Deeper Understanding, Faster Calculation
--Exam FM Insights & Shortcuts

Part I: Theories of Interest

10th Edition

by Yufeng Guo

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Chapter 1 Exam-taking and study strategy

Read this chapter before opening your textbooks!

This chapter requires some knowledge about the time value of money, annuity, and loan amortization. If you don’t know these concepts, don’t worry. Just skip the detailed math calculations and focus on the main ideas in this chapter. Later on, after you understand the time value of money, annuity, and loan amortization, come back to this chapter and go through the math.

It’s critical that you understand the essence of this chapter before you rush to read the textbooks.

A tale of two Exam FM takers, Mr. Busy and Mr. Lazy

It was the best of the times, it was the worst of times, it was the age of being lazy, it was the age of being busy, it was the epoch of passing Exam FM, it was the epoch of failing Exam FM, it was the hope of getting ASA, it was the despair of going nowhere.

Two actuarial students, Mr. Busy and Mr. Lazy, are both preparing for Exam FM. They have the same height and weight. They have the same level of intelligence. As a matter of fact, they are similar about almost everything except that Mr. Busy is very busy and Mr. Lazy is very lazy.

Mr. Busy and Mr. Lazy both would have a worry-free life if they don’t need to amortize a loan with geometrically increasing payments.

Challenge -- loan amortization with geometrically increasing payments

Mr. Busy and Mr. Lazy both love standard annuity problems that require the use of memorized formulas such as $a_{m_i}$ and $\ddot{a}_{m_i}$. They both hate geometrically increasing annuity and loan amortization problems. They would gladly solve one hundred standard annuity problems than amortize a messy loan. Sadly though, SOA loves to test the problems that Mr. Busy and Mr. Lazy hate.
Finally the exam day has come. Mr. Busy and Mr. Lazy walk into the exam room. The 1st problem in the exam is about increasing annuity and loan amortization.

**Problem 1**

<table>
<thead>
<tr>
<th>Date of loan</th>
<th>1/1/2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount of loan</td>
<td>$150,000</td>
</tr>
<tr>
<td>Term of loan</td>
<td>25 years</td>
</tr>
<tr>
<td>Payments</td>
<td>Annual payments with first payment due 12/31/2005. Each subsequent payment is 2% larger than the previous payment.</td>
</tr>
<tr>
<td>Interest rate</td>
<td>8.5% annual effective</td>
</tr>
</tbody>
</table>

Question: What’s the total interest paid during the first 18 payments?

**Mr. Busy:**

Though not fond of this type of problems, Mr. Busy was not a bit worried. When preparing for Exam FM, Mr. Busy bought the solution manual for Broverman’s textbook. He solved all of the practice problems in the solution manual. Some of the problems in the solution manuals are geometric annuity and loan amortization problems.

In addition, Mr. Busy bought another manual with tons of practice problems and solved all the practice problems in the manual.

Mr. Busy felt ready to tackle this problem. He solved over a thousand practice problems. Surely some of the problems he solved were about loan amortization where payments were geometrically increasing. He started to mentally search for how he solved such problems in the past, hoping to recall a quick solution.

To his dismay, Mr. Busy couldn’t remember any quick solutions to loan amortization with geometrically increasing payments. Though he solved many problems before the exam, Mr. Busy was always in a rush to solve the next practice problem. He never had the time to condense his solutions to quickly recallable solutions ready to be used in the exam. He didn’t even have the time to thoroughly understand the basic concept behind loan amortization and behind the present value calculation of a geometrically increasing annuity. He was always in a big hurry to solve more practice problems.

Time seemed to go much faster in the exam room. And the pressure was keen. 5 minutes passed. Mr. Busy was going nowhere. Reluctantly, Mr. Busy abandoned this problem and moved to the next one.
Mr. Lazy:

Unlike Mr. Busy, Mr. Lazy has a lazy approach to loan amortization.

Mr. Lazy realized that loan amortization and geometric annuity problems were repeatedly tested in the past. Year after year, SOA asks candidates to amortize a loan. Sometimes the loan to be amortized has level payments; other times the payments are arithmetically or geometrically increasing or decreasing. Loan amortization and geometric annuity problems are so predictable that Mr. Lazy suspected that SOA would test it again this year when he takes the exam.

Mr. Lazy starts to strategize:

1. SOA loves to test loan amortization. Such a problem is doomed to occur when I take the exam.

2. Loan amortization is nasty, especially when the payments are geometrically increasing or decreasing. It’s hard for me to figure it out from scratch in the heat of the exam.

3. I’m lazy. I want to pass Exam FM with least effort.

Mr. Lazy’s conclusion:

1. Before the exam, I’ll design a standard cookie-cutter solution to loan amortization with geometrically increasing payments. This way, I don’t have to invent a solution from scratch in the exam.

2. I’ll make my solution less than 3 minute long; 3 minutes is pretty much all the time I have per question in the exam.

3. I’ll walk into the exam room with the 3 minute solution script ready in my head. I’ll use this script to solve a nasty loan amortization problem 100% right in 3 minutes under pressure.

Result:

1. Mr. Busy got a 5 in the exam. Though he solved hundreds of practice before the exam, he never tried to build any reusable solution scripts to any of the commonly tested problems in Exam FM. As a result, every repeatable problem tested in the exam became a brand new problem, which he must solve from scratch. This, in turn, makes his solution long and prone to errors. Sorry, Mr. Busy. Good luck to your 2nd try for Exam FM.
2. **Mr. Lazy got a 6 in the exam.** He didn’t bother to solve any practice problems in any textbooks. Nor did he buy old SOA problems dug up from the graveyard. He just downloaded the Sample FM Questions and May and November 2005 FM exam from SOA website. Then for every problem tested in the Sample FM Exam and 2005 FM exam, he built a reusable 3 minutes solution script. Then when he was taking FM exam, he solved all of the repeatable problems using his scripts. Of course, SOA threw in some new problems, to which Mr. Lazy simply guessed the answers. Nice job, Mr. Lazy. See you in Exam M.

**Lessons to be learned from Mr. Busy and Mr. Lazy:**

**Good exam takers solve problems. Great exam takers build 3 minute solution processes (i.e. scripts).** For example, good candidates can solve many integration problems using integration-by-parts:

\[
\int x^2 e^{-x} \, dx, \quad \int x^2 e^{-\frac{x}{3}} \, dx, \quad \int x^2 e^{-\frac{x}{5}} \, dx, \quad \int x^2 e^{-\frac{x}{4}} \, dx, \quad \ldots
\]

However, because the integration-by-parts is a complex and error-prone process, those candidates who use this method often fluster in the heat of the exam.

Great exam takers, on the other hand, focus on building a flawless solution process. Prior to the exam, they built the following generic solution:

\[
\int_a^{\infty} x^2 \left( \frac{1}{\theta} e^{-x/\theta} \right) \, dx = \left[ (a + \theta)^2 + \theta^2 \right] e^{-a/\theta}
\]

After getting this generic process right, great exam takers simply apply this generic solution to every integration problem \( \int_a^{\infty} x^2 \left( \frac{1}{\theta} e^{-x/\theta} \right) \, dx \) and are able to solve similar problems 100% right in a hurry.

**Why building a process is superior to solving problems**

With a generic process, if you have solved one problem, you have solved this type of the problems once and for all. In contrast, if you solve 99 individual problems without a building generic process, you’re never sure that you can solve the 100\textsuperscript{th} problem correctly.
Building process, not solving problems, is the most efficient way to pass SOA exams, especially when you are short of study time. Follow this study method and rigorously build a flawless solution process for each of the previously tested FM problems. Next, test your solution process in the exam condition and solve all the previously tested FM problems 100% right. Use this study method for Exam M and C. You’ll zip through tough exams with a fraction of study time while other busy folks get stuck in one exam for years.

It may take you a little while to get used to this process-oriented study method. Though you may feel insecure when other candidates boast of having solved 1,000 practice problems, please be assured that this approach is far superior.

This manual is written to teach you how to build a generic process to solve SOA FM problems. It’s not a book to give you 500 problems for you to solve.

**Truths about Exam FM**

1. **To pass Exam FM, you need to learn how to solve problems in a hurry under pressure.** In many professions, the difference between an expert and an amateur is that an expert can solve a routine problem flawlessly in a hurry under pressure, while an amateur can solve a problem right only under no time constraints. For example, an expert car mechanic can change a flat tire flawlessly in less than 15 minutes. In comparison, amateurs like me can change a flat tire only after several hours. Similarly, those who pass FM can generally solve a complex problem in 3 minutes under the exam pressure, while those who fail might be able to solve a complex problem perhaps in twenty minutes.

2. **Understand what’s going in the real world.** One common mistake in preparing for SOA exams at all levels is to treat business problems as pure math problems. For example, when studying short sales, many candidates simply solve one short sale problem after another without really understanding what’s going on in a short sale. If you don’t understand the business essence in a short sale, solving problems is garbage in, garbage out and you don’t learn much. When learning a business concept such as short sales, pricing of a bond between two coupon dates, immunization, cash flow matching, try to understand what’s going on the real world. Think through the business meaning. This way, you’ll find that difficult formulas begin to make sense. You’ll be able to solve problems must faster.
3. **Always solve problems systematically.** Research indicates that experts become experts because they always use systematic approaches to problem solving. They never solve problem haphazardly.

4. **When preparing for Exam FM, focus on building a flawless solution process to all of the previously tested problems, not on aimlessly solving one problem after another with a shaky process.** If you solve a great number of problems (including SOA problems) with shaky process, you’ll make the same misstate over and over. In contrast, if you have a correct process, you’ll find the right answer without the need to solve many problems.

5. **Simplify fancy jargon and complex formulas into simple ones.** While talking fancy and thinking fancy may impress lot of amateurs, talking simple and thinking simple are the key to solving thorny problems 100% right in a hurry under pressure. Common sense concepts and simple solutions are always the easiest solutions to remember and use in the heat of the exam. Complex and unintuitive concepts and formulas are prone to errors. For example, many candidates waste their time memorizing the fancy phrase “annuities payable more (or less) frequently than the interest is convertible” and the related complex formulas. What they should have done is to simplify complex annuities into simple annuities and throw away the fancy phrase and complex formulas once for all.

**Example.** The interest rate is \( i^{(12)} = 12\% \), but the annuity payments of $1 are made quarterly in arrears for 2 years. If you need to calculate the present value of this annuity, use the payment frequency (quarterly) as the interest compounding period and calculate the quarterly effective interest rate:

\[
\left[ 1 + \frac{i^{(12)}}{12} \right]^3 - 1 = \left[ 1 + \frac{12\%}{12} \right]^3 - 1 = 3.03\%
\]

After using quarterly as the compounding period, the original annuity becomes a standard immediate annuity with 8 quarterly payments. The present value of this annuity is \( a_{8|3.03\%} \).

This approach is far better than using the following complex formula:
6. Rely on recalling a pre-built 3 minute solution script to solve repeatable problems in FM. Three minutes is like the blink of an eye in the heat of the exam. In three minutes, most people can, at best, only regurgitate solutions to familiar problems. Most likely, they cannot invent a fresh solution to a previously unseen type of problem. Inventing a solution requires too much thinking and too much time. In fact, if you find yourself having to think too much in the exam, prepare to take Exam FM again.

**How to study hard yet fail the exam miserably**

1. Walk into the exam room without a mental 3 minute solution script, hoping to invent solutions on the spot. This is by the far the most common mistake. Let’s look at a few common myths:

<table>
<thead>
<tr>
<th>Myth</th>
<th>Reality Check</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1 I don’t need a script. I</td>
<td>Why putting yourself on the spot when you can easily come up with a script ahead of time? Never take any unnecessary risks.</td>
</tr>
<tr>
<td>think fast on my feet.</td>
<td></td>
</tr>
<tr>
<td>#2 I don’t need a script. I’ll</td>
<td>There’s no such thing as outperforming yourself in the exam. You always underperform and score less than what your knowledge and ability deserve. This is largely due to the tremendous amount of pressure you inevitably feel in the exam. If you don’t have a script ready for an exam problem, don’t count on solving the problem on the spur of the moment.</td>
</tr>
<tr>
<td>I’ll outperform myself in</td>
<td></td>
</tr>
<tr>
<td>the exam.</td>
<td></td>
</tr>
<tr>
<td>#3 I don’t need a script. If</td>
<td>A solution may come to you automatically, but such a solution is often crude, complex and prone to errors. Even if you have solved a great number of practice problems, you still need to reduce your solutions to a easily repeatable 3 minute process.</td>
</tr>
<tr>
<td>I solve hundreds of</td>
<td></td>
</tr>
<tr>
<td>practice problems before</td>
<td></td>
</tr>
<tr>
<td>the exam, a solution will</td>
<td></td>
</tr>
<tr>
<td>automatically come to me</td>
<td></td>
</tr>
<tr>
<td>when I’m taking the exam.</td>
<td></td>
</tr>
</tbody>
</table>

2. Walk into the exam room without mastering SOA problems. A Chief Executive Officer was about to retire. He asked for the three most promising candidates to come to his office for a quiz. After the three candidates arrived, the CEO asked the first candidate, “How much is one plus one?” “One plus one is two, sir,” replied the
first candidate. The CEO shook his head disappointedly. He turned to the second candidate and asked the same question. The second candidate replied, “One plus one is three, sir.” Once again, the CEO shook his head. Finally, he turned to the third candidate and asked again, “How much is one plus one?” The third candidate replied, “How much do you want it to be, sir?” The CEO smiled and appointed the third candidate as the next CEO.

Key points

- To pass Exam FM, you need to tell SOA what it wants to hear. SOA does not want you to be creative. SOA wants you to demonstrate understanding of core concepts through a standard and methodical solution in keeping with SOA format.

- If you master SOA problems, you pass FM; if you do not, you fail.

- While practice problems in textbooks, study manuals, or seminars are useful, always master SOA problems before mastering any other problems.

3. **Solve hundreds of practice problems (even SOA problems) without generating reusable solutions.** In every exam sitting, there are always busy candidates who take great pride in solving hundreds of SOA problems administered many years back. While solving problems are necessary for passing Exam FM, solving too many problems adds little value. Here is why:

- **Solving too many problems encourages “garbage in, garbage out.”** When a candidate is busy solving a great number of practice problems, often the focus is on searching for any solution that magically produces the correct answer provided in the book (often the book merely provides an answer with little explanation). This encourages problem-solving without fully understanding the nuances of the problem. If your goal is to solve 800 practice problems in 3 months, you miss the point of fully understanding core concepts and problems.

- **Solving too many practice problems exaggerates your ability.** When a candidate focuses on solving hundreds of practice problems, he often does not put himself under exam-like conditions. This almost always leads to inefficient solutions, solutions that look good on paper but fall apart in the heat of the exam.
Recommended study method

1. **Sense before study.** Before opening any textbooks, carefully look at Sample FM and get a feel for the exam style. This prevents you from wasting time trying to master the wrong thing.

2. **Quickly go over the textbook and study the fundamentals (the core concepts and formulas).** Do not attempt to master the complex problems in the textbooks. Solve some basic problems to enhance your understanding of the core concepts.

3. **Put yourself under exam conditions and practice the previous Course 2 exams (if the problems are still on the syllabus), the Sample FM Exam, and November 2005 FM problems.**
   - (1) Put yourself under the strict exam condition.
   - (2) Practice one exam at a time.
   - (3) After taking a practice exam, take several days to analyze what you did right and what you did wrong (don’t do this in a hurry).
   - (4) For each problem in the practice exam, build a reusable 3 minute solution script.
   - (5) Take the same practice exam the 2\textsuperscript{nd} time, using your 3 minute solution script. This puts your 3 minute solution script to test. Find which script works and which doesn’t. Improve your 3 minute solution scripts.
   - (6) Take the next SOA practice exam, repeating Step (1) to Step (5) listed above. You will have more and more 3 minute solution scripts. You will continue refine your 3 minute solution scripts.

4. **Work and rework Sample FM Exam and any released FM exams until you can get them right 100%.** Continue refining your 3 minute solution scripts. Mastering Sample FM Exam and newly released FM exams is the foundation for passing FM.

5. **Don’t worry about solving the same FM problems over and over.** No candidates, however intelligent, can over-solve SOA FM exams. Besides, if you really put yourself under the exam condition, it’s highly unlikely that you’ll memorize the answer to a previously solved problem.

6. **Never, never walk into the exam room without being able to solve Sample FM problems and newly released FM exams 100% right.**

7. **Set up a study schedule and follow it through.**
8. In the two weeks prior to the exam date, dry run Sample FM Exam and May 2005 FM exam, even though this may be your 4th or 5th dry run. You can never practice SOA FM problems too much.

