, d^3x
\]

\[
-k^2(r \cdot \mathcal{M})j_i(kr)
\]

\[
(16.92)
\]

These results are the exact multipole coefficients, valid for arbitrary frequency and source size.

For many applications in atomic and nuclear physics the source dimensions are very small compared to a wavelength \((kr_{\text{max}} \ll 1)\). Then the multipole coefficients can be simplified considerably. The small argument limit (16.12) can be used for the spherical Bessel functions. Keeping only the lowest powers in \(kr\) for terms involving \(\rho\) or \(J\) and \(\mathcal{M}\), we find the approximate electric multipole coefficient,

\[
a_E(l, m) \approx \frac{4\pi k^{l+2}}{i(2l+1)!} \left( \frac{l+1}{l} \right)^{\frac{l}{2}} (Q_{lm} + Q_{lm}')
\]

\[
(16.93)
\]

where the multipole moments are

\[
Q_{lm} = \int r^l Y_{lm}^* \rho \, d^3x
\]

and

\[
Q_{lm}' = \int \frac{-ik}{l+1} r^l Y_{lm}^* \nabla \cdot (r \times \mathcal{M}) \, d^3x
\]

\[
(16.94)
\]
The moment $Q_{lm}$ is seen to be the same in form as the electrostatic multipole moment $q_{lm}$ (4.3). The moment $Q'_{lm}$ is an induced electric multipole moment due to the magnetization. It is generally at least a factor $kr$ smaller than the normal moment $Q_{lm}$. For the magnetic multipole coefficient $a_M(l, m)$ the corresponding long-wavelength approximation is

$$a_M(l, m) \approx \frac{4\pi i k^{l+2}}{(2l + 1)!!} \left(\frac{l + 1}{2}\right)^{\frac{1}{2}} (M_{lm} + M'_{lm})$$ (16.95)

where the magnetic multipole moments are

$$M_{lm} = -\frac{1}{l + 1} \int r^l Y_{lm}^* \nabla \cdot \left(\frac{r \times J}{c}\right) d^3x$$

and

$$M'_{lm} = -\int r^l Y_{lm}^* \nabla \cdot \mathcal{M} d^3x$$ (16.96)

In contrast to the electric multipole moments $Q_{lm}$ and $Q'_{lm}$, for a system with intrinsic magnetization the magnetic moments $M_{lm}$ and $M'_{lm}$ are generally of the same order of magnitude.

In the long-wavelength limit we see clearly the fact that electric multipole fields are related to the electric-charge density $\rho$, while the magnetic multipole fields are determined by the magnetic-moment densities, $(1/2c)(r \times J)$ and $\mathcal{M}$.

### 16.6 Multipole Radiation in Atomic and Nuclear Systems

Although a full discussion involves a proper quantum-mechanical treatment of the states involved,* the essential features of multipole radiation in atoms and nuclei can be presented with simple arguments. From (16.74) and the multipole coefficients (16.93) and (16.95), the total power radiated by a multipole of order $(l, m)$ is

$$P_R(l, m) = \frac{2\pi c}{[(2l + 1)!!]^2} \left(\frac{l + 1}{l}\right)^{l/2} k^{2l+2} |Q_{lm} + Q'_{lm}|^2$$

$$P_M(l, m) = \frac{2\pi c}{[(2l + 1)!!]^3} \left(\frac{l + 1}{l}\right)^{l/2} k^{2l+2} |M_{lm} + M'_{lm}|^2$$ (16.97)

* See Blatt and Weisskopf, pp. 597–599, for the quantum-mechanical definitions of the multipole moments. Beware of factors of 2 between our moments and theirs, due to their definitions, (3.1) and (3.2) on p. 590, of the source densities, as compared to our (16.76).
In quantum-mechanical terms we are interested in the transition probability (reciprocal mean life), defined as the power divided by the energy of a photon:

\[
\frac{1}{\tau} = \frac{P}{\hbar \omega}
\]  
(16.98)

Since we are concerned only with order-of-magnitude estimates, we make the following schematic model of the source. The oscillating charge density is assumed to be

\[
\rho(x) = \begin{cases} 
\frac{3e}{a^3} Y_{lm}(\theta, \phi), & r < a \\
0, & r > a
\end{cases}
\]  
(16.99)

Then an estimate of the electric multipole moment \(Q_{lm}\) is

\[
Q_{lm} \sim \frac{3}{l+3} ea^l
\]  
(16.100)

independent of \(m\). Similarly for the divergences of the magnetizations we assume the schematic form:

\[
\nabla \cdot \mathbf{\mathcal{M}} + \frac{1}{l + 1} \nabla \cdot \left( \mathbf{r} \times \mathbf{J} / c \right) = \begin{cases} 
\frac{2g}{a^3} Y_{lm}(\theta, \phi) \left( \frac{eh}{mc} \right), & r < a \\
0, & r > a
\end{cases}
\]  
(16.101)

where \(g\) is the effective \(g\) factor for the magnetic moments of the particles in the atomic or nuclear system, and \(eh/mc\) is twice the Bohr magneton for those particles. Then an estimate of the sum of magnetic multipole moments is

\[
M_{lm} + M_{lm}' \simeq -\frac{2}{l+2} ea^l \left( \frac{gh}{mc^2} \right)
\]  
(16.102)

From the definition of \(Q_{lm}'\) (16.94) we see that

\[
Q_{lm}' \sim g \left( \frac{\hbar \omega}{mc^2} \right) Q_{lm}
\]  
(16.103)

Since the energies of radiative transitions in atoms and nuclei are always very small compared to the rest energies of the particles involved, \(Q_{lm}'\) is always completely negligible compared to \(Q_{lm}\).