How to build a 3 minute solution script

Building a 3 minute solution script is critical to passing Exam FM. Let’s look at a few examples. They are not necessarily the best, but they really work in the heat of the exam. And feel free to create your own solution scripts.

3 minute solution script example #1 -- loan amortization where payments are level

Traditional method (3 minute solution script)

A loan is borrowed at time zero. It is repaid by \( n \) level payments of \( X \), the 1st payment occurring at \( t = 1 \). In other words, a loan is repaid through an \( n \) year annuity immediate with level payments of \( X \).

Question – how to split each level payment \( X \) into a principal portion and the interest portion?

<table>
<thead>
<tr>
<th>Time ( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>( \ldots )</th>
<th>( k )</th>
<th>( \ldots )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>( X )</td>
<td>( X )</td>
<td>( \ldots )</td>
<td>( X )</td>
<td>( X )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Principal</td>
<td>( X v^n )</td>
<td>( X v^{n-1} )</td>
<td>( \ldots )</td>
<td>( X v^{n-1-k} )</td>
<td>( X v )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interest</td>
<td>( X - X v^n )</td>
<td>( X - X v^{n-1} )</td>
<td>( \ldots )</td>
<td>( X - X v^{n-1-k} )</td>
<td>( X - X v )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Please refer to the textbook to understand why the principal portion is indeed \( X v^n, X v^{n-1}, X v^{n-1-k}, X v \) at \( t = 1,2,k,n \) respectively.

You should walk into the exam room with this rule memorized in your head. Then if an exam problem asks you to split a level payment into principal and interest, you don’t need to calculate the answer from scratch. You simply apply this memorized script and quickly find the answer.

Improved 3 minute solution script --- Imaginary cash flow method

The above script has a trouble spot. You have to memorize that the principal is \( X v^{n+1-k} \) at time \( k \). However, memorizing the discount factor \( v^{n+1-k} \) is a pain. In the heat of the exam, you might use a wrong discount
factor such as \( v^{n-k} \) or \( v^k \). How can you remember the correct discount factor? This leads to the imaginary cash flow method.

To find the principle portion of each payment, we add an imaginary cash flow one step after the final payment (i.e. we add a cash flow of $X$ at \( n+1 \)).

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>k</th>
<th>…</th>
<th>n</th>
<th>n+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>X</td>
<td>X</td>
<td>…</td>
<td>X</td>
<td>…</td>
<td>X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
X \cdot v^{n+1-k} & \quad \text{Imaginary Cash flow} \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>Principal</th>
<th>( Xv^n )</th>
<th>( Xv^{n-1} )</th>
<th>…</th>
<th>( Xv^{n+1-k} )</th>
<th>( Xv )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest</td>
<td>( X - Xv^n )</td>
<td>( X - Xv^{n-1} )</td>
<td>…</td>
<td>( X - Xv^{n+1-k} )</td>
<td>( X - Xv )</td>
</tr>
</tbody>
</table>

To find the principal portion of the payment \( X \) occurring at \( t = k \) where \( k \) is a positive integer and \( 1 \leq k \leq n \), we simply discount our imaginary cash flow \$X\ at \( n+1 \) to \( t = k \). The discounted cash flow \( X \cdot v^{n+1-k} \) is the principal portion of the payment.

The interest portion of the payment \( X \) occurring at \( t = k \) is \( X - X \cdot v^{n+1-k} \).

Do we need to come up with an intuitive explanation for this script? We don’t have to. If this method generates the correct answer, we’ll use it as our script, even though we may not have an intuitive explanation for it.

Now assume that you walk into the exam room with this script in your head. And you see the following problem:
Problem 1

<table>
<thead>
<tr>
<th>Date of loan</th>
<th>1/1/2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term of loan</td>
<td>30 years</td>
</tr>
<tr>
<td>Payments</td>
<td>Annual payments of $2,000 at the end of each year</td>
</tr>
<tr>
<td>1st payment</td>
<td>12/31/2005</td>
</tr>
<tr>
<td>Loan interest</td>
<td>6% annual effective</td>
</tr>
</tbody>
</table>

**Question -- What’s the present value of the interest payments at 1/1/2005 over the life of the loan at a 10% annual effective interest rate?**

This is NOT a simple problem. If you have to figure out the solution from scratch using the prospective or retrospective method, you may have to spend five to ten minutes on it. What’s more, you are likely to make an error here and there if you solve a problem from scratch.

Let’s use our imaginary cash flow script to solve this problem.

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>k</th>
<th>…</th>
<th>30</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>…</td>
<td>2</td>
<td>…</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Principal: \(2v^{30} 2v^{29} … 2v^{31-k} 2v\)

Interest: \(2 - 2v^{30} 2 - 2v^{29} … 2 - 2v^{31-k} 2 - 2v\)

First, to simply our calculation, we’ll use $1,000 as one unit of money. So $2,000 annual payment is 2 units of money.

Next, we’ll add an imaginary cash flow of 2 at \(t = 31\). Then we find the principal portion of each level payment by discounting this imaginary cash flow to \(t = 1,2,\ldots,30\).

The present value at 1/1/2005 of the interest payments over the life of the loan at a 10% annual effective interest rate:

\[
P V = (2 - 2v^{30})V + (2 - 2v^{29})V^2 + (2 - 2v^{28})V^3 + \ldots + (2 - 2v)V^{30}
\]
In the above expression, \( v = \frac{1}{1 + 6\%} \) and \( V = \frac{1}{1 + 10\%} \).

\[ \Rightarrow PV = 2(V + V^2 + V^3 + \ldots + V^{30}) - 2(v^{30}V + v^{29}V^2 + \ldots + vV^{30}) \]

\[ V + V^2 + V^3 + \ldots + V^{30} = a_{30\%}^{10\%} = 9.4269147 \]

\[ v^{30}V + v^{29}V^2 + v^{28}V^3 + \ldots + vV^{30} = \frac{v^{30}V - V^{31}}{1 - \frac{V}{v}} = \frac{1.06^{-30}1.1^{-1} - 1.1^{-31}}{1 - \frac{1.1}{1.06}} = 2.92003944 \]

\[ \Rightarrow PV = 2(9.42691447) - 2(2.92003944) \cdot 13.01375 = $13,013.75 \]

Please note that the imaginary cash flow method also works for a loan repaid by an annuity due. I'll let you prove it.

A loan is borrowed at time zero. It is repaid by \( n \) level payments of \( X \), the 1st payment occurring at \( t = 0 \). In other words, a loan is repaid through an \( n \) year annuity due with level payments of \( X \).

**Question – how to split each level payment \( X \) into a principal portion and the interest portion?**

<table>
<thead>
<tr>
<th>Time ( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>( k )</th>
<th>\ldots</th>
<th>( n-1 )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Imaginary cash flow</td>
<td>$Xv^{n-k}$</td>
<td>( \overbrace{\ldots}^{\text{( \ldots )( \ldots )( \ldots )( \ldots )}} )</td>
<td>$X$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Principal</td>
<td>( Xv^n )</td>
<td>( Xv^{n-1} )</td>
<td>\ldots</td>
<td>( Xv^{n-k} )</td>
<td>\ldots</td>
<td>( Xv )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interest</td>
<td>( X - Xv^n )</td>
<td>( X - Xv^{n-1} )</td>
<td>\ldots</td>
<td>( X - Xv^{n-k} )</td>
<td>\ldots</td>
<td>( X - Xv )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3 minute solution script example #2 -- loan amortization where payments are NOT level

Because SOA can ask us to split non-level payments into principal and interest, we need to build a 3 minute solution script for this.

A loan is borrowed at time zero. It is repaid by $n$ payments of $X_1, X_2, \ldots, X_n$ at $t = 1, 2, \ldots, n$ respectively. Of the total payment $X_1 + X_2 + \ldots + X_k$ made during the first $k$ payments, how much is the principal payment? How much is the interest payment?

Time $t$ 0 1 2 $\ldots$ $k$ $\ldots$ $n$
Cash flow $X_1$ $X_2$ $\ldots$ $X_k$ $X_n$

3 minute solution script:

Step 1 – Calculate $P_0$, the outstanding balance of the loan at $t = 0$.

Step 2 – Calculate $P_k$, the outstanding balance of the loan at $t = k$ immediately after the $k-th$ payment is made.

Step 3 – Calculate $P_0 - P_k$, the reduction of the outstanding balance between $t = 0$ and $t = k$. $P_0 - P_k$ should be the total principal repaid during the first $k$ payments.

Step 4 – Calculate $(X_1 + X_2 + \ldots + X_k) - (P_0 - P_k)$. This should the total interest paid during the first $k$ payments.

The core logic behind this script:

$$\frac{X_1 + X_2 + \ldots + X_k}{\text{Total payments made during the first } k \text{ payments}} = \frac{P_0 - P_k}{\text{Principal reduction}} + \text{Interest Payment}$$
Problem 2

<table>
<thead>
<tr>
<th>Date of loan</th>
<th>1/1/2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount of loan</td>
<td>$150,000</td>
</tr>
<tr>
<td>Term of loan</td>
<td>25 years</td>
</tr>
<tr>
<td>Payments</td>
<td>Annual payments with first payment due 12/31/2005. Each subsequent payment is 2% larger than the previous payment.</td>
</tr>
<tr>
<td>Interest rate</td>
<td>8.5% annual effective</td>
</tr>
</tbody>
</table>

What’s the total interest paid during the first 18 payments?

Solution

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>…</th>
<th>18</th>
<th>19</th>
<th>…</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>(X)</td>
<td>(1.02X)</td>
<td>(1.02^2X)</td>
<td>…</td>
<td>(1.02^{17}X)</td>
<td>(1.02^{18}X)</td>
<td>…</td>
<td>(1.02^{24}X)</td>
<td></td>
</tr>
</tbody>
</table>

**Step 1** – calculate the outstanding balance at \(t = 0\). \(P_0 = 150,000\)

**Step 2** – calculate the outstanding balance at \(t = 18\) immediately after the 18th payment is made (we discount future cash flows to \(t = 18\)):

\[
P_{18} = 1.02^{18}Xv + 1.02^{19}Xv^2 + \ldots + 1.02^{24}Xv^7 = X \left( 1.02^{18}v + 1.02^{19}v^2 + \ldots + 1.02^{24}v^7 \right)
\]

So we need to calculate \(X\). Because the present value of the loan at \(t = 0\) is 150,000, we have:

\[
Xv + 1.02Xv^2 + 1.02^2Xv^3 + \ldots + 1.02^{24}Xv^{25} = 150,000
\]

\[
\Rightarrow X \left( v + 1.02v^2 + 1.02^2v^3 + \ldots + 1.02^{24}v^{25} \right) = 150,000
\]

\[
v + 1.02v^2 + 1.02^2v^3 + \ldots + 1.02^{24}v^{25} = \frac{v - 1.02^{25}v^{26}}{1 - 1.02} = \frac{1.085^{-1} - 1.02^{25}1.085^{-26}}{1 - 1.02(1.085^{-1})} = 12.10103628
\]

\[
\Rightarrow X = \frac{150,000}{12.10103628} = 12,395.6326
\]

The outstanding balance at \(t = 18\) immediately after the 18th payment is:

\[
P_{18} = X \left( 1.02^{18}v + 1.02^{19}v^2 + \ldots + 1.02^{24}v^7 \right)
\]
Step 3 – Calculate the reduction of the outstanding balance between 
t = 0 and t = 18.

\[ P_0 - P_{18} = 150,000 - 95,622.7113 = 54,377.2887 \]

This is the total principal payment during the first 18 payments.

Step 4 - Calculate \((X_1 + X_2 + \ldots + X_{18}) - (P_0 - P_{18})\). This is the total interest paid during the first 18 payments.

\[
X_1 + X_2 + \ldots + X_{18} = X \left(1 + 1.02^1 + 1.02^2 + \ldots + 1.02^{18}\right) = X_{\frac{18}{1.02}}
\]

\[
= 12,395.6326 \times \frac{18}{1.02} = 265,419.1574
\]

\[
(X_1 + X_2 + \ldots + X_{18}) - (P_0 - P_{18}) = 265,419.1574 - 54,377.2887 = 211,041.8687
\]

Extend our script  What if we want to calculate the total principal and interest during the 3rd, 4th, and 5th payments? In other words, how can we split \(X_3 + X_4 + X_5\) into principal and interest?

\[
\begin{align*}
\text{Step 1} & \quad \text{Calculate } P_2, \text{ the outstanding balance of the loan at } t = 2 \\
\text{Step 2} & \quad \text{Calculate } P_5, \text{ the outstanding balance of the loan at } t = 5 \\
\text{Step 3} & \quad \text{Calculate } P_2 - P_5, \text{ the reduction of the outstanding balance between } t = 2 \text{ and } t = 5. \text{ This is the total principal repaid during the 3rd, 4th, and 5th payments.} \\
\text{Step 4} & \quad \text{Calculate } (X_3 + X_4 + X_5) - (P_2 - P_5). \text{ This should be the total interest paid during the 3rd, 4th, and 5th payments.}
\end{align*}
\]

The core logic behind this script:

\[
\underbrace{X_3 + X_4 + X_5}_{\text{Total payments made during 3rd, 4th, and 5th payments}} = \underbrace{P_2 - P_5}_{\text{Principal reduction}} + \underbrace{\text{Interest payment}}_{\text{Principal reduction}}
\]
Similarly, if you are asked to split the 3\textsuperscript{rd} payment into principal and interest, you can create a script using the following logic:

\[
X_3 = P_3 - P_3 + \text{Interest payment}
\]

\[\text{3rd payment} \quad \text{Principal reduction}\]

\textbf{3 minute solution script example #3 – present value of a geometrically increasing annuity}

Let’s look at our solution to Problem 2, which involves a geometrically increasing annuity. In our solution, we calculated the present value of several geometrically increasing annuities from scratch. Each time, we use the sum rule of a power series:

\[
a + aq + aq^2 + \ldots + aq^{n-1} = \frac{a - aq^n}{1 - q} \quad \text{where } q \neq 1
\]

We ask ourselves, “Why not create a script for a geometrically increasing annuity to avoid calculating its present value from scratch?” Luckily, we find a script:

\[
\frac{1}{1 + k} \sum_{j=1}^{n} i^{-k} = \frac{1}{1 + k} \sum_{j=1}^{n} \left(1+k\right)^{j-1}
\]

\[
\frac{1}{1 + k} \sum_{j=1}^{n} i^{-k} = \frac{1}{1 + k} \sum_{j=1}^{n} \left(1+k\right)^{j-1}
\]

For a geometrically increasing annuity where

1. \(n\) geometrically increasing payments are made at a regular interval;
2. the 1\textsuperscript{st} payment is $1;
3. the next payment is always \((1+k)\) times the previous payment.

Then
(1) The present value one step before the 1st payment is \( \frac{1}{1+k} a_{\frac{i}{1+k}} \). This value has a factor of \( \frac{1}{1+k} \) because the geometric payment pattern at one interval prior to the 1st payment is \( \frac{1}{1+k} \). This value also has an annuity factor of \( a_{\frac{i}{1+k}} \), where \( j \) is the adjusted interest rate.

(2) The present value at the 1st payment time is \( (1) \frac{i}{1+k} \). The present value has a factor of 1 because the 1st payment is 1. This value has an annuity factor of \( \frac{i}{1+k} \), where \( j \) is the adjusted interest rate.

To find the proof of this script, see the chapter on geometrically increasing annuity. For now, let’s focus on how to use this script.

From this script, we see that the present value of a geometrically increasing annuity is always the product of a payment factor and an annuity factor:

\[
PV \text{ of geometric annuity} = \text{Payment Factor} \times \text{Annuity Factor} @ \ j = \frac{i}{1+k}
\]

The above script is simple yet powerful. Let’s redo Problem 2 using the geometric annuity script.

**Problem 2**

<table>
<thead>
<tr>
<th>Date of loan</th>
<th>1/1/2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount of loan</td>
<td>$150,000</td>
</tr>
<tr>
<td>Term of loan</td>
<td>25 years</td>
</tr>
<tr>
<td>Payments</td>
<td>Annual payments with first payment due 12/31/2005. Each subsequent payment is 2% larger than the previous payment.</td>
</tr>
<tr>
<td>Interest rate</td>
<td>8.5% annual effective</td>
</tr>
</tbody>
</table>

Q: What’s the total interest paid during the first 18 payments?

**Solution**
Step 1 – calculate the outstanding balance at $t = 0$. $P_0 = 150,000$

Step 2 – calculate the outstanding balance at $t = 18$ immediately after the $18^{th}$ payment is made.

$$P_{18} = \left(\text{payment factor } @ t = 18\right) \times \left(\text{annuity factor } @ j = \frac{i-k}{1+k}\right)$$

The payment factor $@ t = 18$ is $1.02^{17}$ X ;

The annuity factor $@ j = \frac{i-k}{1+k}$ is:

$$a_{\frac{8.5\%-2\%}{1.02}} = a_{\frac{6.5\%}{1.02}} = 6.372549$$

Then:

$$P_{18} = 1.02^{17} X a_{\frac{6.5\%}{1.02}} = 1.02^{17} X 6.372549$$

Next, we’ll calculate $X$. Because the present value of the loan at $t = 0$ is 150,000, we have:

$$150,000 = PV \text{ of geometric annuity}= \left(\text{payment factor } @ t = 0\right) \times \left(\text{annuity factor } @ j = \frac{i-k}{1+k}\right)$$

The payment factor $@ t = 0$ is $\frac{X}{1.02}$.

If we extend the payment pattern to time zero, we’ll have a payment of $\frac{X}{1.02}$ at time zero.

The annuity factor $@ j = \frac{i-k}{1+k}$ is:

$$a_{\frac{6.372549}{1.02}} = a_{\frac{6.372549}{1.02}} = 6.372549$$

Then:

$$150,000 = \frac{X}{1.02} a_{\frac{6.372549}{1.02}} = \frac{X}{1.02} 6.372549$$

Using BA II Plus/BA II Plus Professional, we quickly find that

$$a_{\frac{6.372549}{1.02}} = 12.34305703$$
\[ X = \frac{150,000(1.02)}{a_{18}^{6.372549\%}} = 12,395.6326 \]

\[ P_{18} = 1.02^{17} X a_{18}^{6.372549\%} = 1.02^{17} (12,395.6326)(5.50921141) = 95,622.7113 \]

**Step 3** – Calculate the reduction of the outstanding balance between \( t = 0 \) and \( t = 18 \).

\[ P_0 - P_{18} = 150,000 - 95,622.7113 = 54,377.2887 \]

This is the total principal payment during the first 18 payments.

**Step 4** - Calculate \((X_1 + X_2 + ... + X_{18}) - (P_0 - P_{18})\). This is the total interest paid during the first 18 payments.

\[ X_1 + X_2 + ... + X_{18} = X \left(1 + 1.02 + 1.02^2 + ... + 1.02^{17}\right) = X a_{18}^{6.372549\%} \]
\[ = 12,395.6326 a_{18}^{6.372549\%} = 265,419.1574 \]

\((X_1 + X_2 + ... + X_{18}) - (P_0 - P_{18}) = 265,419.1574 - 54,377.2887 = 211,041.8687 \)

You see that our calculation is much faster if we use the geometric annuity script.

**3 minute solution script example #4 – present value of an arithmetically increasing/decreasing annuity**

Among all the annuities, an arithmetically increasing or decreasing annuity is most prone to errors. Errors associated with an arithmetically increasing or decreasing annuity often come from two sources:

- **Incorrectly identify the cash flow pattern.** In an increasing or decreasing annuity, we can easily find the precise cash flow at each time point. However, to calculate the present value of an increasing or decreasing annuity, knowing the precise cash flow at each time is not enough. The present value formula forces us to find the pattern by which the cash flows change over time. Identifying the cash flow pattern is much harder than identifying cash flows.
• Incorrectly apply the present value formula. Even if we find the cash flow pattern, we still have to use the complex and awkward formula to calculate the present value. Any mistake in using the formula will destroy our previous work.

If calculating the present value of an increasing or decreasing annuity sounds complex, imagine solving such a problem in the exam condition where your heat beats fast and your hands tremble. How can we find a magic script that will enable us to calculate the present value of an arithmetically increasing or decreasing annuity 100% in a hurry under pressure?

The bad news is that we can't always find a script. The good news is that we can find a script that works most of the time.

3 minute solution script -- PV of increasing/decreasing annuity

If the # of cash flows is NOT enormous (fewer than 50 for example), simply enter the cash flows into BA II Plus/BA II Plus Professional Cash Flow Worksheet.

### Problem 3

| Year 1 | $50,000 |
| Year 2 to Year 11 | Annually increasing by $5,000. |
| Year 11 to Year 15 | Level |
| Year 15 to Year 22 | Annually decreasing by $6,000 |

| Interest rate | 8% annual effective |

Calculate the present value of the fund.

### Solution

We don't want to spend lot of time analyzing the cash flow pattern or using the complex formula. We just identify all of the cash flows:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$0</td>
<td>$50</td>
<td>$55</td>
<td>$60</td>
<td>$65</td>
<td>$70</td>
<td>$75</td>
<td>$80</td>
<td>$85</td>
<td>$90</td>
<td>$95</td>
<td>$100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time t</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$100</td>
<td>$100</td>
<td>$100</td>
<td>$100</td>
<td>$100</td>
<td>$94</td>
<td>$88</td>
<td>$82</td>
<td>$76</td>
<td>$70</td>
<td>$64</td>
<td>$58</td>
</tr>
</tbody>
</table>
In the above table, we use $1,000 as one unit of money to simplify our calculation.

Next, we enter the above cash flows into Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>Cash flow</th>
<th>CF0</th>
<th>C01</th>
<th>C02</th>
<th>C03</th>
<th>C04</th>
<th>C05</th>
<th>C06</th>
<th>C07</th>
<th>C08</th>
<th>C09</th>
<th>C10</th>
<th>C11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0</td>
<td>$50</td>
<td>$55</td>
<td>$60</td>
<td>$65</td>
<td>$70</td>
<td>$75</td>
<td>$80</td>
<td>$85</td>
<td>$90</td>
<td>$95</td>
<td>$100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency</th>
<th>F01</th>
<th>F02</th>
<th>F03</th>
<th>F04</th>
<th>F05</th>
<th>F06</th>
<th>F07</th>
<th>F08</th>
<th>F09</th>
<th>F10</th>
<th>F11</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cash flow</th>
<th>C12</th>
<th>C13</th>
<th>C14</th>
<th>C15</th>
<th>C16</th>
<th>C17</th>
<th>C18</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$94</td>
<td>$88</td>
<td>$82</td>
<td>$76</td>
<td>$70</td>
<td>$64</td>
<td>$58</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency</th>
<th>F12</th>
<th>F13</th>
<th>F14</th>
<th>F15</th>
<th>F16</th>
<th>F17</th>
<th>F18</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Set I=8 (i.e. the interest rate is 8%).

You should get: NPV = 797.80666333 (units) = $797,806.66333

**Problem 4**

<table>
<thead>
<tr>
<th>Annuity #1</th>
<th>An amount payable at the end of each quarter beginning with a $2,000 payment on 3/31/2005. Each subsequent payment is 1.5% larger than the previous payment. The annuity pays for 15 years.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annuity #2</td>
<td>An amount payable at the end of each year beginning with an $X$ payment on 12/31/2005. Each subsequent payment is $30 less than the previous payment. The annuity pays for 20 years.</td>
</tr>
<tr>
<td>interest rate</td>
<td>9% annual effective</td>
</tr>
</tbody>
</table>

On 1/1/2007, the present value of the remaining Annuity #1 equals the present value of the remaining annuity #2.

Calculate $X$.

**Solution**

This problem is conceptually simple. However, tracking down the timings and amounts of the cash flows of each annuity is a nightmare. Unless you use a systematic approach, you'll make errors here and there.
First, we draw a cash flow diagram:

**Annuity #1**

<table>
<thead>
<tr>
<th>Time t (Year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1/4</td>
<td>2/4</td>
<td>3/4</td>
<td>1/4</td>
<td>2/4</td>
<td>3/4</td>
</tr>
<tr>
<td>← 2005</td>
<td>→</td>
<td>←</td>
<td>←</td>
<td>←</td>
<td>→</td>
<td></td>
</tr>
</tbody>
</table>

15 – 2 = 13 years remaining
13 1/4 = 52 payments remaining

Payment factor = $2,000(1.015)^7$
Annuity factor = $a_{52|j}$

\[
j = \frac{(1 + 9\%)^{1/4} - 1}{1 + 1.5\%}
\]

PV of remaining payments = $2,000(1.015)^7 a_{52|j}$

**Annuity #2**

<table>
<thead>
<tr>
<th>Time t (Year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>← 2005</td>
<td>→</td>
<td>←</td>
<td>←</td>
<td>←</td>
<td>→</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

20 – 2 = 18 years remaining
18 payments remaining
We’ll break down the remaining payments of Annuity #2 into two streams:

\[
\begin{array}{ccccccccccc}
\text{Time } t & 0 & 1 & 2 & 3 & 4 & \ldots & 20 \\
\text{(Year)} & \hline
\end{array}
\]

\[
\begin{array}{ccccccccccc}
2005 & \leftarrow & 2006 & \rightarrow & 2007 & \leftarrow & 2008 & \rightarrow & X & X & \ldots & X \\
\end{array}
\]

\[
\begin{array}{ccccccccccc}
2005 & \leftarrow & 2006 & \rightarrow & 2007 & \leftarrow & 2008 & \rightarrow & 2(30) & 3(30) & \ldots & 19(30) \\
\end{array}
\]

\[x_{a_{10\%}}\]

\[\text{PV of an increasing annuity} \]

\[\Rightarrow \text{PV of the remaining payments} = x_{a_{10\%}} - \text{PV of a increasing annuity}\]

To pin down the PV of the increasing annuity, we’ll use BA II Plus/BA II Plus Professional Cash Flow Worksheet. Enter the following cash flows:

\[
\begin{array}{ccccccccccc}
\text{Cash flow} & \text{CF0} & \text{C01} & \text{C02} & \text{C03} & \text{C04} & \text{C05} & \text{C06} & \text{C07} & \text{C08} & \text{C09} \\
\text{Frequency} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccccccccccc}
\text{Cash flow} & \text{C10} & \text{C11} & \text{C12} & \text{C13} & \text{C14} & \text{C15} & \text{C16} & \text{C17} & \text{C18} \\
\text{Frequency} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccccccccccc}
\text{Cash flow} & \text{F01} & \text{F02} & \text{F03} & \text{F04} & \text{F05} & \text{F06} & \text{F07} & \text{F08} & \text{F09} \\
\text{Frequency} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccccccccccc}
\text{Cash flow} & \text{F10} & \text{F11} & \text{F12} & \text{F13} & \text{F14} & \text{F15} & \text{F16} & \text{F17} & \text{F18} \\
\text{Frequency} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

In the above cash flow table, we used $30 as one unit of money to simplify our data entry.
Next, in Cash Flow Worksheet, set the interest rate to 9%. You should get: \( \text{NPV}=72.39722563 \). Because one unit of money represents $30, so the real NPV is

\[
(30) \times 72.39722563 = 2,191.916769
\]

You might groan at such a solution as ugly. However, this quick and dirty solution has its beauty: it provides a mechanic solution to a complex problem. It really works in the heat of the exam where you are stressed.

If you really want to use the PV formula, you can add a cash flow of $30 at \( t = 0 \). This gives you a clean and nice increasing annuity.

<table>
<thead>
<tr>
<th>Time ( t ) (Year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>← 2005</td>
<td>← 2006</td>
<td>← 2007</td>
<td>← 2008</td>
<td>→</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>2(30)</td>
<td>3(30)</td>
<td>…</td>
<td>19</td>
<td>(30)</td>
</tr>
<tr>
<td>–30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ PV = 30\left( {\bar{a}}_{19|9} \right) - 30 \]

So the present value of the decreasing annuity at time zero is:

\[
30\left( {\bar{a}}_{19|9} \right) - 30 = 30 \left[ \frac{\ddot{a}_{19|9} - 19\ddot{v}^{19}}{d} - 1 \right] = 2,171.916769 \quad @ \ i = 9\%
\]

Finally, we are ready to solve for \( X \):

\[
2,000(1.015)^7a_{\overline{19}|j} = Xa_{\overline{18}|9\%} - 2,171.916769
\]

where \( j = \left[ \frac{(1+9\%)^{1.5} - 1}{1+1.5\%} \right]^{-1.5\%} \)

Solving the equation, we have:

\[ X = 11,354.44396 \]
Key points to remember about 3 minute solution scripts:

- Scripts prevent us from reinventing a wheel again and again. Using scripts, we invent a wheel once; next time, we simply reuse the wheel invented before.

- Scripts cut to the chase and quickly get to the core of the problem at hand, enabling us to solve a complex problem 100% right in a hurry under pressure.

- Don’t blindly copy solutions from SOA exams, textbooks, study manuals, or seminars. Customize those solutions so they are simpler, faster, and less prone to errors. The scripts created by you are the best scripts.

- Keep refining your scripts.

How to eliminate errors

Of all the SOA exams, Exam FM requires us to have the least amount of creativity. The textbooks have given us a myriad of formulas for virtually every concept from level annuity, to varying annuity, to loan amortization, to bond evaluation, to duration, to convexity. We need only to memorize these formulas and apply them to specific situations.

Then why can’t everyone pass Exam FM? Why didn’t everyone pass Course 2?

Answer: errors.

Even if we have memorized all the formulas, mastered all the concepts, and created powerful 3 minute solution scripts, we may still not be able to solve every problem right. After all, we are fallible. We are especially fallible when we have to solve a problem in a hurry and under stress.

To convince yourself that you are fallible, put yourself under the exam condition and take a Course 2 exam administered in the past. See how many incorrect answers are caused by your silly mistakes.

**Truth #1**: To pass Exam FM, you need to do two tasks: (1) learn new things and expand your knowledge, (2) reduce errors in areas you already know and defend your knowledge.
Truth #2: Most candidates spend too much time learning new things; they spend too little time reducing errors.

Truth #3: If you are a borderline student knowing the minimum amount knowledge on FM, your chance to pass Exam FM solely depends on whether you can eliminate errors. Say you solved only 70% of the exam leaving the remaining problems blank. If you made zero error in your solutions, you surely passed FM if the passing standard was solving 66% of the problems right.

How can we reduce errors?

Let’s look at an everyday situation.

<table>
<thead>
<tr>
<th>Situation</th>
<th>Grocery shopping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>Forgetting to buy certain items</td>
</tr>
<tr>
<td>Error elimination strategy</td>
<td>Don’t trust your memory. Rigorously use a shopping list. Check off an item from your list once you have bought it. Before leaving the store, make sure that you have checked off all the items in your list.</td>
</tr>
</tbody>
</table>

How to eliminate calculation errors – Use a systematic approach

1. Don’t think that “Others will make mistakes but I won’t” or “If I just solve hundreds of practice problems, I will never make mistakes again.” Remember that mistakes will surely creep in like a thief unless you use a systematic approach to problem solving.

2. Trust that if you understand the concept and if you use a systematic approach, the right solution will surely emerge.

3. Under a systematic approach, always move from general principles to specific answers. For example, when solving a cash flow problem, you first set up the fundamental equation such as the present value of the cash inflows is equal to the present value of the cash outflows. Then you calculate the missing item (the interest, a specific cash flow, or the number of payments, etc) by solving this equation.

4. Solve a problem step by step; don’t jump steps. For example, if you have a choice of solving a problem in one giant step or solving the problem in three small steps, use the three-step approach to avoid errors.
5. **Unless a problem is overly simple, always draw a cash flow diagram.** This avoids the common off-by-one error (forgetting one payment or adding a payment that does not exist).

6. **When taking the exam, use calculators to their full power and delegate calculations to your calculators as much as you can.** When solving practice problems, however, solve them in two ways. First, don’t use a calculator’s built in functionalities (such as BA II Plus Professional’s TVM or modified duration) and solve the problem manually; this sharpens your conceptual thinking. Next, maximize the use of your calculator and minimize your work; this quickens your solution.

7. **Don’t do mental math; use your calculator for even the simplest calculations.** In the heat of the exam, people tend to make silly mistakes for even the simplest calculations. For example, if you need to calculate 11 - 9.4 (eleven minus nine point four) in the exam. Is the answer 1.6 or 2.6? If you do mental math, you are likely to miscalculate. So don’t be creative. Just let your calculator do the math for you.

8. **When the payment frequency differs from the interest compounding frequency, use the payment frequency as the interest compounding frequency.** This greatly reduces the number of formulas you have to memorize.

9. **Never transfer intermediate values between the scrap paper and your calculator.** Always store the intermediate values in your calculator’s memories. Use symbols to keep track which number is in which memory.