For electric multipole transitions of order \(l\), estimate (16.100) leads to a transition probability (16.98):

\[
\frac{1}{\tau_{le}(l)} \sim \left( \frac{e^2}{\hbar c} \right) \frac{2\pi}{[(2l + 1)!!]^2} \left( \frac{l + 1}{l} \right) \left( \frac{3}{l + 3} \right)^3 (ka)^{2l+1} \omega
\]  
(16.104)
Apart from factors of the order of unity, the transition probability for magnetic multipoles is, according to (16.102),

$$\frac{1}{\tau_M(l)} \approx \left( \frac{g\hbar}{mc a} \right)^2 \frac{1}{\tau_E(l)}$$

(16.105)

The presence of the factor \((ka)^2\) in the transition probability (16.104) means that in the long-wavelength limit \((ka \ll 1)\) the transition rate falls off rapidly with increasing multipole order, for a fixed frequency. Consequently in an atomic or nuclear transition the lowest nonvanishing multipole will generally be the only one of importance. The ratio of transition probabilities for successive orders of either electric or magnetic multipoles of the same frequency is

$$\frac{\tau(l + 1)}{\tau(l)}^{-1} \sim \frac{(ka)^2}{4l^2}$$

(16.106)

where we have omitted numerical factors of relative order \((1/l)\).

In atomic systems the electrons are the particles involved in the radiation process. The dimensions of the source can be taken as \(a \sim (a_0/Z_{eff})\), where \(a_0\) is the Bohr radius and \(Z_{eff}\) is an effective nuclear charge \((Z_{eff} \sim 1)\) for transitions by valence electrons; \(Z_{eff} \ll Z\) for X-ray transitions). To estimate \(ka\) we note that the atomic transition energy is generally of the order

$$\hbar \omega \ll \frac{Z_{eff}^2 e^2}{a_0}$$

(16.107)

so that

$$ka \ll \frac{Z_{eff}}{137}$$

(16.108)

From (16.106) we see that successive multipoles will be in the ratio \((Z_{eff}/137)^2\). The ratio of magnetic to electric multipole transition rates can be estimated from (16.105). The \(g\) factor is of the order of unity for electrons. With \(a \sim a_0/Z_{eff} = 137(h/mc Z_{eff})\), we see that the magnetic \(l\)th multipole rate is a factor \((Z_{eff}/137)^2\) smaller than the corresponding electric multipole rate. We conclude that in atoms electric dipole transitions will be most intense, with electric quadrupole and magnetic dipole transitions a factor \((Z_{eff}/137)^2\) weaker. Only for X-ray transitions in heavy elements is there the possibility of competition from other than the lowest-order electric multipole.

We now turn to the question of radiative transitions in atomic nuclei. Because nuclear radiative transition energies vary greatly (from \(\sim 10\) Kev to several Mev), the values of \(ka\) cover a wide range. This means that for a given multipole order the transition probabilities (or mean lifetimes)
Fig. 16.2 Estimated lifetimes of excited nuclear states against emission of electric multipole radiation as a function of the photon energy for \( l = 1, 2, 3, 4 \).

will range over many powers of 10, depending on the energy release, overlapping the multipoles on either side. In spite of this, rough estimates (16.104) and (16.105) are useful in cataloging nuclear multipole transitions, because for a fixed energy release the estimates for different multipoles differ greatly.

Figure 16.2 shows a log-log plot of estimate (16.104) for lifetimes of electric multipole transitions, using \( e \) as the protonic charge and \( a \approx 5.6 \times 10^{-13} \text{ cm} \). This is a nuclear radius appropriate to mass number \( A \sim 100 \). We see that, although the curves tend to converge at high energies, the lifetimes for different multipoles at the same energy differ by
factors typically of order $10^5$. This means that the actual multipole moments in individual transitions can deviate widely from our simple estimates without vitiating the usefulness of those estimates as a guide in assigning multipole orders. Experimentally, the lifetime-energy diagram shows broad, but well-defined, bands lying in the vicinity of the straight lines in Fig. 16.2. There is a general tendency for estimate (16.104) to serve as a lower bound on the lifetime, corresponding to (16.100) being an upper bound on the multipole moment, but for certain so-called “enhanced” electric quadrupole transitions the lifetimes can be as much as 100 times shorter than shown in Fig. 16.2.

Magnetic and electric multipoles of the same order can be compared using (16.105). For nucleons the effective $g$ factor is typically of the order of $g \sim 3$ because of their anomalous magnetic moments. Then, with the source size estimate $a \sim R = 1.2A^{1/2} \times 10^{-13}$ cm, we find

$$\frac{1}{\tau_M(l)} \sim \frac{0.3}{A^{3/2}} \frac{1}{\tau_E(l)}$$

(16.109)

The numerical factor ranges from $4 \times 10^{-2}$ to $0.8 \times 10^{-2}$ for $20 < A < 250$. We thus anticipate that for a given multipole order electric transitions will be 25–120 times as intense as magnetic transitions. For most multipoles this is generally true. But for $l = 1$ there are special circumstances in nuclei (strongly attractive, charge-independent forces) which inhibit electric dipole transitions (at least at low energies). Then estimate (16.109) fails; magnetic dipole transitions are far commoner and just as intense as electric dipole transitions.

In Section 16.3 the parity and angular-momentum selection rules were discussed, and it was pointed out that in a transition between two quantum states a mixture of multipoles, such as magnetic $l$, $(l + 2)$, ... pole and electric $(l + 1)$, $(l + 3)$, ... pole, could occur. In the long-wavelength limit we need consider only the lowest multipole of each type. Ratios (16.105) and (16.106) can be combined to yield the relative transition rates of electric $(l + 1)$ pole to magnetic $l$ pole (most commonly used for $l = 1$),

$$\frac{[\tau_E(l + 1)]^{-1}}{[\tau_M(l)]^{-1}} \sim \left(\frac{A^{1/2}E}{200l}\right)^2$$

(16.110)

where $E$ is the photon energy in Mev. For energetic transitions in heavy elements the electric quadrupole amplitude is $\sim 5$ per cent of the magnetic dipole amplitude. If, as actually occurs in the rare earth and transuranic elements, there is an enhancement of the effective quadrupole moment by a factor of 10, the electric quadrupole transition competes favorably with the magnetic dipole transition.
For a mixture of magnetic \((l + 1)\) pole and electric \(l\) pole, the ratio of transition rates is

\[
\frac{[\tau_M(l + 1)]^{-1}}{[\tau_E(l)]^{-1}} \sim \left(\frac{E}{600l}\right)^2
\]

(16.111)

Even for energetic transitions, a magnetic \((l + 1)\) pole never comes close to competing with an electric \(l\) pole.

16.7 Radiation from a Linear, Center-fed Antenna

As an illustration of the use of a multipole expansion for a source whose dimensions are comparable to a wavelength, we consider the radiation from a thin, linear, center-fed antenna, as shown in Fig. 16.3. We have already given in Chapter 9 a direct solution for the fields when the current distribution is sinusoidal. This will serve as a basis of comparison to test the convergence of the multipole expansion. We assume the antenna to lie along the \(z\) axis from \(-d/2 \leq z \leq d/2\), and to have a small gap at its center so that it can be suitably excited. The current along the antenna vanishes at the end points and is an even function of \(z\). For the moment we will not specify it more than to write

\[
I(z, t) = I(|z|)e^{-i\omega t}, \quad I\left(\frac{d}{2}\right) = 0
\]

(16.112)

Since the current flows radially, \((r \times J) = 0\). Furthermore there is no

![Fig. 16.3 Linear, center-fed antenna.](image)
intrinsic magnetization. Consequently all magnetic multipole coefficients \( a_M(l, m) \) vanish. To calculate the electric multipole coefficient \( a_E(l, m) \) (16.91) we need expressions for charge and current densities. The current density \( \mathbf{J} \) is a radial current, confined to the \( z \) axis. In spherical coordinates this can be written for \( r < (d/2) \)

\[
\mathbf{J}(x) = e_r \frac{I(r)}{2\pi r^2} \left[ \delta(\cos \theta - 1) - \delta(\cos \theta + 1) \right]
\]

where the delta functions cause the current to flow only upwards (or downwards) along the \( z \) axis. From the continuity equation (16.78) we find the charge density,

\[
\rho(x) = \frac{1}{i\omega} \frac{dI(r)}{dr} \left[ \frac{\delta(\cos \theta - 1) - \delta(\cos \theta + 1)}{2\pi r^2} \right]
\]

These expressions for \( \mathbf{J} \) and \( \rho \) can be inserted into (16.91) to give

\[
a_k(l, m) = \frac{2k^2}{\sqrt{l(l + 1)}} \int_0^{a/2} dr \left( \frac{k}{c} r j_l(kr) I(r) - \frac{1}{\omega} \frac{dI}{dr} [r j_l(kr)] \right)
\]

\[
\times \int d\Omega Y_{lm}^* [\delta(\cos \theta - 1) - \delta(\cos \theta + 1)]
\]

The integral over angles is

\[
\int d\Omega = 2\pi \delta_{m,0} [Y_{l0}(0) - Y_{l0}(\pi)]
\]

showing that only \( m = 0 \) multipoles occur. This is obvious from the cylindrical symmetry of the antenna. The Legendre polynomials are even (odd) about \( \theta = \pi/2 \) for \( l \) even (odd). Hence, the only nonvanishing multipoles have \( l \) odd. Then the angular integral has the value,

\[
\int d\Omega = \sqrt{4\pi(2l + 1)}, \quad l \text{ odd, } m = 0
\]

With slight manipulation (16.115) can be written

\[
a_E(l, 0) = \frac{2k}{c} \left[ \frac{4\pi(2l + 1)}{l(l + 1)} \right]^{1/2} \int_0^{a/2} dr \left[ - \frac{d}{dr} \left( r j_l(kr) \frac{dI}{dr} \right) + r j_l(kr) \left( \frac{d^2I}{dr^2} + k^2I \right) \right]
\]

To evaluate (16.118) we must specify the current \( I(z) \) along the antenna. If no radiation occurred, the sinusoidal variation in time at frequency \( \omega \) would imply a sinusoidal variation in space with wave number \( k = \omega/c \).
The emission of radiation modifies this somewhat. To find the correct spatial variation of the current along the antenna we would have to consider the whole boundary-value problem of antenna plus radiation fields. This is a difficult task which must be faced if one wishes precise answers to antenna problems. Fortunately neglect of the effects of the radiation on the current distribution is not serious. Reasonably good answers are obtained with the sinusoidal approximation. Accordingly we assume

\[ I(z) = I \sin \left( \frac{kd}{2} - k |z| \right) \]  

(16.119)

where \( I \) is the peak current, and the phase is so chosen that the current vanishes at the ends of the antenna. With a sinusoidal current the second part of the integrand in (16.118) vanishes. The first part is a perfect differential. Consequently we immediately obtain, with \( I(z) \) from (16.119),

\[ a_E(l, 0) = \frac{4I}{cd} \left[ \frac{4\pi(2l + 1)}{l(l + 1)} \right]^{1/2} \left( \frac{k \lambda}{2} \right)^{2l} j_l \left( \frac{k \lambda}{2} \right) \], \quad l \text{ odd} \quad (16.120)

Since we wish to test the multipole expansion when the source dimensions are comparable to a wavelength, we consider the special cases of a half-wave antenna \((kd = \pi)\) and a full-wave antenna \((kd = 2\pi)\). For these two values of \(kd\) the \(l = 1\) coefficient is tabulated, along with the relative values for \(l = 3, 5\). From the table it is evident that (a) the coefficients decrease rapidly in magnitude as \(l\) increases, and (b) higher \(l\) coefficients are more important the larger the source dimensions. But even for the full-wave antenna it is probably adequate to keep only \(l = 1\) and \(l = 3\) in the angular distribution and certainly adequate for the total power (which involves the squares of the coefficients).