**Example of using a systematic approach**

**Problem 1**

John buys the following bond:

<table>
<thead>
<tr>
<th>Face amount</th>
<th>$1,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term to maturity</td>
<td>20 years</td>
</tr>
<tr>
<td>Coupons</td>
<td>8% payable semi-annually</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>10%</td>
</tr>
</tbody>
</table>

With 5 years to maturity and immediately after receiving the 30th coupon, John sells the bond at a price yielding 9% annual effective to the buyer.

Calculate John’s annual effective return on his investment in the bond.
**Solution**

This problem seems difficult at first glance. However, let’s solve it step by step using a systematic approach. First, we draw a cash flow diagram to help us visualize the information given to us. Because the cash flows occur every 6 months, we will use 6 months as the interest compounding period. We’ll convert an annual effective interest rate to a 6-month effective rate.

![Cash Flow Diagram]

We are still not sure how to calculate John’s return for investing in the bond; we don’t have a formula ready. What should we do?

In a systematic approach, we move from general principles to specifics. How do we calculate an investment return in general?

**General principle**

We deposit $X(0)$ at time zero in a fund. At time $t$ the fund grows to $X(t)$. Our return for investing in the fund during the time horizon $[0,t]$ is $r$. We can solve for $r$ as follows:

$$X(0)(1+r)^t = X(t) \quad \Rightarrow r = \left[ \frac{X(t)}{X(0)} \right]^\frac{1}{t} - 1$$
Let’s apply this general principle to John’s investment. To find John’s return, we need three data:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X(0))</td>
<td>John’s initial out-of-pocket expense to buy the investment</td>
</tr>
<tr>
<td>(t)</td>
<td>John’s investment horizon</td>
</tr>
<tr>
<td>(X(t))</td>
<td>John’s total wealth at the end of the investment horizon</td>
</tr>
</tbody>
</table>

John’s cost for buying the bond is the bond’s market price. We know that a bond’s fair market price is the present value of the bond’s cash flows discounted at YTM (yield to maturity).

\[
X(0) = 40a_{\overline{30|}}^{10\%} + 1,000v^{40} \quad \text{@} \ i = (1 + 10\%)^{\frac{1}{2}} - 1
\]

How long did John investment his money in the bond? He bought the bond at time zero. He got out of the bond at time 30. So his investment horizon is \( t = 30 \). Please note that the time unit is still 6 months.

Finally, we need to find \( X(30) \), John’s total wealth at time 30. If John counts his money at time 30, how much does he have? John’s total wealth at time 30 comes from two sources: (1) reinvesting 30 coupons, (2) selling 10 cash flows at 9%.

\[
X(30) = \left( \frac{40s_{\overline{30|}}^{10\%}}{30} \right) \quad \text{@} \ i = (1 + 10\%)^{\frac{1}{2}} - 1 + \left( 40a_{\overline{10|}}^{9\%} + 1,000v^{10} \right) \quad \text{@} \ j = (1 + 9\%)^{\frac{1}{2}} - 1
\]

In the above equation, we use 10% as John’s annual effective reinvest rate. Though the reinvestment rate can differ from the YTM, this problem doesn’t specifically give us John’s reinvestment rate. As a result, we assume that John reinvests his 30 coupons at YTM of 10%.

Finally, we are ready to calculate John’s return \( r \). However, we need to proceed cautiously. We’ll perform many messy calculations to get \( r \). If we make a silly mistake in one calculation, all the good work we have done so far is ruined.

To flawless perform messy calculations, we will follow 3 rules:
• **Rule #1**   In the exam, delegate work to our calculator as much as we can. Our calculator is our obedient servant, who always executes our orders precisely and flawlessly. As long as our orders are clear and unambiguous, we can rest assured that our calculator will get the job done 100% right.

• **Rule #2**   In doing practice problems, solve the problems twice. First time we solve the problem manually using formulas. Second time we let the calculator do most of the work (just as we do in the exam).

• **Rule #3**   Don’t transfer intermediate values between our calculator and the scrap paper. Store all the intermediate values in our calculator’s memories and systematically track which memory stores which value.

Let’s apply these rules. We’ll solve the problem with two methods. We’ll systematically track down our intermediate values in each method.

**The calculation procedure for Method #1 - use BA II Plus/BA II Plus Professional TVM Worksheet**

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Store in memory</th>
<th>Track down values stored in memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1 Convert annual 10% into 6-month rate</td>
<td>$i = \left(1 + \frac{1}{2} \cdot 10% \right)^\frac{1}{2} - 1$</td>
<td>$M0$</td>
<td>$100i = M0$</td>
</tr>
</tbody>
</table>

There’s a slight difference between $i$ and $100i$. BA II Plus/BA II Plus Professional displays a number up to 8 decimal places. If a number has more than 8 decimal places, the calculator’s rounds the number to 8 decimal places in the display. However, internally BA II Plus/BA II Plus Professional stores a number in 13 decimal places, even though it displays up to 8 decimal places. Please note that the display does NOT affect how a number is stored internally in the calculator.
Calculate $X(0)$ using TVM.

\[ X(0) = 40\alpha_{\overline{40}|} + 1,000v^{40} \quad @ \ i \]
\[ \Rightarrow X(0) = 846.3502152 \]

Convert annual 9% into 6-month rate

\[ j = (1 + 9\%)^{\frac{1}{2}} - 1 = 4.40306509\% \]
\[ \Rightarrow 100j = 4.40306509 \]

Calculate \( 40x_{\overline{30}|i} \) using TVM

\[ \Rightarrow 40x_{\overline{30}|i} = 2,603.829665 \]

Calculate \( 40a_{\overline{10}|} + 1,000v^{10} \) using TVM

\[ 40a_{\overline{10}|} + 1,000v^{10} \quad @ \ j = 967.9540424 \]

Calculate \( X(30) \)

\[ 40x_{\overline{30}|i} + \left(40a_{\overline{10}|} + 1,000v^{10}\right) \quad @ \ j = 3,571.783707 \]
\[ \Rightarrow M3 + M4 = 3,571.783707 \]

(Press “RCL 3” and “RCL 4” to recall M3 and M4 respectively)

Calculate \( r \), John’s return in a 6-month period

\[ r = \left[ \frac{X(30)}{X(0)} \right]^{\frac{1}{30}} - 1 \]
\[ \Rightarrow r = 4.91667086\% \]

(Press “RCL 5” and “RCL 1” to recall M5 and M1 respectively)

Convert \( r \) into an annual rate \( R \)

\[ R = (1 + r)^2 - 1 = (1 + M6)^2 - 1 \]
\[ \Rightarrow R = 10.07507825\% \]

(Press “RCL 6” to recall M6)

So John’s annual effective return for investing in the bond is 10.08%.
The calculation procedure for Method #2 (formula driven approach)
- use general functions of BA II Plus/BA II Plus Professional

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Store in memory</th>
<th>Track down values stored in memory</th>
</tr>
</thead>
</table>
| #1 Convert annual 10% into 6-month rate | \( i = \left(1 + \frac{10\%}{2}\right) - 1 \)  
\( \Rightarrow \ i = 4.88088482\% \) |  | \( i = M0 \) |
| #2 Calculate \( a_{40} \) | \( a_{40} = \frac{1 - \nu^{40}}{i} \)  
\( \Rightarrow \ a_{40} = \frac{1 - (1 + M0)^{-40}}{M0} \)  
\( \Rightarrow \ a_{40} = 17.44266468 \) | \( M1 \) | \( a_{40} = M1 \) |
| #3 Calculate \( X(0) \) | \( X(0) = 40a_{40} + 1,000\nu^{40} \)  
\( \Rightarrow \ X(0) = 40M1 + 1,000(1 + M0)^{-40} \)  
\( \Rightarrow \ X(0) = 846.3502152 \) | \( M2 \) | \( X(0) = M2 \) |
| #4 Convert annual 9% into 6-month rate | \( j = \left(1 + \frac{9\%}{2}\right) - 1 = 4.40306509\% \) | \( M3 \) | \( j = M3 \) |
| #5 Calculate \( s_{30} \) | \( s_{30} = \frac{(1+i)^{30} - 1}{i} = \frac{(1 + M0)^{30} - 1}{M0} \)  
\( \Rightarrow \ s_{30} = 65.09574162 \) | \( M4 \) | \( s_{30} = M4 \) |
| #6 Calculate \( a_{10} \) | \( a_{10} = \frac{1 - \nu^{10}}{j} = \frac{1 - (1 + M3)^{-10}}{M3} \)  
\( \Rightarrow \ a_{10} = 7.95056640 \) | \( M5 \) | \( a_{10} = M5 \) |
| #7 Calculate \( 40a_{10} + 1,000\nu^{10} \) | \( 40a_{10} + 1,000\nu^{10} \)  
\( \Rightarrow \ 40M5 + 1,000(1 + M3)^{-10} \)  
\( \Rightarrow \ 967.9540424 \) | \( M6 \) | \( 40a_{10} + 1,000\nu^{10} \)  
\( j = M6 \) |
| #8 Calculate \( X(30) \) | \( X(30) = 40s_{30} + \left(40a_{10} + 1,000\nu^{10}\right) \)  
\( \Rightarrow \ 40M4 + M6 = 3,571.783707 \) | \( M7 \) | \( X(30) = M7 \) |
| #9 Calculate \( r \), John’s return in a 6-month period | \( r = \left[ \frac{X(30)}{X(0)} \right]^{\frac{1}{2}} - 1 \)  
\( \Rightarrow \ r = \left[ \frac{M7}{M2} \right]^{\frac{1}{2}} - 1 = 4.91667086\% \) | \( M8 \) | \( r = M8 \) |
| #10 Convert \( r \) into an annual rate \( R \) | \( R = (1+r)^2 - 1 = (1 + M8)^2 - 1 \)  
\( \Rightarrow \ R = 10.07507825\% \) | \( M9 \) | \( R = M9 \) |
In the two methods above, we painstakingly track down which number is stored in which memory. This tracking system has the following advantages:

- Eliminate the need to transfer numbers back and forth between a calculator and a scrap paper.
- Eliminate the errors caused by transferring numbers between a calculators and a scrap paper.
- Eliminate the loss of precisions caused by transferring a fraction numbers between a calculators and a scrap paper. For example, if we have to transfer $i = 4.88088482\%$ back and forth between a calculator and a scrap paper, we feel compelled to round $i$ to $i = 4.88\%$. However, if we store $i = 4.88088482\%$ in a calculator’s memory and recall it whenever we use it, BA II Plus/BA II Plus Professional will store $i$ in 13 decimal places in its internal calculations, yielding results with good precision.
- Leave an audit trail, isolating good calculations from bad. For example, after arriving the final answer of $R$, we realized that our Step #1 calculation was wrong. To fix the error, we simply redo Step 1 calculation and reload the newly calculated $i$ to $M0$. Next, we redo all the calculations that use $M0$, leaving the calculations that do not involve $M0$ intact. In contrast, if we don’t have such a good tracking system in place, one single error will blow up all the calculations, forcing us to recalculate everything from scratch.

**Problem 2 (EA-1 #1  2003)**

Over a 3-year period, a series of deposits are made to a savings account. All deposits within a given year are equal in size ad are made at the beginning of each relevant period. Deposits for each year are total $1,200. The following chart shows the frequency of deposits and the interest rate credited for each year.

<table>
<thead>
<tr>
<th>Year</th>
<th>Frequency of deposits</th>
<th>Interest rate credited during the year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Semi-annually</td>
<td>$d^{(12)} = 6%$</td>
</tr>
<tr>
<td>2</td>
<td>Quarterly</td>
<td>$i^{(3)} = 8%$</td>
</tr>
<tr>
<td>3</td>
<td>Every 2 months</td>
<td>$\delta = 7%$</td>
</tr>
</tbody>
</table>

Calculate the value of the account at the end of the 3\textsuperscript{rd} year.
Solution

This problem is tricky. To solve it right, we need to systematically track down the timing and the amount of each deposit.

To simplify our calculation, we’ll set 1 unit of money = $100. So each year, the total amount of deposits made each year is 12.

<table>
<thead>
<tr>
<th>Time (year)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>1.75</th>
<th>2</th>
<th>2.25</th>
<th>2.5</th>
<th>2.75</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode</td>
<td>Semi-annual</td>
<td>Quarterly</td>
<td>Once every 2 months</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deposit</td>
<td>$6</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Interest rate</td>
<td>$d^{(12)} = 6%$</td>
<td>$i^{(3)} = 8%$</td>
<td>$\delta = 7%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Getting this table right is half the battle. If you can set up a table like this, you are on the right track.

Next, we need to accumulate deposits year by year; the interest rates earned are different year by year.

Find FV @ $t=1$ of the 1st year deposits:

During Year 1, the monthly discounting factor is \(1 - \frac{d^{(12)}}{12}\); the monthly accumulating factor is just the reverse of the discount factor: \(1 - \frac{d^{(12)}}{12}^{-1}\).

In other words, if you have $1 at the beginning of a month, then this $1 will accumulate to \(1 - \frac{d^{(12)}}{12}^{-1}\) at the end of the month. You should learn this technique and quickly convert a nominal discount rate $d^{(n)}$ into an accumulating factor.
FV @ $t = 1$ of the 1st year deposits:

\[
6 \left[ 1 - \frac{d^{(12)}}{12} \right]^{-12} + 6 \left[ 1 - \frac{d^{(12)}}{12} \right]^{-6} = 6 \left( 1 - \frac{6\%}{12} \right)^{-12} + 6 \left( 1 - \frac{6\%}{12} \right)^{-6} = 6 \left( 0.995^{-12} + 0.995^{-6} \right) = 12.55517
\]

Next, we need to accumulate this amount through Year 2 and Year 3. The FV @ $t = 3$ of Year 1 deposits:

\[
12.55517 \left( 1 + \frac{i^{(3)}}{3} \right)^3 e^{\delta} = 12.55517 \left[ 1 + \frac{8\%}{3} \right]^3 e^{7\%} = 14.57175
\]

**Find FV @ $t = 2$ of the 2nd year deposits:**

Compounding period = 0.25 (quarterly)
# of compounding periods = 4
The quarterly interest rate $j$.

To find $j$, let's accumulate $1$ at the beginning of Year 2 to the end of Year 2. If we use the quarterly interest rate $j$, the accumulated value @ the end of Year 2 is $(1 + j)^4$. On the other hand, if we accumulate $1$ using $i^{(3)}$, the accumulate value is:

\[
\left[ 1 + \frac{i^{(3)}}{3} \right]^3 = \left[ 1 + \frac{8\%}{3} \right]^3
\]

\[
\Rightarrow (1 + j)^4 = \left[ 1 + \frac{8\%}{3} \right]^3, \quad j = \left[ 1 + \frac{8\%}{3} \right]^\frac{3}{4} - 1 = 1.993406\%
\]

\[
\Rightarrow \quad \text{FV @ } t = 2 \text{ of the 2nd year deposits: } 3\ddot{x}_{1.993406\%} = 12.610062
\]

Next, we accumulate this amount through Year 3
The FV @ \( t = 3 \) of Year 2 deposits:

\[
12.610062 e^{0.07} = 12.82162 e^{0.07} = 13.52439
\]

Find FV @ \( t = 3 \) of the 3rd year deposits:

Compounding period = two months
# of compounding periods = 6
Interest rate per period: \( e^{\left(\frac{0.02}{12}\right)} - 1 \approx e^{\left(\frac{0.02}{12}\right)} - 1 = 1.1734988\%
\)

FV @ \( t = 3 \) of the 3rd year deposits: \( 2 \bar{s}_{\frac{1.1734988\%}{2}} = 12.50262 \)
Finally, we sum up the FV @ \( t = 3 \) of each year’s deposits:

\[
14.57175 + 13.52439 + 12.50262 = 40.59876
\]

Remember that early on we set one unit of money=$100. So the account value @ the end of Year 3 is $4,059.88.

**Problem 3**

<table>
<thead>
<tr>
<th>Bond face amount</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term to maturity</td>
<td>30 years</td>
</tr>
<tr>
<td>Coupons</td>
<td>10% semiannual</td>
</tr>
<tr>
<td>Redemption</td>
<td>Par</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>12.36% annual effective</td>
</tr>
</tbody>
</table>

You are also given the following formula to calculate a bond’s Macaulay duration:

\[
D_{MAC} = \frac{\sum_{t=1}^{n} t CF(t) v^t}{\sum_{t=1}^{n} CF(t) v^t}
\]

In the above formula:

- \( CF(t) = \) the cash flow at time \( t \) (measured in # of years)
- \( v = \frac{1}{1+i} \), where \( i \) is the annual effective yield of the bond.
Calculate $D_{MAC}$ using a systematic approach.