With only dipole and octupole terms in the angular distribution we find that the power radiated per unit solid angle (16.70) is

\[ \frac{dP}{d\Omega} = \frac{c |a_E(1, 0)|^2}{16\pi k^2} \left| L Y_{1,0} - \frac{a_E(3, 0)}{\sqrt{6} a_E(1, 0)} L Y_{3,0} \right|^2 \]  

(16.121)
The various factors in the absolute square are

\[
|L_{Y_{1,0}}|^2 = \frac{3}{4\pi} \sin^2 \theta
\]

\[
|L_{Y_{3,0}}|^2 = \frac{63}{16\pi} \sin^2 \theta (5 \cos^2 \theta - 1)^2
\]

\[
(L_{Y_{1,0}})^* \cdot (L_{Y_{3,0}}) = \frac{3\sqrt{21}}{8\pi} \sin^2 \theta (5 \cos^2 \theta - 1)
\]

With these angular factors (16.121) becomes

\[
\frac{dP}{d\Omega} = \frac{\lambda}{\pi^2 c} \left( \frac{3}{8\pi} \sin^2 \theta \right) \left| 1 - \sqrt{\frac{7}{8}} a_E(3,0) \frac{5 \cos^2 \theta - 1}{\sin^2 \theta} \right|^2
\]

(16.123)

where the factor \( \lambda \) is equal to 1 for the half-wave antenna and \( \pi^2/4 \) for the full wave. The coefficient of \( 5 \cos^2 \theta - 1 \) in (16.123) is 0.0463 and 0.304 for the half-wave and full-wave antenna, respectively.

From Chapter 9 the exact angular distributions (for sinusoidal driving currents) are

\[
\cos^2 \left( \frac{\pi}{2} \cos \theta \right) \sin^2 \theta, \quad kd = \pi
\]

(16.124)

\[
4 \cos^4 \left( \frac{\pi}{2} \cos \theta \right) \sin^3 \theta, \quad kd = 2\pi
\]

A numerical comparison of the exact and approximate angular distributions is shown in Fig. 16.4. The solid curves are the exact results, the dashed curves the two-term multipole expansions. For the half-wave case (Fig. 16.4a) the simple dipole result [first term in (16.123)] is also shown as a dotted curve. The two-term multipole expansion is almost indistinguishable from the exact result for \( kd = \pi \). Even the lowest-order approximation is not very far off in this case. For the full-wave antenna (Fig. 16.4b) the dipole approximation is evidently quite poor. But the two-term multipole expansion is reasonably good, differing by less than 5 per cent in the region of appreciable radiation.

The total power radiated is, according to (16.75),

\[
P = \frac{c}{8\pi k^3} \sum_{l, \text{odd}} |a_E(l, 0)|^2
\]

(16.125)

For the half-wave antenna the coefficients in the table on p. 564 show that the power radiated is a factor 1.00245 times larger than the simple dipole
Comparison of exact radiation patterns (solid curves) for half-wave \((kd = \pi)\) and full-wave \((kd = 2\pi)\) center-fed antennas with two-term multipole expansions (dashed curves). For the half-wave pattern, the dipole approximation (dotted curve) is also shown. The agreement between the exact and two-term multipole results is excellent, especially for \(kd = \pi\).

result, \((12I^2/\pi^2c)\). For the full-wave antenna, the power is a factor 1.114 times larger than the dipole form \((3I^2/c)\).

16.8 Spherical Wave Expansion of a Vector Plane Wave

In discussing the scattering or absorption of electromagnetic radiation by spherical objects, or localized systems in general, it is useful to have an expansion of a plane electromagnetic wave in spherical waves.

For a scalar field \(\psi(x)\) satisfying the wave equation the necessary expansion can be obtained by using the orthogonality properties of the basic spherical solutions \(f_i(kr)Y_{lm}(\theta, \phi)\). An alternative derivation makes
use of the spherical wave expansion (16.22) of the Green's function \(e^{ikR/4\pi R}\). We let \(|x'| \to \infty\) on both sides of (16.22). Then we can put \(|x - x'| \approx r' - \mathbf{n} \cdot \mathbf{x}\) on the left-hand side, where \(\mathbf{n}\) is a unit vector in the direction of \(x'\). On the right side \(r_+ = r'\) and \(r_- = r\). Furthermore we can use the asymptotic form (16.13) for \(h_1^{(1)}(kr')\). Then we find

\[
e^{ikr'} e^{-ik\mathbf{n} \cdot \mathbf{x}} = i k e^{ikr'} \sum_{l, m} (-i)^{l+1} j_l(kr) Y^*_l(\theta', \phi') Y_{lm}(\theta, \phi) \tag{16.126}
\]

Canceling the factor \(e^{ikr'/r'}\) on either side and taking the complex conjugate, we have the expansion of a plane wave,

\[
e^{ik \cdot \mathbf{x}} = 4\pi \sum_{l=0}^{\infty} i j_l(kr) \sum_{m=-l}^{l} Y^*_l(\theta, \phi) Y_{lm}(\theta', \phi') \tag{16.127}
\]

where \(\mathbf{k}\) is the wave vector with spherical coordinates \(k, \theta', \phi'\). The addition theorem (3.62) can be used to put this in a more compact form,

\[
e^{ik \cdot \mathbf{x}} = \sum_{l=0}^{\infty} i^l(2l + 1) j_l(kr) P_l(\cos \gamma) \tag{16.128}
\]

where \(\gamma\) is the angle between \(\mathbf{k}\) and \(\mathbf{x}\). With (3.57) for \(P_l(\cos \gamma)\), this can also be written as

\[
e^{ik \cdot \mathbf{x}} = \sum_{l=0}^{\infty} i^l\sqrt{4\pi(2l + 1)} j_l(kr) Y_{l, 0}(\gamma) \tag{16.129}
\]