**Solution**

First, we draw a cash flow diagram. Because the coupons are paid semiannually but the interest given is an annual rate, we’ll use the coupon payment frequency as the unit time.

**Unit time = 6 months**

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>59</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$5</td>
<td>$5</td>
<td>$5</td>
<td>...</td>
<td>$5</td>
<td>$5</td>
</tr>
</tbody>
</table>

The interest rate per unit time is:

$$i = \sqrt{1 + 12.36\%} - 1 = 6\% \quad \Rightarrow \quad v = \frac{1}{1+i} = \frac{1}{1.06}$$

Next, we expand, step by step, the formula $D_{MAC} = \frac{\sum t CF(t) v^t}{\sum CF(t) v^t}$.

$$\Rightarrow \quad D_{MAC} = \frac{\sum_{t=1}^{60} \left( \frac{1}{2} t \right) CF(t) v^t}{\sum_{t=1}^{60} CF(t) v^t} = \frac{1}{2} \frac{\sum_{t=1}^{60} t CF(t) v^t}{\sum_{t=1}^{60} CF(t) v^t}$$

We add a factor of $\frac{1}{2}$ in the right hand side because 6 months = $\frac{1}{2}$ year.

$$\sum_{t=1}^{60} t CF(t) v^t = 1(5)v + 2(5)v^2 + 3(5)v^3 + ... + 60(5)v^{60} + 60(100)v^{60}$$

$$= 5(Ia)_{60|} + 60(100)v^{60}$$

$$\sum_{t=1}^{60} CF(t) v^t = 5a_{60|} + 100v^{60}$$

$$\Rightarrow \quad D_{MAC} = \frac{1}{2} \left[ \frac{5(Ia)_{60|} + 60(100)v^{60}}{5a_{60|} + 100v^{60}} \right]$$

We’ll use BA II Plus/BA II Plus Professional to solve this problem.
To avoid transferring intermediate values back and forth between a calculator and the scrap paper, we’ll track down what value is stored in which memory.

\[ v = \frac{1}{1+i} = \frac{1}{1.06} = M0 \quad \text{(so we store } v \text{ in } M0) \]

\[(Ia)_{60}\!\!\!\!|_i = \frac{\ddot{a}_{60}}{6\%} - 60v^{60} \]

\[ \ddot{a}_{60} = \frac{1-v^n}{d} = \frac{1-v^{60}}{1-v} = \frac{1-(M0)^{60}}{1-M0} = M1 \]

\[ (Ia)_{60}\!\!\!\!|_i = \frac{\ddot{a}_{60}}{6\%} - 60v^{60} = \frac{M1-60(M0^{60})}{6\%} = M2 \]

\[ a_{60}\!\!\!\!|_i = \frac{1-v^n}{6\%} = \frac{1-(M0)^{60}}{6\%} = M3 \]

\[ D_{MAC} = \frac{1}{2} \left[ \frac{5(Ia)_{60}\!\!\!\!|_i + 60(100)v^{60}}{5a_{60}\!\!\!\!|_i + 100v^{60}} \right] = \frac{1}{2} \left[ \frac{5(M2) + 60(100)(M0)^{60}}{5(M3) + 100(M0)^{60}} \right] = M4 \]

After doing the calculation, we get \( D_{MAC} = 8.69472773 \)

**Problem 4**

<table>
<thead>
<tr>
<th>Bond face amount</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term to maturity</td>
<td>30 years</td>
</tr>
<tr>
<td>Coupons</td>
<td>10% semiannual</td>
</tr>
<tr>
<td>Redemption</td>
<td>Par</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>12.36% annual effective</td>
</tr>
</tbody>
</table>

Calculate the bond’s duration.

**Solution**

When taking the exam, we want to delegate calculations to our calculators as much as we can. The solution to Problem 2 is complex. We want to avoid this hard core calculation in the exam if we can.
Before taking Exam FM, we have thoroughly researched SOA approved calculators. We know that BA II Plus Professional has a bond worksheet, which can calculate the modified duration of a bond. So we’ll let BA II Plus Professional Bond Worksheet find the modified duration of the bond. Then we convert the modified duration into Macaulay duration.

Please note that Bond Worksheet in BA II Plus does NOT have functionality for directly calculating a bond’s modified duration.

To calculate the modified duration of the bond, we enter the following in Bond Worksheet.

Key strokes in BA II Plus Professional (duration functionality not available in BA II Plus Bond Worksheet):

<table>
<thead>
<tr>
<th>2nd Bond</th>
<th>This activates Bond Worksheet.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Enter SDT=1.0100</td>
<td>SDT = settlement date (i.e. purchase date of the bond)</td>
</tr>
<tr>
<td>This sets SDT=1-01-2000</td>
<td>We arbitrarily set the purchase date of the bond is to 1/1/2000. You can set the purchase date to another date such as 6/1/1988. However, 1/1/2000 is an easy number.</td>
</tr>
<tr>
<td>Enter CPN=10</td>
<td>Set coupon is 10% of par.</td>
</tr>
<tr>
<td>Enter RDT=1.0130</td>
<td>RDT = redemption date (or bond’s maturity)</td>
</tr>
<tr>
<td>This sets RDT=1-01-2030</td>
<td>We set RDT=1-01-2030 (the bond has a 30 year maturity).</td>
</tr>
<tr>
<td>Enter RV=100</td>
<td>Redemption value. Because the bond is redeemed at par and the par=100, we set RV=100.</td>
</tr>
<tr>
<td>Day counting method</td>
<td>Use 360 counting method (i.e. assume every year has 360 days and every month has 30 days). Don’t use the actual counting method.</td>
</tr>
<tr>
<td>Coupon frequency</td>
<td>2/Y (i.e. twice a year)</td>
</tr>
<tr>
<td>Enter YLD=2(6)=12</td>
<td>Set bond’s yield to maturity to 12%.</td>
</tr>
</tbody>
</table>

We need to be careful here. The yield to maturity in Bond Worksheet must be a nominal interest rate compounding at the same frequency as the coupons are paid. Because we set coupon frequency as semiannual (remember we set by 2/Y as coupon frequency), we need to enter a nominal yield.
compounding semiannually. So we double the 6% semiannual effective interest rate to get the nominal rate compounding semiannually. Don’t enter the 6-month effective interest rate of 6% as the yield to maturity. You need to remember this in order to use Bond Worksheet.

CPT PRI (compute price of the bond)

We get PRI=83.83857229. This is the PV of the bond cash flows discounted at 12.36% annual effective (or 6% per 6 months).

AI=0

Accumulated Interest (AI) is zero if bond is sold exactly at a coupon date.

When a bond is sold between two coupon payments, the bond buyer must pay the original bond holder a portion of the coupon. AI is calculated as follows:

\[ AI = \text{Coupon} \times \frac{\text{Days between settlement and last coupon payment}}{\text{Total days between two coupon payments}} \]

This is a minor function of Bond Worksheet. Don’t worry about this.

DUR

DUR=8.20257333

Remember this is the modified duration calculated using the Wall Street method. This is different from the textbook definition.

\[ D_{MOD}^{Wall Street} = 8.20257333 \]

Please note that the bond worksheet in BA II Plus/BA II Plus Professional uses the Wall Street convention in quoting a bond. In Wall Street, a bond’s yield to maturity is quoted as a nominal yield compounding as frequently as coupons are payable per year. In contrast, in Exam FM, the yield to maturity of a bond is almost always quoted as an annual effective interest rate. As a result, if you use Bond Worksheet to do any calculations, you’ll need to enter the nominal yield to maturity, not the effective yield. This is an annoying detail to remember. So I recommend that you use Bond Worksheet only to calculate the bond’s duration. Don’t use it to calculate the bond price; you can calculate the bond price using TVM Worksheet.
Please also note that BA II Plus Professional uses the Wall Street convention to calculate the bond’s modified duration:

\[
D_{\text{MOD}}^{\text{Wall Street}} = \frac{D_{\text{MAC}}}{1 + \frac{YLD^{(m)}}{m}}
\]

(using nominal yield)

In the above equation, \( m \) is the coupon frequency and \( YLD^{(m)} \) is the nominal yield to maturity compounding at the coupon frequency.

In contrast, the textbook by Broverman use the following method:

\[
D_{\text{MOD}}^{\text{Textbook}} = \frac{D_{\text{MAC}}}{1 + r}
\]

(where \( r \) is the annual effective yield)

Please note that \( r \) is really the opportunity cost of capital (i.e., the interest rate earned by investors if they invest in other securities). As a result, the yield to maturity is really the prevailing market interest rate; the investors can earn the prevailing market interest rate if they invest in other securities.

Finally, we’ll convert the Wall Street modified duration into Macaulay duration.

\[
D_{\text{MAC}} = D_{\text{MOD}} \left[ 1 + \frac{YLD^{(m)}}{m} \right] = 8.20257333 \left[ 1 + \frac{12\%}{2} \right] = 8.69472773
\]

If you don’t understand the Macaulay duration and the modified duration, that’s OK; I will explain them later. For now, you just need to know that you can calculate Macaulay duration using BA II Plus Professional Bond Worksheet.

Because BA II Plus does not have the modified duration functionality, you might want to buy a BA II Plus Professional calculator.

Please note that the modified duration function in BA II Plus Professional Bond Worksheet works only for a bond. A bond has a neat cash flow patterns. If an exam problem gives you a stream of random cash flows (such as $100 at \( t=1 \), $104 at \( t=2 \), $200 at \( t=3 \),...), then you can’t use Bond Worksheet to calculate the duration. Later in this book, I’ll explain how to calculate the duration of a stream of random cash flows.
Lesson learned from Problem 5

Calculators can save you time. However, you need to know how to properly use a calculator. In this problem, if you don’t know that you need enter a nominal yield, if you don’t know that the duration generated by Bond Worksheet is a Wall Street version of the modified, you’ll get a wrong result.

There is extra complexity in using BA II Plus Professional Bond Worksheet to calculate the duration of bond when a bond is not redeemed at par. Don’t worry about it now. We’ll pick up this topic later when we study, in more depths, the concept of duration and convexity.
Chapter 2  Getting started

To best use this study manual, please have the following items ready:

1. **Textbooks recommended by SOA.** Only one textbook of the four books is required:
   - *Mathematics of Investment and Credit*
   - *Mathematical interest theory*
   - *Theory of interest*
   - *Financial Mathematics – A Practical Guide for Actuaries and other Business Professionals*

2. **Have a BA II Plus or BA II Plus Professional calculator.** BA II Plus Professional is preferable because it has several new features (such as calculating modified duration) not found in BA II Plus

3. **Download Sample FM Questions, Sample questions for Derivatives Markets, May 2005, and November 2005 FM Exam, all from the SOA website.**

4. **Review online discussion forums about FM.** There are two major discussion forums: [www.actuarialoutpost.com](http://www.actuarialoutpost.com) and [http://www.actuary.com](http://www.actuary.com). Here are some discussion threads:


Chapter 3  FM Fundamental

Time value of money
youtube video:
http://www.youtube.com/watch?v=BXm5mZqMp6Y
http://www.youtube.com/watch?v=ks33lMoxst0
http://www.youtube.com/watch?v=4LSktB7Pk
http://www.youtube.com/watch?v=3SgVULEcOBU
http://www.youtube.com/watch?v=6WCfVjUTTEY

- The value of money depends on not only the amount, but also when we receive it.

- We all intuitively understand the time value of money. If someone owes us money, we want the money now; if we have to pay bills, we want to put them off as late as we can.

- $100 today is worthy more than $100 tomorrow. If we have $100 today, we can spend it for pleasure today. Alternatively, we can lend $100 out today and earn interest on it.

Principal

- The principal is the initial amount borrowed and yet to be repaid.

Interest rate
http://www.youtube.com/watch?v=GtaoP0skPWc
http://www.youtube.com/watch?v=t4zfiBw0hwM

- An interest rate is the rental price of money. When you borrow someone’s money, you pay a rent for using the money for a period of time.

- An interest rate is the rent paid by the borrower to the lender per unit of money per unit of time. An annual interest rate of 5% means that for every $1 borrowed for a year, the borrower pays a $0.05 annual rent to the lender.
Simple interest rate

Interest is paid only on the original amount borrowed (called principal) for the length of time the borrower has the use of the money.

You deposit $100 into a bank account and earn 5% simple interest per year. You are actually lending $100 to your bank. The bank pays an annual rent equal to 5% of the principal ($100).

At the end of Year 1, your account grows to \(100(1+5%) = 105\). $100 is your original principal and $5 is the interest.

At the end of Year 2, your accounts grows to \(100(1+5%+5%) = 110\). $100 is the original principal. You have earned $5 interest in Year 1 and $5 interest in Year 2. Notice that the $5 interest you have earned in Year 1 does not earn any additional interest in Year 2 (sitting idle in the bank for 1 year).

At the end of Year 3, your accounts grows to \(100(1+5%+5%+5%) = 115\). $100 is the original principal. You have earned $5 interest in Year 1, $5 interest in Year 2, and $5 interest in Year 3. Notice that the $5 interest you have earned in Year 1 doesn’t earn any interest in Year 2 or Year 3 (sitting idle for 2 years). The $5 interest you have earned during Year 2 does not earn any interest in Year 3 (sitting idle for 1 year).

At the end of Year \(n\), your account grows to \(100(1+n5%)\). $100 is your original principal. You have earned $5 interest per year for \(n\) years. All the interest you have earned year after year is sitting idle.

Compound interest rate

Both the principal and the accrued interest earn interest.

Interest is paid on the original principal plus all interest accrued to that point in time.
You deposit $100 into a bank account and earn 5% compound interest per year. This means that for each dollar in your account in the beginning of the year, you will earn $0.05 at the end of the year.

At the end of Year 1, your account grows to $100(1+5%)=$105. At the end of Year 2, your account grows to $100(1+5%)^2=110.25$. At the end of Year $n$, your account grows to $100(1+5%)^n$. At the end of each year, you earn 5% interest on your account balance at the beginning of the year.

### Comparison of simple interest rate and compound interest rate

<table>
<thead>
<tr>
<th>Time $T$</th>
<th>Simple interest rate 5%</th>
<th>Compound interest rate 5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>$t = 1$</td>
<td>$100(1+5%) = 105$</td>
<td>$100(1+5%) = 105$</td>
</tr>
<tr>
<td>$t = 2$</td>
<td>$100(1+5%+5%) = 110$</td>
<td>$105(1+5%) = 100(1+5%)^2 = 110.25$</td>
</tr>
<tr>
<td>$t = 3$</td>
<td>$100(1+5%+5%+5%) = 115$</td>
<td>$110.25(1+5%) = 100(1+5%)^3 = 115.7625$</td>
</tr>
<tr>
<td>$t = 4$</td>
<td>$100(1+5%+5%+5%+5%) = 120$</td>
<td>$115.7625(1+5%) = 100(1+5%)^4 = 121.550625$</td>
</tr>
<tr>
<td>$t = n$</td>
<td>$100(1+n5%)$</td>
<td>$100(1+5%)^n$</td>
</tr>
</tbody>
</table>

If $n=1$, simple interest rate and compounding interest rate give you the same amount of wealth; if $n>1$, you accumulate more money under a compounding interest rate; if $n<1$, a simple interest rate gives you more wealth.

To stay competitive in business, many banks offer a compound interest rate to their customers. To see why, imagine you deposit $100 on January 1, 2003 in Bank A, which offers you 5% simple interest rate. On December 31, 2003, your account grows to $105. If you keep your money in Bank A for another year and do nothing, your account will grow to $110 on December 31, 2004. But you are not happy with the simple interest rate system where you are forced to give up interest on the interest you've already earned. What can you do so you can earn interest on the interest?

Obviously, you could move your money to another bank which offers you a compound interest rate of 5%.

The second option is stay in Bank A but opens an additional account.
On January 1, 2004, you deposit $100 in Bank A, which offers you a 5% simple interest rate. On December 31, 2004, your account grows to $105. If you do nothing, your account will grow to $110 on December 31, 2005.

On December 31, 2004, you open another account at Bank A and you deposit $5 into this second account. As a result, you have two accounts at Bank A—one account has $100 and the other has $5. At the end of December 31, 2005, the total money you have in Bank A will be:

\[
\frac{100(1+5\%)}{1} + \frac{5(1+5\%)}{2} = 110.25
\]

By opening another account, you have created a compound interest rate in Bank A and beat the bank at its own game. Because of this, banks offer you the compound interest rate. If a bank doesn’t offer you a compound interest rate, you simply open more and more accounts each year (this is going to drive both you and the bank crazy).