We now wish to make an equivalent expansion for a circularly polarized plane wave incident along the \(z\) axis,

\[
\mathbf{E}(\mathbf{x}) = (\mathbf{e}_1 \pm i \mathbf{e}_2) e^{ikz} \\
\mathbf{B}(\mathbf{x}) = \mathbf{e}_3 \times \mathbf{E} = \mp \mathbf{i} \mathbf{E} \tag{16.130}
\]

Since the plane wave is finite everywhere, we can write its multipole expansion (16.47) involving only the regular radial functions \(j_l(kr)\):

\[
\mathbf{E}(\mathbf{x}) = \sum_{l, m} \left[ a_\pm(l, m) j_l(kr) \mathbf{X}_{lm} + \frac{i}{k} b_\pm(l, m) \nabla \times j_l(kr) \mathbf{X}_{lm} \right] \\
\mathbf{B}(\mathbf{x}) = \sum_{l, m} \left[ \frac{-i}{k} a_\pm(l, m) \nabla \times j_l(kr) \mathbf{X}_{lm} + b_\pm(l, m) j_l(kr) \mathbf{X}_{lm} \right] \tag{16.131}
\]

To determine the coefficients \(a_\pm(l, m)\) and \(b_\pm(l, m)\) we utilize the orthogonality properties of the vector spherical harmonics \(\mathbf{X}_{lm}\). For reference
purposes we summarize the basic relation (16.46), as well as some other useful relations:

\[
\int [f_i(r)X_{l'm'}]^* \cdot [g_i(r)X_{lm}] \, d\Omega = f_i^* g_i \delta_{ii'} \delta_{mm'}
\]

\[
\int [f_i(r)X_{l'm'}]^* \cdot [\nabla \times g_i(r)X_{lm}] \, d\Omega = 0
\]

\[
\frac{1}{k^2} \int [\nabla \times f_i(r)X_{l'm'}]^* \cdot [\nabla \times g_i(r)X_{lm}] \, d\Omega = \delta_{ii'} \delta_{mm'} \left( f_i^* g_i + \frac{1}{k^2 r^2} \frac{\partial}{\partial r} \left[ r f_i^* \frac{\partial}{\partial r} (r g_i) \right] \right)
\]

(16.132)

In these relations \( f_i(r) \) and \( g_i(r) \) are linear combinations of spherical Bessel functions, satisfying (16.5). The second and third relations can be proved using the operator identity (16.49), the representation (16.37) for the gradient operator, and the radial differential equation (16.5).

To determine the coefficients \( a_{\pm}(l, m) \) and \( b_{\pm}(l, m) \) we take the scalar product of both sides of (16.131) with \( X_{lm}^* \) and integrate over angles. Then with the first and second orthogonality relations in (16.132) we obtain

\[
a_{\pm}(l, m) j_l(kr) = \int X_{lm}^* \cdot E(x) \, d\Omega
\]

(16.133)

and

\[
b_{\pm}(l, m) j_l(kr) = \int X_{lm}^* \cdot B(x) \, d\Omega
\]

(16.134)

With (16.130) for the electric field, (16.133) becomes

\[
a_{\pm}(l, m) j_l(kr) = \int \frac{(L_{\pm} Y_{lm})^*}{\sqrt{l(l+1)}} e^{ikz} \, d\Omega
\]

(16.135)

where the operators \( L_{\pm} \) are defined by (16.26), and the results of their operating by (16.28). Thus we obtain

\[
a_{\pm}(l, m) j_l(kr) = \frac{\sqrt{(l \pm m)(l \mp m + 1)}}{\sqrt{l(l+1)}} \int Y_{l,m \mp 1}^* e^{ikz} d\Omega
\]

(16.136)

If expansion (16.129) for \( e^{ikz} \) is inserted, the orthogonality of the \( Y_{lm} \)s evidently leads to the result,

\[
a_{\pm}(l, m) = i^l \sqrt{4\pi(2l+1)} \delta_{m, \pm 1}
\]

(16.137)

From (16.134) and (16.130) it is clear that

\[
b_{\pm}(l, m) = \mp ia_{\pm}(l, m)
\]

(16.138)
Then the multipole expansion of the plane wave (16.130) is

\[ E(x) = \sum_{i=1}^{\infty} i^l \sqrt{4\pi(2l + 1)} \left[ \pm \frac{1}{k} \nabla \times j_i(kr)X_{l, \pm 1} \right] \]

\[ B(x) = \sum_{i=1}^{\infty} i^l \sqrt{4\pi(2l + 1)} \left[ \pm \frac{i}{k} \nabla \times j_i(kr)X_{l, \pm 1} \right] \]

(16.139)

For such a circularly polarized wave the \( m \) values of \( m = \pm 1 \) have the obvious interpretation of \( \pm 1 \) unit of angular momentum per photon parallel to the propagation direction. This has already been established in Problem 6.12.

16.9 Scattering of Electromagnetic Waves by a Conducting Sphere

If a plane wave of electromagnetic radiation is incident on a spherical obstacle, as indicated schematically in Fig. 16.5, it is scattered so that far away from the scatterer the fields are represented by a plane wave plus outgoing spherical waves. There may be absorption by the obstacle as well as scattering. Then the total energy flow away from the obstacle will be less than the total energy flow towards it, the difference being absorbed. We will consider the simple example of scattering by a sphere of radius \( a \) and infinite conductivity.