**Force of interest**

This is a difficult concept for many. The idea, however, is really simple. Money doesn’t sleep; it works on the 7×24 basis. Consequently, money should generate interest continuously and instantly. The force of interest is an instantaneous interest rate (the interest rate you earn during an instant of time).

Another way to see how your money can earn interest instantly and continuously, imagine that the interest compounding period keeps shrinking. Now instead of compounding on the annual basis, your money compounds every second. As the compounding frequency approaches the infinity, your money indeed earns interest instantly.

At time t, your account has \( A(t) \) dollars. At time \( t + \Delta t \), your account has \( A(t + \Delta t) \) dollars. What’s the interest rate you’ve earned during the instant \( \Delta t \)?

\[
\delta(t) = \lim_{\Delta t \to 0} \frac{\text{interest earned during } \Delta t}{\text{your beginning account } \times \Delta t} = \lim_{\Delta t \to 0} \frac{A(t + \Delta t) - A(t)}{A(t) \Delta t}
\]

In the above equation, we divide interest earned by the beginning account \( A(t) \) and by the length of time \( \Delta t \). This is because an interest rate is $ earned per unit of time per unit of money lent out.
\[
\delta(t) = \lim_{\Delta t \to 0} \frac{A(t + \Delta t) - A(t)}{A(t) \Delta t} = \lim_{\Delta t \to 0} \frac{\frac{A(t + \Delta t) - A(t)}{\Delta t}}{A(t)} = \frac{1}{A(t)} \frac{dA(t)}{dt} = \frac{d}{dt} \ln A(t)
\]

By integrating the above equation, we have
\[
A(t) = A(0) \exp \left[ \int_0^t \delta(x) \, dx \right]
\]

**Example 1.**
Your $100 deposit grows under the following force of interest:
\[
\delta(t) = \begin{cases} 2\% & 0 \leq t \leq 5 \\ 5\% + 0.3\%t & t > 5 \end{cases}
\]

Calculate
1. Your account value after 10 years.
2. The equivalent simple interest you have earned during the 10 years.
3. The equivalent compound interest you have earned during the 10 years.

**Solution**
The account value after 10 years:
\[
A(10) = A(0) \exp \left[ \int_0^{10} \delta(t) \, dt \right] = 100 \exp \left[ \int_0^5 (2\%) \, dt + \int_5^{10} (5\% + 0.3\%t) \, dt \right]
\]
\[
= 100 \exp \left[ \left( (2\%) \cdot t \right)_0^5 + \left[ 5\%t + \frac{1}{2} 0.3\%t^2 \right]_5^{10} \right] = 100e^{46.25\%} = $158.8039
\]

Find the equivalent simple interest rate:
\[
A(10) = A(0) \left( 1 + 10i \text{ simple} \right) \Rightarrow 158.8039 = 100 \left( 1 + 10i \text{ simple} \right), \quad i \text{ simple} = 5.88\%
\]

Find the equivalent compound interest rate:
\[
A(10) = A(0) \left( 1 + i \text{ compound} \right)^{10} \Rightarrow 158.8039 = 100 \left( 1 + i \text{ compound} \right)^{10}, \quad i \text{ compound} = 4.73\%
\]
Example 2 (May 2004 SOA EA-1 #2)
On 1/1/2004, Smith purchases an annuity certain that has three semi-annual payments of $500 each, with the first payment to be made 7/1/2009. The force of interest at time t is given by:

\[
\delta(t) = \frac{1}{50 + 2t} \quad \text{where} \quad t \geq 0; \quad \text{t is measured in years from 1/1/2004}
\]

In what range is the present value of the annuity on 1/1/2004?
[A] Less than $1,350
[B] $1,350 but less than $1,355
[C] $1,355 but less than $1,360
[D] $1,360 but less than $1,365
[E] $1,365 or more

Solution
The correct answer is A

<table>
<thead>
<tr>
<th>Time</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2004</td>
<td>$500</td>
</tr>
<tr>
<td>7/1/2009</td>
<td>$500</td>
</tr>
<tr>
<td>1/1/2010</td>
<td>$500</td>
</tr>
<tr>
<td>7/1/2010</td>
<td>$500</td>
</tr>
</tbody>
</table>

Generally, for a given the force of the interest \( \delta(t) \), the present value of a cash flow \( X \) occurring at time \( T \) is

\[
PV = X \exp\left(-\int_0^T \delta(s) \, ds\right).
\]

Applying this general rule, we find that the present value of the annuity is:

\[
500 \left[ \exp\left(-\int_0^{5.5} \delta(t) \, dt\right) + \exp\left(-\int_0^6 \delta(t) \, dt\right) + \exp\left(-\int_0^{6.5} \delta(t) \, dt\right) \right]
\]
\[= 500 \left[ \exp \left( \int_{0}^{5} \frac{1}{50 + 2t} \, dt \right) + \exp \left( \int_{0}^{6} \frac{1}{50 + 2t} \, dt \right) + \exp \left( \int_{0}^{6.5} \frac{1}{50 + 2t} \, dt \right) \right] \]

The tricky part is doing the integration.

Let's find \( \int_{0}^{x} \frac{1}{50 + 2t} \, dt \). Set \( 50 + 2t = y, \) \( \Rightarrow \) \( t = \frac{y}{2} - 25 \) and \( dt = \frac{1}{2} \, dy \)

\[
\int_{0}^{x} \frac{1}{50 + 2t} \, dt = \int_{50}^{50 + 2x} \frac{1}{y} \, dy = \left[ \frac{1}{2} \ln y \right]_{50}^{50 + 2x} = \frac{1}{2} \ln \frac{50 + 2x}{50} = \frac{1}{2} \ln \left( 1 + \frac{x}{25} \right)
\]

\[\Rightarrow \exp \left( -\int_{0}^{x} \delta(t) \, dt \right) = \exp \left( \int_{0}^{x} \frac{1}{50 + 2t} \, dt \right) = \exp \left( -\frac{1}{2} \ln \left( 1 + \frac{x}{25} \right) \right)\]

\[= \exp \left[ \ln \left( 1 + \frac{x}{25} \right) \right]^{\frac{1}{2}} = \left( 1 + \frac{x}{25} \right)^{\frac{1}{2}}\]

The present value is:

\[500 \left[ \left( 1 + \frac{5.5}{25} \right)^{\frac{1}{2}} + \left( 1 + \frac{6}{25} \right)^{\frac{1}{2}} + \left( 1 + \frac{6.5}{25} \right)^{\frac{1}{2}} \right] = 1,347.14274\]

**Example 3 (SOA May 2005 EA-1 #25)**

Loan amount: $10,000

Payment Terms: Two payments:
- End of Year 1: \( X \)
- End of Year 2: \( 1.1X \)

Force of interest: \( 0.06 + 0.01t \), for \( t \leq 2 \)

In what range is \( X \) ?

(A) Less than $5,210
(B) $5,210 but less than $5,280
(C) $5,280 but less than $5,380
(D) $5,380 but less than $5,480
(E) 45,420 or more
Solution
Generally, for a given the force of the interest \( \delta(t) \), the present value of a cash flow \( X \) occurring at time \( T \) is \( PV = X \exp\left( -\int_0^T \delta(s) \, ds \right) \).

\[
\begin{array}{c|c|c}
\text{Time} & 0 & T \\
\hline
PV & X \exp\left( -\int_0^T \delta(s) \, ds \right) & \text{X}
\end{array}
\]

Then it follows:
\[
X \exp\left[ -\int_0^1 (0.06 + 0.01 t) \, dt \right] + 1.1X \exp\left[ -\int_0^2 (0.06 + 0.01 t) \, dt \right] = 10,000
\]
\[
X = \frac{10,000}{e^{-0.065} + 1.1e^{-0.14}} = 5,281.61
\]

So the answer is C.

Example 4
You are given the following force of interest:

\[
\delta(t) = \begin{cases} 
0.09 - 0.005t & \text{for } t \leq 6 \\
0.08 & \text{for } t > 6
\end{cases}
\]

(1) Calculate the accumulation at \( t = 10 \) of \$100\) invested at \( t = 0\)
(2) Calculate the present value at \( t = 0 \) of a continuous payment stream at the rate of \( e^{0.2t} \) from \( t = 10 \) to \( t = 15 \).

Solution
(1) The accumulation value is:

\[
100 \exp\left[ \int_0^{10} \delta(t) \, dt \right] = 100 \exp\left[ \int_0^6 (0.09 - 0.005t) \, dt + \int_6^{10} 0.08 \, dt \right]
\]
\[
= 100 \exp\left[ 0.09t - \frac{1}{2} \times 0.005t^2 \right]_0^6 + [0.08t]_6^{10} = 100 \exp(0.45 + 0.32) = 215.98
\]
(2) If we have $1 at time $t$ where $t>6$, then its PV at time zero is
\[ e^{-\int_6^t \delta(s)ds} = e^{-\int_6^t (0.09-0.005 s)ds + \int_0^6 0.08ds}. \]

If we have $e^{0.2t}$ at time $t>6$, then its PV at time zero is
\[ e^{0.2t} e^{-\int_6^t (0.09-0.005 s)ds + \int_0^6 0.08ds}. \]

If we have a continuous cash flow of $e^{0.2t}$ from $t=10$ to $t=15$, then the PV of the cash flow stream is
\[
\int_{10}^{15} e^{0.2t} e^{-\int_6^t (0.09-0.005 s)ds + \int_0^6 0.08ds} dt = \int_{10}^{15} e^{0.2t} e^{-0.08t+0.03} dt
\]
\[= e^{0.03} \int_{10}^{15} e^{0.12t} dt = e^{0.03} \left[\frac{e^{0.12t}}{0.12}\right]_{10}^{15} = e^{0.03} \left(\frac{e^{1.8} - e^{1.2}}{0.12}\right) = 23.44\]

**Example 5**
The force of interest $\delta(t) = a + bt^2$ where $a$ and $b$ are constants. An amount of $100 invested at $t=0$ accumulates to $135 at $t=6$ and $200 at $t=9$. Calculate $a$ and $b$.

**Solution**
\[135 = 100 e^{\int_0^6 (a + bt^2) dt} = 100 e^{\left[ at + \frac{1}{3} bt^3 \right]_0^6} = 100 e^{6a + 72b}\]
\[200 = 100 e^{\int_0^9 (a + bt^2) dt} = 100 e^{\left[ at + \frac{1}{3} bt^3 \right]_0^9} = 100 e^{9a + 243b}\]
\[\ln 1.35 = 6a + 72b, \quad \ln 2 = 9a + 243b\]
\[\Rightarrow a = 0.0284, \quad b = 0.0018\]

**Nominal interest rate**
http://www.youtube.com/watch?v=wzvpD5eaunk

Many banks calculate and pay interest on the quarterly, monthly, or even daily basis. When the interest is calculated more frequently than annually, the interest rate, however, is often still quoted on the annual basis.
For example, you deposit $100 in a bank with a nominal interest rate of 6% compounded monthly. Here the compounding frequently is monthly, but the interest rate is still quoted as an annual rate.

To calculate you account value, you'll need to convert the annual interest rate to the monthly interest rate. The monthly interest rate is 6%/12=0.5%. Your interest is 100(0.5%)=$0.5 at the end of Month 1.

At the end of Month 1, your bank account grows to $100.5.

At the end of Month 2, your bank account grows to 
\[
100(1+0.5\%)^2 = $101.0025.
\]

At the end of Month 3, your bank account grows to 
\[
100(1+0.5\%)^3 = $101.5075125.
\]

If the interest rate is quoted on the annual basis but is actually applied to shorter intervals (such as quarterly, monthly, daily), then it is called a nominal interest rate. A nominal interest rate is often expressed as \(i^{(p)}\), meaning that interest rate is compounded \(p\)-thly.

If you deposit $100 into a bank account with a nominal interest rate \(i^{(2)} = 6\%\), then the interest you have earned is calculated twice a year, using a 6 month interest rate of 6%/2=3%. Your bank account at the end of Month 6 is $100(1+3\%) = $103; your bank account at the end of Month 12 is $100(1+3\%)^2 = $106.09.

If you deposit $100 into a bank account with a nominal interest rate \(i^{(4)} = 6\%\), then the interest you have earned is calculated four times a year, using a quarterly interest rate of 6%/4=1.5%. Your bank account at the end of Month 3 is $100(1+1.5\%) = $101.5; at the end of Month 6 it is $100(1+1.5\%)^2 = $103.0225; at the end of Month 9 it is $100(1+1.5\%)^3; at the end of Month 12 it is $100(1+1.5\%)^4.

**APR**

APR stands for annual percentage rate. APR calculation considers the fact that a borrower often pays various fees in getting a loan and pays a higher interest rate than the stated interest rate.
For example, you borrow $150,000 from a bank to finance your new home. The interest rate is nominal 5% compounded monthly. You plan to pay off your mortgage in 20 years.

The followings are the fees your bank charges you when you get your mortgage:

<table>
<thead>
<tr>
<th>Stated Principal</th>
<th>$150,000</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fees charged:</strong></td>
<td></td>
</tr>
<tr>
<td>Loan origination fees (1% of the principal)</td>
<td>1%(150,000)=$1,500</td>
</tr>
<tr>
<td>Underwriting fees</td>
<td>$400</td>
</tr>
<tr>
<td>Credit report</td>
<td>$50</td>
</tr>
<tr>
<td>Processing, documentation, and miscellaneous fees</td>
<td>$500</td>
</tr>
<tr>
<td><strong>Total fees</strong></td>
<td>1,500+400+50+500 = $2,450</td>
</tr>
<tr>
<td><strong>Net amount borrowed</strong></td>
<td>150,000-2,450= $147,550</td>
</tr>
</tbody>
</table>

To calculate the APR, we need to first find the monthly payment using the stated principal of $150,000. In other words, when you sign the mortgage contract with your bank, your bank calculates your monthly payment using the stated principal of $150,000. So the question is this: You borrow $150,000 at a nominal interest rate of 5%; you are paying off the loan in 20 years. How much do you need to pay per month (assuming that you pay your monthly mortgage at the end of each month)?

Right now, you probably haven’t learned how to calculate the monthly payment. That’s OK. Let me give you the answer -- $989.93. I calculated this number using BA II Plus TVM.

Next, you ask the question, “I’m paying $989.93 per month for 20 years. However, the net amount I borrowed is really $147,550. What’s the monthly interest rate am I paying?” We do some math and find the monthly interest rate is 0.433%. Then, the nominal interest rate compounded monthly is 0.433%(12)= 5.20%. The APR is 5.20%. The stated interest rate is 5%; the APR is 5.2%. The APR is higher than the stated interest rate.

**Annual effective interest rate**

If you deposit $100 into a bank account with a nominal interest rate $i^{(2)} = 6\%$, then the interest you earned is calculated twice a year, using a 6 month interest rate of $6\%/2=3\%$. Your bank account at the end of Month 12 is $100(1+3\%)^2 = $106.09. As a result, the actual interest rate on
the annual basis is \((1+3\%)^2-1=6.09\%\). 6.09% is the effective annual interest rate.

If the nominal interest rate is \(i^{(p)}\), then the annual effective rate is

\[
\left[1 + \frac{i^{(p)}}{p}\right]^{p} - 1.
\]

**Continuous compounding**
You deposit $1 into a bank account. The interest is calculated instantly at a nominal rate of \(i^{(+\infty)} = \delta\). At the end of the year, $1 will become

\[
\left[1 + \frac{i^{(+\infty)}}{+\infty}\right]^{+\infty} = \lim_{p \to +\infty}\left(1 + \frac{\delta}{p}\right)^{p} = e^{\delta}
\]

At the end of \(t\) years (\(t\) can be fractional), $1 will become

\[
\lim_{p \to +\infty}\left\{\left[1 + \frac{\delta}{p}\right]^{p}\right\}^{t} = \lim_{p \to +\infty}\left\{1 + \frac{\delta}{p}\right\}^{p/\delta} = e^{\delta t}
\]

**Example.**
The annual effective interest rate is 10%. What’s the continuously compounded annual interest rate?

**Solution**
If we invest $1 at time zero, then we’ll have $1.1 at time one.

\[e^{\delta t} = 1.1 \text{ where } t=1 \quad \Rightarrow \quad \delta = \ln(1+i) = \ln1.1 = 9.531\%
\]

**Effective annual rate of discount**

- Interest payable in advance for borrowing $1 for a unit time

**Example 1.** 5% effective annual rate of discount means that for every dollar you borrow for one year, you need to pay $0.05 interest in advance (i.e. at the beginning of the year). Effective annual rate of discount is just another way to calculate the interest a borrower needs to pay to the lender.
**Example 2.** If you borrow $100 from a bank one year. At the end of Year 1, you need to return the principal to the bank. The bank charges you a 5% annual effective rate of discount.