The fields outside the sphere can be written as a sum of incident and scattered waves:

\[ E(x) = E_{\text{inc}} + E_{\text{sc}} \]

\[ B(x) = B_{\text{inc}} + B_{\text{sc}} \]

(16.140)

where \( E_{\text{inc}} \) and \( B_{\text{inc}} \) are given by (16.139). Since the scattered fields are outgoing waves at infinity, their expansions must be of the form,

\[ E_{\text{sc}} = \frac{1}{2} \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l + 1)} \left[ \pm \frac{\alpha_{\pm}(l)h_l^{(1)}(kr)X_{l, \pm 1}}{k} \nabla \times h_l^{(1)(kr)X_{l, \pm 1}} \right] \]

\[ B_{\text{sc}} = \frac{1}{2} \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l + 1)} \left[ \pm \frac{i\alpha_{\pm}(l)}{k} \nabla \times h_l^{(1)(kr)X_{l, \pm 1}} \right] \]

(16.141)

The coefficients \( \alpha_{\pm}(l) \) and \( \beta_{\pm}(l) \) will be determined by the boundary conditions on the surface of the sphere.
For a perfectly conducting sphere the boundary conditions at the surface \( r = a \) are
\[
\mathbf{n} \times \mathbf{E} = 0, \quad \mathbf{n} \cdot \mathbf{B} = 0 \quad (16.142)
\]
In order to apply these boundary conditions we must know the vectorial character of the two types of terms in (16.141). We already know that \( \mathbf{X}_{lm} \) is perpendicular to the radius vector. The other type of term is
\[
\nabla \times f_i(r) \mathbf{X}_{lm} = \frac{\mathbf{i} \sqrt{l(l+1)}}{r} f_i(r) Y_{lm} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r f_i(r) \right] \mathbf{n} \times \mathbf{X}_{lm} \quad (16.143)
\]
where \( f_i \) is any spherical Bessel function satisfying (16.5). Applying the boundary conditions (16.142) to the total fields (16.140), we find conditions on the coefficients \( \alpha_{\pm}(l) \) and \( \beta_{\pm}(l) \):
\[
\begin{cases}
\tfrac{1}{2} \alpha_{\pm}(l) h_i^{(1)}(ka) + j_i(ka) = 0 \\
\tfrac{1}{2} \beta_{\pm}(l) \frac{\partial}{\partial r} \left[ r h_i^{(1)}(kr) \right] + \frac{\partial}{\partial r} \left[ r j_i(kr) \right] r=a = 0
\end{cases} \quad (16.144)
\]
We note that the coefficients are the same for both states of circular polarization. Since \( 2j_i(kr) = h_i^{(1)}(kr) + h_i^{(2)}(kr) \), the coefficients can be written
\[
\begin{align*}
\alpha_{\pm}(l) &= -\left[ \frac{h_i^{(2)}(ka)}{h_i^{(1)}(ka)} + 1 \right] \\
\beta_{\pm}(l) &= -\left\{ \frac{d}{dr} \left[ r h_i^{(2)}(kr) \right]_{r=a} + 1 \right\}
\end{align*} \quad (16.145)
\]
The ratios are ratios of complex conjugate quantities and so are complex numbers of absolute value unity. It is convenient to define two angles,
called phase shifts, as follows:
\[ e^{2i\delta_i} = - \frac{h_i^{(2)}(ka)}{h_i^{(1)}(ka)} \]
\[ e^{2i\delta_i'} = - \left. \frac{d}{dr} \left[ r h_i^{(2)}(kr) \right] \right|_{r=a} - \left. \frac{d}{dr} \left[ r h_i^{(1)}(kr) \right] \right|_{r=a} \]

or alternatively,
\[ \tan \delta_i = \left. \frac{j_i(ka)}{n_i(ka)} \right| \]
\[ \tan \delta_i' = \left. \frac{d}{dr} \left[ r j_i(kr) \right] \right|_{r=a} - \left. \frac{d}{dr} \left[ r n_i(kr) \right] \right|_{r=a} \]

Then the coefficients are
\[ \alpha_{\pm}(l) = (e^{2i\delta_i} - 1), \quad \beta_{\pm}(l) = (e^{2i\delta_i'} - 1) \]

The asymptotic forms (16.12) and (16.13) can be used to find limiting values of these phase shifts for \( ka \ll 1 \) and \( ka \gg 1 \):

\( ka \ll 1 \):
\[ \delta_i \to - \frac{(ka)^{2l+1}}{(2l + 1)[(2l - 1)!]^2} \]
\[ \delta_i' \to - \left( \frac{l + 1}{l} \right) \delta_i \]

\( ka \gg 1 \):
\[ \delta_i \to \frac{l\pi}{2} - ka \]
\[ \delta_i' \to (l + 1) \frac{\pi}{2} - ka \]

The coefficient \( \alpha(l) \) and the phase shift \( \delta_i \) can be termed magnetic parameters since they relate to the magnetic multipole fields in (16.141). Similarly \( \beta(l) \) and \( \delta_i' \) are electric parameters.

With coefficients (16.148) the magnetic induction of the scattered wave therefore becomes
\[ B_{sc} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l + 1)} \left[ \frac{e^{i\delta_i} \sin \delta_i}{k} \nabla \times h_i^{(1)}(l_{i,\pm}) \pm e^{i\delta_i'} \sin \delta_i' h_i^{(1)}(l_{i,\pm}) \right] \]

(16.151)
with the asymptotic form \((kr \to \infty)\),

\[
B_{sc} \to e^{ikr} \frac{1}{kr} \sum_{l=1}^{\infty} \sqrt{4\pi(2l + 1)} \left[ e^{i\delta_l} \sin \delta_l (n \times X_{l,\pm 1}) \mp ie^{i\delta_{l'}} \sin \delta_{l'} X_{l,\pm 1} \right]
\]  
(16.152)

The scattered field (16.152) corresponds in general to elliptically polarized radiation. Only if the electric and magnetic phases were equal would it represent circularly polarized radiation. This means that, if linearly polarized radiation is incident, the scattered radiation will be elliptically polarized; and, if the incident radiation is unpolarized, the scattered radiation will exhibit partial polarization depending on the angle of observation.