Identify your cash flows and the bank's cash flows. Calculate the actual interest rate the bank actually charges you.

**Solution**

<table>
<thead>
<tr>
<th>Time t (year)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Your cash flow</td>
<td>$100 - 100(5%) = $95</td>
<td>-$100</td>
</tr>
<tr>
<td>Bank's cash flow</td>
<td>-100 + 100(5%) = -$95</td>
<td>$100</td>
</tr>
</tbody>
</table>

At the time of borrowing \((t = 0)\), the bank gives $100. Simultaneously, the bank charges you \(100(5\%) = $5\) interest. So you really get $95 net at \(t = 0\). At \(t = 1\), you pay back $100 to the bank.

To calculate the interest rate your loan is charged, note that you actually borrow $95 at \(t = 0\) and pays back $100 at \(t = 1\). So the borrowing rate is

\[
\frac{100 - 95}{95} = 5.263\%
\]

We can also calculate the effective interest rate from the bank's point of view. The bank lends you $95 at \(t = 0\) but gets $100 at \(t = 1\). So the bank's earning rate is:

\[
\frac{100 - 95}{95} = 5.263\%
\]

**Example 3.** If you borrow $1 from a bank for one year. At the end of Year 1, you return the principal to the bank. The annual effective rate of discount is \(d\). Identify your cash flows. Convert the effective annual discount rate to the effective annual interest rate.

**Solution**

<table>
<thead>
<tr>
<th>Time t (year)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Your cash flow</td>
<td>$(1-d)</td>
<td>-$1</td>
</tr>
<tr>
<td>Bank's cash flow</td>
<td>-$(1-d)</td>
<td>$1</td>
</tr>
</tbody>
</table>
The bank invests \((1-d)\) at \(t=0\) but gets back $1 at \(t=1\). So the equivalent annual effective interest rate is:

\[
i = \frac{\text{ending fund value} - \text{beginning fund value}}{\text{beginning fund value}} = \frac{1-(1-d)}{1-d} = \frac{1}{1-d} - 1
\]

\[
\Rightarrow 1+i = \frac{1}{1-d}, \quad 1-d = \frac{1}{1+i}
\]

If we set \(v = \frac{1}{1+i}\), then \(1-d = v\)

**Example 4.** Derive the following equation and explain the meaning of the equation.

\[
d = \frac{i}{1+i}
\]

**Solution**

\[
d = 1 - \frac{1}{1+i} = \frac{i}{1+i} = iv
\]

The above equation says that if we borrow $1 for one unit of time, then the advance interest \(d\) paid at the time of the borrowing (i.e. \(t=0\)) is simply the present value of the interest \(i\) due at \(t=1\). This makes intuitive sense.

**Example 5.** Derive the following equation and explain the meaning of the equation.

\[
i = \frac{d}{1-d} = d(1-d)^{-1}
\]

**Solution**

\[
1+i = \frac{1}{1-d} \quad \Rightarrow \quad i = \frac{1}{1-d} - 1 = \frac{d}{1-d} = d(1-d)^{-1}
\]

To understand the meaning of \(i = d(1-d)^{-1}\), please note that \((1-d)^{-1}\) is the accumulating factor for a unit time. If we invest $1 at \(t=0\), then this $1 will grow into \((1-d)^{-1}\) at \(t=1\).
$i = d (1-d)^{-1}$ means that if we borrow $1$ at $t=0$, then the interest due at $t=1$ is simply the accumulated value of the advance interest $d$ paid at $t=0$. This makes lot of sense.

**Simple annual rate of discount**

Just as interest rate can be a simple rate or an annual effective rate, a discount rate can be a simple rate or an annual effective rate. It’s hard to think intuitively what a simple discount rate really means. So don’t worry about interpreting the simple discount rate. Just need to remember the following key formula:

If a simple discount rate is $d$, then $1$ at time $t$ is worth $1-dt$ at $t=0$; $1$ at time $t=0$ is worth $(1-dt)^{-1}$ at time $t$.

**Nominal annual rate of discount**

This is similar to nominal annual rate of interest rate. The discount rate is quoted on the annual basis but is actually applied to shorter periods (such as monthly or quarterly).

The formula is: 

$$1-d = \left(1-\frac{d^{(m)}}{m}\right)^m$$
**Relationship between** \( i, d, \delta, i^{(m)}, d^{(m)} \)

Example. If \( d^{(2)} = 10\% \). Calculate \( d, i, \delta, i^{(12)} \).

**Strategy:** Don’t just memorize a host of formulas. Understand the meaning of each symbol. Understand how money travels over time. You should do fine.

Let’s look at each symbol.

First, \( i \). This symbol means the effective interest rate per unit of time (typically per year). You are most familiar with this symbol. If \( i = 10\% \), then if you borrow $1 at \( t = 0 \), you’ll need to pay \( 10\% \times 1 = 0.1 \) at the end of the year.

Next, memorize the following diagram:

<table>
<thead>
<tr>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \quad \text{←} \quad (1+i) \quad \text{→} \quad 1 ||</td>
<td></td>
</tr>
<tr>
<td>$v = \frac{1}{1+i} \quad \text{←} \quad 1 \quad \text{→} \quad \frac{1}{1+i} |</td>
<td></td>
</tr>
</tbody>
</table>

$1$ at \( t = 0 \) is worth $(1+i)$ at \( t = 1 \). $1$ at \( t = 1 \) is worth \( v = \frac{1}{1+i} \) at \( t = 0 \).

\[ \text{Wealth Ratio} = \frac{\text{Wealth at } t = 1}{\text{Wealth at } t = 0} = 1 + i \]
\( d \) is the interest paid in advance for every $1 you borrow. If you borrow $1 at time zero. Then immediately you are charged the interest \( d \) for each dollar you borrow. If \( d = 0.09 \) and you borrow $1 at time zero, the lender immediately takes away \( 0.09 \times 1 = 0.09 \). You walk away with $0.91 in your pocket. Generally, if you borrow $1 at \( t = 0 \), you walk away with \((1-d)\) in your pocket; you need to pay $1 at \( t = 1 \) to pay off your loan.

Next, memorize the following diagram:

\[
\begin{array}{c}
\text{Wealth at } t = 0 \quad \text{Wealth at } t = 1 \\
(1-d) \quad \text{at } t = 0 \quad \text{at } t = 1 \\
1 \quad \frac{1}{1-d} \\
\end{array}
\]

\( (1-d) \) at \( t = 0 \) is worth $1 at \( t = 1 \). \( \frac{1}{1-d} \) at \( t = 1 \) is worth $1 at \( t = 0 \).

\[
\text{Wealth Ratio} = \frac{\text{Wealth at } t = 1}{\text{Wealth at } t = 0} = \frac{1}{1-d}
\]
\( \delta \) is the force of the interest, the interest rate you earn during a tiny interval such as one second. If you deposit $1 into a bank account that earns a constant force of interest \( \delta \), then for a tiny interval of time \([t, t + dt]\), you’ll earn \( \delta dt \) interest. If you deposit $1 at \( t = 0 \), then this $1 will grow into \( e^{\delta} = e^\delta \) at \( t = 1 \) (here we assume a constant \( \delta \)).

Next, memorize the following diagram:

\[
\begin{align*}
\text{Wealth at } t=0 & \quad \text{Wealth at } t=1 \\
$1 & \quad \int_0^1 e^{\delta s} = e^\delta \\
$e^{-\delta} & \quad $1
\end{align*}
\]

$1 at \( t = 0 \) is worth \( $e^\delta \) at \( t = 1 \). $1 at \( t = 1 \) is worth \( $e^{-\delta} \) at \( t = 0 \).

\[ \text{Wealth Ratio} = \frac{\text{Wealth at } t=1}{\text{Wealth at } t=0} = e^\delta \]
i(2) means that instead of paying the interest once a year at t = 1, you are paying the interest twice a year, 1st time at t = 0.5 (end of 6-th month) and the 2nd time at t = 1 (end of 12th month). Your effective interest rate per 6 month is \( \frac{i(2)}{2} \). Similarly, \( i^{(12)} \) means that instead of paying the interest once a year at t = 1, you are paying the interest monthly at \( t = \frac{1}{12} \) (end of Month 1), \( \frac{2}{12} \) (end of Month 1), …, \( \frac{11}{12} \) (end of Month 11), and \( \frac{12}{12} \) (end of Month 12). Your effective interest rate per month is \( \frac{i^{(12)}}{12} \). \( i^{(m)} \) means that interest is paid \( m \)-thly per year at \( t = \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, \frac{m}{m} \).

Next, memorize the following diagram:

\[
\begin{align*}
&\text{\$1} \rightarrow \left[1 + \frac{i^{(m)}}{m}\right] \rightarrow \left[1 + \frac{i^{(m)}}{m}\right]^2 \rightarrow \left[1 + \frac{i^{(m)}}{m}\right]^{m-1} \rightarrow \left[1 + \frac{i^{(m)}}{m}\right]^m \\
&\quad \rightarrow \left[1 + \frac{i^{(m)}}{m}\right]^{-(m-1)} \rightarrow \left[1 + \frac{i^{(m)}}{m}\right]^{-(m-2)} \rightarrow \left[1 + \frac{i^{(m)}}{m}\right]^{-1} \rightarrow \text{\$1}
\end{align*}
\]

\$1 at t = 0 is worth \( \left[1 + \frac{i^{(m)}}{m}\right] \) at \( t = \frac{1}{m} \), \( \left[1 + \frac{i^{(m)}}{m}\right]^2 \) at \( t = \frac{2}{m} \), …, and \( \left[1 + \frac{i^{(m)}}{m}\right]^m \) at \( t = 1 \).

\$1 at t = 1 is worth \( \left[1 + \frac{i^{(m)}}{m}\right]^{-(m-1)} \) at \( t = 0 \).

\[
\text{Wealth Ratio} = \frac{\text{Wealth at } t = 1}{\text{Wealth at } t = 0} = \left[1 + \frac{i^{(m)}}{m}\right]^m
\]
Imagine that $1 at \ t = 0 \text{ travels through the time going through } m \text{ consecutive doors.}

When going through each door, $1 expands to \ \left[ 1 + \frac{i(m)}{m} \right]. \text{ After going through } m \text{ doors, it}

lands at \ t = 1 \text{ becoming } \left[ 1 + \frac{i(m)}{m} \right]^m.

Finally, let’s look at the most difficult symbol \ d^{(m)}. \text{ You walk into a bank to borrow } $1. \text{ The bank charges you, } d^{(2)}, \text{ a nominal discount rate convertible semiannually. } d^{(2)} \text{ means that cash is deducted twice from your declining principal before it is lent to you.}

Imagine a loan officer puts $1 worth of coins into a jar. This $1 is the amount you want to borrow. You really want to take away this $1 and go home, but the loan officer says, “Wait a minute. Let me deduct your interest payment twice. Then you can take away the rest.” The 1\text{st} time, the loan officer takes out \ \frac{d^{(2)}}{2} \text{ portion of coins out of the jar. Because the initial principal is just $1, } \frac{d^{(2)}}{2} \text{ portion of $1 is } \frac{d^{(2)}}{2} \times 1 = \frac{d^{(2)}}{2}. \text{ This is your 1\text{st} interest payment in advance. After this payment, your principal shrinks from $1 to } 1 - \frac{d^{(2)}}{2}. \text{ So now the jar has only } 1 - \frac{d^{(2)}}{2} \text{ worth of coins. You really want to take } 1 - \frac{d^{(2)}}{2} \text{ and go home, the loan officer says, “Wait a minute. I need to deduct your interest payment the 2\text{nd} time.” So he takes out } \frac{d^{(2)}}{2} \text{ portion what’s left in the jar. So } \frac{d^{(2)}}{2} \left[ 1 - \frac{d^{(2)}}{2} \right] \text{ worth of coins is taken out from the jar, leaving only } \left[ 1 - \frac{d^{(2)}}{2} \right] - \frac{d^{(2)}}{2} \left[ 1 - \frac{d^{(2)}}{2} \right] = \left[ 1 - \frac{d^{(2)}}{2} \right]^2 \text{ in the jar. Your 2\text{nd} interest payment in advance is } \frac{d^{(2)}}{2} \left[ 1 - \frac{d^{(2)}}{2} \right]. \text{ Now your principle shrinks from } 1 - \frac{d^{(2)}}{2} \text{ to } \left[ 1 - \frac{d^{(2)}}{2} \right]^2.

Your total interest payment in advance is:

\[ \frac{d^{(2)}}{2} + \frac{d^{(2)}}{2} \left[ 1 - \frac{d^{(2)}}{2} \right] = d^{(2)} - \frac{d^{(2)}}{4} = 1 - \left[ 1 - \frac{d^{(2)}}{2} \right]^2 \]
The result should make intuitive sense. Originally, the jar has $1 worth of coins. After two deductions, the jar has only $1 − \left(1 - \frac{d^{(2)}}{2}\right)^2$ worth of coins left. So the total coins deducted (i.e. your total interest payment in advance) is simply $1 - \left(1 - \frac{d^{(2)}}{2}\right)^2$. So you walk away with $\left(1 - \frac{d^{(2)}}{2}\right)^2$ in your pocket.

At $t = 1$, you pay $1 back to the bank, paying off the loan. Now you owe your bank nothing. Let’s draw a diagram to describe your loan:

\[
\begin{align*}
\text{at } t = 0 & \quad \text{is worth } $1 \text{ at } t = 1. \\
\left[1 - \frac{d^{(2)}}{2}\right]^2 & \quad \text{at } t = 0 \quad \text{is equivalent to } \$1 \quad \text{at } t = 1.
\end{align*}
\]

\[
\begin{align*}
\left[1 - \frac{d^{(2)}}{2}\right]^2 & \quad \text{at } t = 0 \quad \text{is equivalent to } \$1 \quad \text{at } t = 1. \text{ Then it follows that } $1 \quad \text{at } t = 0 \quad \text{is equivalent to } \left[1 - \frac{d^{(2)}}{2}\right]^2 \quad \text{at } t = 1. \text{ Now we can change the diagram into:}
\end{align*}
\]
Cash flow diagram under $d^{(2)}$

$\frac{1}{1 - \left(1 - \frac{d^{(2)}}{2}\right)^2}$ at $t = 0$ is worth $1$ at $t = 1$.

$\frac{1}{1 - \left(1 - \frac{d^{(2)}}{2}\right)^2}$ at $t = 1$ is worth $1$ at $t = 0$.

\[
\text{Wealth Ratio} = \frac{\text{Wealth at } t = 1}{\text{Wealth at } t = 0} = \frac{1}{1 - \left(1 - \frac{d^{(2)}}{2}\right)^2}
\]

We can easily extend the above reasoning to $d^{(m)}$. If you want to borrow $1$ from a bank under a nominal discount rate $d^{(m)}$, then your loan officer immediately deducts interest payments in advance $m$ times from your initial principal of $1$. In the 1st deduction, $\frac{d^{(m)}}{m}$ portion of your initial $1$ is deducted, leaving $1 - \frac{d^{(m)}}{m}$ left. In the 2nd deduction, $\frac{d^{(m)}}{m}$ portion of the remaining $1 - \frac{d^{(m)}}{m}$ is deducted, leaving $\left[1 - \frac{d^{(m)}}{m}\right]^2$ left. After $m$
deductions, only \( \left[ 1 - \frac{d^{(m)}}{m} \right]^m \) is left. You walk away with \( \left[ 1 - \frac{d^{(m)}}{m} \right]^m \) in your pocket.

Then one year later at \( t = 1 \), you pay $1 back to the bank. After this payment, you owe the bank nothing.

Cash flow diagram under \( d^{(m)} \):

\[
\begin{array}{c}
\text{\$1} & \text{\( \left[ 1 - \frac{d^{(m)}}{m} \right]^m \)} & \text{\( \left[ 1 - \frac{d^{(m)}}{m} \right]^{-m} \)} & \text{\$1} \\
\end{array}
\]

\( \left[ 1 - \frac{d^{(m)}}{m} \right]^m \) at \( t = 0 \) is worth \$1 at \( t = 1 \).