In discussing the scattered intensity it is convenient to use the concept of a scattering cross section. This has already been defined in (14.101). The scattered power per unit solid angle is

\[
\frac{dP_{sc}}{d\Omega} = \frac{c}{8\pi} |rB_{sc}|^2
\]  
(16.153)

The incident flux is

\[
S = \frac{c}{8\pi} \text{Re} (E_{\text{inc}} \times B_{\text{inc}}^*) = \frac{c}{4\pi} e_3
\]  
(16.154)

Consequently the scattering cross section is

\[
\frac{d\sigma}{d\Omega} = \frac{2\pi}{k^2} \sum_{l=1}^{\infty} \sqrt{2l + 1} \left| e^{i\delta_l} \sin \delta_l (n \times X_{l,\pm 1}) \mp ie^{i\delta_{l'}} \sin \delta_{l'} X_{l,\pm 1} \right|^2
\]  
(16.155)

This angular distribution is rather complicated, except in the long-wavelength limit (see below). But the total cross section can be calculated directly. From the second orthogonality relation in (16.132) and (16.143) it is evident that the cross terms in (16.155) vanish on integration over angles. Then the total cross section is easily found to be

\[
\sigma = \frac{2\pi}{k^2} \sum_{l=1}^{\infty} (2l + 1) \left[ \sin^2 \delta_l + \sin^2 \delta_{l'} \right]
\]  
(16.156)

The electric and magnetic multipole parts of the wave contribute incoherently to the total cross section.

In the long-wavelength limit \((ka \ll 1)\) the scattering cross section becomes relatively simple because the phase shifts (16.149) decrease rapidly with increasing \(l\). Keeping only \(l = 1\) terms in the expansion, we find

\[
\frac{d\sigma}{d\Omega} \approx \frac{2\pi}{3} q^2 (ka)^4 |n \times X_{1,\pm 1} \pm 2iX_{1,\pm 1}|^2
\]  
(16.157)
Fig. 16.6 Angular distribution of radiation scattered by a perfectly conducting sphere in the long-wavelength limit ($ka \ll 1$).

From the table on p. 551 we obtain the absolute squared terms,

$$|n \times X_{1, \pm 1}|^2 = |X_{1, \pm 1}|^2 = \frac{3}{16\pi} (1 + \cos^2 \theta)$$

(16.158)

The cross terms can be easily worked out:

$$\text{Re} [\pm i(n \times X_{1, \pm 1})^* \cdot X_{1, \pm 1}] = \frac{-3}{8\pi} \cos \theta$$

(16.159)

Thus the long-wavelength limit of the differential scattering cross section is

$$\frac{d\sigma}{d\Omega} \simeq a^2(ka)^4 \left[ \frac{2}{3}(1 + \cos^2 \theta) - \cos \theta \right]$$

(16.160)

independent of the state of polarization of the incident radiation. The angular distribution of scattered radiation is shown in Fig. 16.6. The scattering is predominantly backwards, the marked asymmetry about $90^\circ$ being caused by the electric dipole-magnetic dipole interference term.

The total cross section in the long-wavelength limit is

$$\sigma = \frac{10\pi}{3} a^2(ka)^4$$

(16.161)

This is a well-known result, first obtained by Mie and Debye (1908–1909). The dependence of the cross section on frequency as $\omega^4$ is known as Rayleigh's law, and is characteristic of all systems which possess a dipole moment.
16.10 Boundary-Value Problems with Multipole Fields

The scattering of radiation by a conducting sphere is an example of a boundary-value problem with multipole fields. Other examples are the free oscillations of a conducting sphere, the spherical resonant cavity, and scattering by a dielectric sphere. The possibility of resistive losses in conductors adds problems such as $Q$ values of cavities and absorption cross sections to the list. The general techniques for handling these problems are the same ones as have been met in Section 16.9 and in Chapter 8. The necessary mathematical apparatus has been developed in the present chapter. We will leave the discussion of these examples to the problems at the end of the chapter.

REFERENCES AND SUGGESTED READING

The theory of vector spherical harmonics and multipole vector fields is discussed thoroughly by
- Blatt and Weisskopf, Appendix B,
- Morse and Feshbach, Section 13.3.

Applications to nuclear multipole radiation are given in
- Blatt and Weisskopf, Chapter XII,
- Siegbahn, Chapter XIII by S. A. Moszkowski and
- Chapter XVI (II) by M. Goldhaber and A. W. Sunyar.

A number of books on antennas were cited at the end of Chapter 9. None of them discusses multipole expansions in a rigorous way, however.

The scattering of radiation by a perfectly conducting sphere is treated briefly by
- Morse and Feshbach, pp. 1882-1886,
- Panofsky and Phillips, Section 12.9.

Much more elaborate discussions, with arbitrary dielectric and conductive properties for the sphere, are given by
- Born and Wolf, Section 13.5,
- Stratton, Section 9.25.

Mathematical information on spherical Bessel functions, etc., will be found in
- Morse and Feshbach, pp. 1573-1898.

PROBLEMS

16.1 Three charges are located along the $z$ axis, a charge $+2q$ at the origin and charges $-q$ at $z = \pm a \cos \omega t$. Determine the lowest nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated. Assume that $ka \ll 1$.

16.2 An almost spherical surface defined by

$$R(\theta) = R_0 [1 + \beta P_2(\cos \theta)]$$
has inside of it a uniform volume distribution of charge totaling $Q$. The small parameter $\beta$ varies harmonically in time at frequency $\omega$. This corresponds to surface waves on a sphere. Keeping only lowest-order terms in $\beta$ and making the long-wavelength approximation, calculate the nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated.

16.3 The uniform charge density of Problem 16.2 is replaced by a uniform density of intrinsic magnetization parallel to the $z$ axis and having total magnetic moment $M$. With the same approximations as above calculate the nonvanishing radiation multipole moments, the angular distribution of radiation, and the total power radiated.

16.4 An antenna consists of a circular loop of wire of radius $a$ located in the $x$-$y$ plane with its center at the origin. The current in the wire is

$$I = I_0 \cos \omega t = \text{Re} I_0 e^{-i\omega t}$$

(a) Find the expressions for $E$, $B$ in the radiation zone without approximations as to the magnitude of $ka$. Determine the power radiated per unit solid angle.

(b) What is the lowest nonvanishing multipole moment ($Q_{l,m}$ or $M_{l,m}$)? Evaluate this moment in the limit $ka \ll 1$.

16.5 Two fixed electric dipoles of dipole moment $p$ are located in a plane a distance $2a$ apart, their axes parallel and perpendicular to the plane, but their moments directed oppositely. The dipoles rotate with constant angular velocity $\omega$ about a parallel axis located halfway between them ($\omega \ll c/a$).

(a) Calculate the components of the quadrupole moment.

(b) Show that the angular distribution of radiation is proportional to

$$1 - 3 \cos^2 \theta + 4 \cos^4 \theta$$

and that the total power radiated is

$$P = \frac{2cp^2a^2}{5} \left( \frac{\omega}{c} \right)^6.$$

16.6 In the long-wavelength limit evaluate the nonvanishing electric multipole moments for the charge distribution:

$$\rho = Cr^3 e^{-5r^6} Y_{1,1}(\theta, \phi) Y_{2,0}(\theta, \phi) e^{-i\omega t}$$

and determine the angular distribution and total power radiated for each multipole. This charge distribution is appropriate to a transition between the states $n = 3$, $l = 2$ ($3d$) and $n = 2$, $l = 1$ ($2p$) in a hydrogen atom.

16.7 The fields representing a transverse magnetic wave propagating in a cylindrical wave guide of radius $R$ are:

$$E_z = J_m(yr) e^{im\phi} e^{-i\omega t}, \quad H_z = 0$$

$$E_\phi = -\frac{m\beta}{\gamma^2} \frac{E_z}{r}, \quad H_r = -\frac{k}{\beta} E_\phi$$

$$E_r = \frac{i\beta}{\gamma^2} \frac{\partial E_z}{\partial r}, \quad H_\phi = \frac{k}{\beta} E_r.$$
where \( m \) is the index specifying the angular dependence, \( \beta \) is the propagation constant, \( \gamma^2 = k^2 - \beta^2 \) \( (k = \omega/c) \), where \( \gamma \) is such that \( J_m(\gamma R) = 0 \). Calculate the ratio of the \( z \) component of the electromagnetic angular momentum to the energy in the field. It may be advantageous to perform some integrations by parts, and to use the differential equation satisfied by \( E_z \), in order to simplify your calculations.

16.8 A spherical hole of radius \( a \) in a conducting medium can serve as an electromagnetic resonant cavity.

(a) Assuming infinite conductivity, determine the transcendental equations for the characteristic frequencies \( \omega_{\text{in}} \) of the cavity for TE and TM modes.

(b) Calculate numerical values for the wavelength \( \lambda_{\text{in}} \) in units of the radius \( a \) for the four lowest modes for TE and TM waves.

(c) Calculate explicitly the electric and magnetic fields inside the cavity for the lowest TE and lowest TM mode.

16.9 The spherical resonant cavity of Problem 16.8 has nonpermeable walls of large, but finite, conductivity. In the approximation that the skin depth \( \delta \) is small compared to the cavity radius \( a \), show that the \( Q \) of the cavity, defined by equation (8.82), is given by

\[
Q = \frac{a}{2\pi\delta}, \quad \text{for all TE modes}
\]

and

\[
Q = \frac{a}{2\pi\delta} \left( 1 - \frac{l(l+1)}{\chi_{ln}^2} \right), \quad \text{for TM modes}
\]

where \( \chi_{ln} = (a/c)\omega_{\text{in}} \) for TM modes.

16.10 Discuss the normal modes of oscillation of a perfectly conducting solid sphere of radius \( a \) in free space.

(a) Determine the characteristic equations for the eigenfrequencies for TE and TM modes of oscillation. Show that the roots for \( \omega \) always have a negative imaginary part, assuming a time dependence of \( e^{-i\omega t} \).

(b) Calculate the eigenfrequencies for the \( l = 1 \) and \( l = 2 \) TE and TM modes. Tabulate the wavelength (defined in terms of the real part of the frequency) in units of the radius \( a \) and the decay time (defined as the time taken for the energy to fall to \( e^{-1} \) of its initial value) in units of the transit time \( (a/c) \) for each of the modes.

16.11 A circularly polarized plane wave of radiation of frequency \( \omega = ck \) is incident on a nonpermeable, conducting sphere of radius \( a \).

(a) Assuming that the conductivity of the sphere is infinite, write down explicit expressions for the electric and magnetic fields near and at the surface of the sphere in the long-wavelength limit, \( ka \ll 1 \).

(b) Using the techniques of Chapter 8, calculate the power absorbed by the sphere from the incident wave, assuming that the conductivity is large but finite. Express your result as an absorption cross section in terms of the wave number \( k \), the radius \( a \), and the skin depth \( \delta \). Assume \( ka \ll 1 \).

16.12 Discuss the scattering of a plane wave of electromagnetic radiation by a nonpermeable, dielectric sphere of radius \( a \) and dielectric constant \( \varepsilon \).

(a) By finding the fields inside the sphere and matching to the incident
plus scattered wave outside the sphere, determine the multipole coefficients in the scattered wave. Define suitable phase shifts for the problem.

(b) Consider the long-wavelength limit \((ka \ll 1)\) and determine explicitly the differential and total scattering cross sections. Sketch the angular distribution for \(\epsilon = 2\).

(c) In the limit \(\epsilon \to \infty\) compare your results to those for the perfectly conducting sphere.