\( \left[ 1 - \frac{d^{(m)}}{m} \right]^{-m} \) at \( t = 1 \) is worth \$1 at \( t = 0 \).

\[
\text{Wealth Ratio} = \frac{\text{Wealth at } t = 1}{\text{Wealth at } t = 0} = \left[ 1 - \frac{d^{(m)}}{m} \right]^{-m}
\]
Now, we are ready to derive the relationship between $i, d, \delta, i^{(m)}, d^{(m)}$. You should memorize the following diagram:

![Diagram showing the relationship between $i, d, \delta, i^{(m)}, d^{(m)}$.](image)

Immediately, we see that

$$1 + i = \frac{1}{1 - d} = e^\delta = \left[ 1 + \frac{i^{(m)}}{m} \right]^m = \left[ 1 - \frac{d^{(m)}}{m} \right]^{-m}$$

From here you can derive all sorts of formulas. For example,

$$\frac{1}{1 + i} = 1 - d, \quad \Rightarrow \quad d = 1 - \frac{1}{1 + i} = 1 - v$$

$$1 + i = e^\delta, \quad \Rightarrow \quad \delta = \ln(1 + i)$$

$$1 + i = \left[ 1 + \frac{i^{(m)}}{m} \right]^m, \quad \Rightarrow \quad i = \left[ 1 + \frac{i^{(m)}}{m} \right]^m - 1$$

$$1 + i = \left[ 1 - \frac{d^{(m)}}{m} \right]^{-m}, \quad \Rightarrow \quad i = \left[ 1 - \frac{d^{(m)}}{m} \right]^{-m} - 1$$

$$\frac{1}{1 - d} = \left[ 1 - \frac{d^{(m)}}{m} \right]^{-m}, \quad \Rightarrow \quad 1 - d = \left[ 1 - \frac{d^{(m)}}{m} \right]^m$$
You can also derive the relationships using the following diagram:

\[
\begin{align*}
\frac{1}{1+i} &= v \\
1 - d &= e^{-\delta} \\
\left[1 + \frac{i^{(m)}}{m}\right]^{-m} &= 1 \\
\left[1 - \frac{d^{(m)}}{m}\right]^m &= 1
\end{align*}
\]

Now you should clearly see that:

\[
\frac{1}{1+i} = v = 1 - d = e^{-\delta} = \left[1 + \frac{i^{(m)}}{m}\right]^{-m} = \left[1 - \frac{d^{(m)}}{m}\right]^m
\]

From here, you can derive all sorts of relationship formulas.

Yet, there’s the third approach – to use the wealth ratio. The wealth ratio should the same under different measurements of interest. Hence:

\[
Wealth\ Ratio = \frac{\text{Wealth at } t = 1}{\text{Wealth at } t = 0} = 1 + i = \frac{1}{1 - d} = e^\delta = \left[1 + \frac{i^{(m)}}{m}\right]^{-m} = \left[1 - \frac{d^{(m)}}{m}\right]^{-m}
\]
Finally, let’s solve the problem: If $d^{(2)} = 10\%$. Calculate $d, i, \delta, i^{(12)}$.

\[
1 - d = \left(1 - \frac{d^{(2)}}{2}\right)^2 = \left(1 - \frac{10\%}{2}\right)^2, \quad d = 9.75\%
\]

\[
\frac{1}{1+i} = 1 - d = 1 - 9.75\%, \quad i = 10.8\% \quad e^\delta = 1 + i, \quad \delta = \ln(1 + i) = \ln 1.108 = 0.10259
\]

\[
\left[1 + \frac{i^{(12)}}{12}\right]^{12} = 1 + i = 1.108, \quad i^{(12)} = 10.3\%
\]

Example 2. You invest $150 at time zero in an account that earns the following interest:

During Year 1, $i^{(4)} = 10\%$.
During Year 2, $d^{(12)} = 10\%$.
During Year 3, $\delta = 10\%$
During Year 4, $\delta(t) = 2 + 0.1t$

Calculate your account value at the end of Year 4.

**Solution**

Let $A(t)$ represent your account value at time $t$.

\[
A(0) = 100 \quad A(1) = A(0)\left(1 + \frac{10\%}{4}\right)^4\quad A(2) = A(1)\left(1 - \frac{10\%}{12}\right)^{12}
\]

\[
A(3) = A(2)e^{10\%} \quad A(4) = A(3)\exp\left(\int_3^4 (2 + 0.1t)\,dt\right)
\]

\[
\int_3^4 (2 + 0.1t)\,dt = \left(2t + 0.05t^2\right)_3^4 = 2(4 - 3) + 0.05(4^2 - 3^2) = 2.35 \quad A(4) = A(3)e^{2.35}
\]

\[
A(4) = 100\left(1 + \frac{10\%}{4}\right)^4\left(1 - \frac{10\%}{12}\right)^{12}e^{10\%}\exp\left(\int_3^4 (2 + 0.1t)\,dt\right)
\]

\[
= 100\left(1 + \frac{10\%}{4}\right)^4\left(1 - \frac{10\%}{12}\right)^{12}e^{10\%}e^{2.35} = 1,414.26
\]
**Future value**

Future value is tomorrow’s value of today’s deposit. You deposit $100 into a bank account and your money grows at a compound interest rate of 6%. Then at the end of Year 1, your original deposit of $100 will grow into $100(1+6%)=$106. $106 is the future (1 year from now) value of your original $100 deposit.

Similarly, at the end of Year 2, your original deposit of $100 will become $100(1+6%)^2=$112.36. So the future (2 years from now) value of your original deposit is $112.36.

Generally, if you deposit $A(0)$ at $t=0$ and your deposit grows at a compound interest rate of $i$, then the future value of your original deposit $t$ years from now is $A(t)=A(0)(1+i)^t$.

**Present value**

Present value is today’s value of tomorrow’s money. It is the deposit you must make today in order to receive some money in the future. For example, one year from now you will receive $100. Assume the interest rate is 6%. How much money do you need to deposit into a bank account in order for you to receive $100 one year from now?

Let $A(0)$ represent the amount of money you must deposit into a bank account now. Then

$$A(0)(1+6%) = $100 \quad \Rightarrow \quad A(0) = \frac{$100}{1+6\%} \approx $94.34$$

So to receive $100 one year from now, you must deposit $94.34 today and have it grow at 6%. $94.34 is the present value of receiving $100 one year from now.

You are receiving $A(t)$ amount of money in $t$ years. The interest rate is $i$. What’s the present value of $A(t)$ amount of money $t$ years from now?

$$A(t)=A(0)(1+i)^t \quad \Rightarrow \quad A(0) = \frac{A(t)}{(1+i)^t} = A(t)(1+i)^{-t}$$
So the present value of receiving $A(t)$ amount of money in $t$ years is $A(t)(1+i)^{-t}$.

**Example 1**
Calculate the present value of $100 at $t = 10$ at the following rates:
1. a rate of interest rate of 6% per year convertible monthly
2. a rate of discount of 6% convertible monthly
3. a force of interest rate of 6%

**Solution**

1. $100 \left(1 + \frac{6\%}{12}\right)^{-12\times10} = 54.96$
2. $100 \left(1 - \frac{6\%}{12}\right)^{12\times10} = 54.80$

3. $100e^{-10\delta} = 100e^{-10\times6\%} = 54.88$

**Example 2 (SOA May 2000 EA-1 #2)**
A loan of $1,800 is to be repaid by a single payment of $2,420.8 two years after the date of the loan. The terms of the loan are quoted using a nominal annual interest rate of 15%.

What’s the frequency of compounding?
(A) monthly
(B) every two months
(C) quarterly
(D) semiannually
(E) annually

**Solution**
First, let’s calculate the effective annual interest rate.

$1,800(1+i)^2 = 2,420.8 \Rightarrow i = 15.9693\%$

We are given that the nominal interest rate is 15%. One way to find the frequency of compounding is to test different frequencies.

**The general formula:**

Given $i^{(p)}$, the nominal interest rate compounding $p$-thly a year, the equivalent annual effective rate is: $i = \left[1 + \frac{i^{(p)}}{p}\right]^p - 1$
First, let try an interest rate compounding monthly. The equivalent annual effective rate of 15% compounding monthly is \( p = 12 \):

\[
\left(1 + \frac{15\%}{12}\right)^{12} - 1 = 16.075\% \neq 15.9693\% 
\]

So the monthly compounding is not good.

Next, let's try an interest rate compounding every two months. The equivalent annual effective rate of 15% compounding every two months is \( p = 6 \):

\[
\left(1 + \frac{15\%}{6}\right)^{6} - 1 = 15.9693 \approx 15.9693\% 
\]

This is good. So the answer is B.

**Convert interest rate to discount rate or vice versa**

**Problem 1 (EA-1 #1 2001)**

Selected values:

\[
\begin{align*}
1,000d^{(m)} &= 85.256 \\
1,000d^{(2m)} &= 85.715
\end{align*}
\]

Calculate \( 1,000d^{(3m)} \)

**Solution**

Instead of memorizing complex formulas, let's use the “discount $1” method. If we have $1 at time one, what’s its value at time zero?

**Discounting Method 1**

\[
\text{discount } \frac{1}{m} \text{ thly from time zero to time 1.}
\]

The total # of discounting periods is \( m \). Per period discount factor is

\[
1 - \frac{d^{(m)}}{m}.
\]

\[
PV = \left[1 - \frac{d^{(m)}}{m}\right]^{m} = \left[1 - \frac{85.256}{1,000}\right]^{m} = \left[1 - \frac{0.085256}{m}\right]^{m}
\]
Discounting Method 2  
Discount $\frac{1}{2m}$thly from time zero to time $1$. The total # of periods is $2m$. Per period discount factor is $\left[1 - \frac{d^{(2m)}}{2m}\right]^2$.

$$PV = \left[1 - \frac{d^{(2m)}}{2m}\right]^{2m} = \left[1 - \frac{85.715}{1,000}\right]^{2m} = \left[1 - \frac{0.085715}{2m}\right]^{2m}$$

The PV should be the same:

$$\Rightarrow \left[1 - \frac{0.085256}{m}\right]^m = \left[1 - \frac{0.085715}{2m}\right]^{2m} = \left[1 - \frac{0.085256}{m}\right] = \left[1 - \frac{0.085715}{2m}\right]^2$$

Let $x = \frac{1}{m}$

$$\Rightarrow 1 - 0.085256x = (1 - 0.0428575x)^2 = 0.0428575^2x^2 - 2(0.0428575)x + 1$$

$$\Rightarrow -0.085256x = 0.0428575^2x^2 - 2(0.0428575)x$$

$$\Rightarrow -0.085256 = 0.0428575^2x - 2(0.0428575)x$$

$$\Rightarrow x = 0.25 \Rightarrow m = \frac{1}{x} = 4$$

Discounting Method 3  
Discount $\frac{1}{3m}$thly from time one to time zero. The total # of periods is $3m$. Per period discount factor is $\left[1 + \frac{i^{(3m)}}{3m}\right]^{-1}$.

$$PV = \left[1 + \frac{i^{(3m)}}{3m}\right]^{-3m}$$

Once again, we should get the same PV:

$$\Rightarrow \left[1 + \frac{i^{(3m)}}{3m}\right]^{-3m} = \left[1 - \frac{0.085256}{m}\right]^m \Rightarrow \left[1 + \frac{i^{(3m)}}{3m}\right]^{-3} = \left[1 - \frac{0.085256}{m}\right]$$

$$\Rightarrow \left[1 + \frac{i^{(3m)}}{3\times 4}\right]^{-3} = 1 - \frac{0.085256}{4} \Rightarrow 1 + \frac{i^{(3m)}}{3\times 4} = \left(1 - \frac{0.085256}{4}\right)^{\frac{1}{3}}$$
\[ i^{(3m)} = 0.08648788, \quad \Rightarrow 1,000i^{(3m)} \approx 86.49 \]

**Problem 2**
If the annual effective rate \( i = 8\% \), calculate \( i^{(12)} \), \( d^{(4)} \), and the force of interest \( \delta \).

**Solution**
We’ll use “the accumulate $1 method.” Assume we have $1 at \( t = 0 \).

What’s the FV of this $1?

If we accumulate $1 using \( i = 8\% \), then \( FV = 1 + i \).

If we accumulate $1 using \( i^{(12)} \), then
\[
FV = \left[ 1 + \frac{i^{(12)}}{12} \right]^{12}
\]

If we accumulate $1 using \( d^{(4)} \), then
\[
FV = \left[ 1 - \frac{d^{(4)}}{4} \right]^{-4}
\]

If we accumulate $1 using \( \delta \), then \( FV = e^\delta \)

We should accumulate the same amount of wealth.

\[ 1 + i = \left[ 1 + \frac{i^{(12)}}{12} \right]^{12}, \quad 1 + \frac{i^{(12)}}{12} = (1 + i)^{\frac{1}{12}} \]

\[ i^{(12)} = 12 \left( (1 + i)^{\frac{1}{12}} - 1 \right) = 12 \left( 1.08^{\frac{1}{12}} - 1 \right) = 7.7208\% \]

\[ \left[ 1 - \frac{d^{(4)}}{4} \right]^{-4} = 1 + i, \quad 1 - \frac{d^{(4)}}{4} = (1 + i)^{\frac{1}{4}} \]

\[ 1 - \frac{d^{(4)}}{4} = (1 + i)^{\frac{1}{4}}, \quad d^{(4)} = 4 \left[ 1 - (1 + i)^{\frac{1}{4}} \right] = 4 \left[ 1 - 1.08^{\frac{1}{4}} \right] = 7.6225\% \]

\[ e^\delta = 1 + i, \quad \delta = \ln(1 + i) = \ln1.08 = 7.696\% \]
**Problem 3** Prove the following equation

\[
\frac{1}{d^{(m)}} = \frac{1}{m} + \frac{1}{i^{(m)}}
\]

**Solution**

Assume we have $1 at \ t = 0$ and we want to accumulate this $1 to \ t = \frac{1}{m}.

Accumulate $1$ from \( t = 0 \) to \( t = \frac{1}{m} \) using \( d^{(m)} \):

\[
FV = \left[ 1 - \frac{d^{(m)}}{m} \right]^{-1}
\]

Accumulate $1$ from \( t = 0 \) to \( t = \frac{1}{m} \) using \( i^{(m)} \):

\[
FV = 1 + \frac{i^{(m)}}{m}
\]

\[
\Rightarrow \left[ 1 - \frac{d^{(m)}}{m} \right]^{-1} = 1 + \frac{i^{(m)}}{m}, \quad 1 - \frac{d^{(m)}}{m} = \frac{1}{m + i^{(m)}}
\]

\[
\Rightarrow \frac{d^{(m)}}{m} = 1 - \frac{m}{m + i^{(m)}} = \frac{i^{(m)}}{m + i^{(m)}}
\]

\[
\Rightarrow \frac{m}{d^{(m)}} = \frac{m + i^{(m)}}{i^{(m)}} = \frac{m}{i^{(m)}} + 1 \quad \Rightarrow \frac{1}{d^{(m)}} = \frac{1}{m} + \frac{1}{i^{(m)}}
\]

**Problem 4** Prove \( \lim_{m \to \infty} d^{(m)} = \lim_{m \to \infty} i^{(m)} = \delta \)

**Solution**

Based on Problem 3, we have:

\[
\lim_{m \to \infty} \frac{1}{d^{(m)}} = \lim_{m \to \infty} \frac{1}{m} + \lim_{m \to \infty} \frac{1}{i^{(m)}} = \lim_{m \to \infty} \frac{1}{i^{(m)}} = \frac{1}{\delta}
\]

\[
\Rightarrow \lim_{m \to \infty} d^{(m)} = \lim_{m \to \infty} i^{(m)} = \delta
\]
Problem 5

List the relationship among $\delta, i, v, d$.

Solution

<table>
<thead>
<tr>
<th></th>
<th>$\delta$</th>
<th>$i$</th>
<th>$v$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = \delta$</td>
<td>$\ln(1+i)$</td>
<td>$-\ln v$</td>
<td>$-\ln(1-d)$</td>
<td></td>
</tr>
<tr>
<td>$i = e^\delta - 1$</td>
<td>$i$</td>
<td>$\frac{1}{v} - 1$</td>
<td>$\frac{d}{1-d}$</td>
<td></td>
</tr>
<tr>
<td>$v = e^{-\delta}$</td>
<td>$\frac{1}{1+i}$</td>
<td>$v$</td>
<td>$1-d$</td>
<td></td>
</tr>
<tr>
<td>$d = 1 - e^{-\delta}$</td>
<td>$\frac{i}{1+i}$</td>
<td>$1-v$</td>
<td>$d$</td>
<td></td>
</tr>
</tbody>
</table>

No need to memorize the above table. However, make sure you can derive the table.
PV of a stream of cash flows

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>......</th>
<th>k</th>
<th>......</th>
<th>n</th>
</tr>
</thead>
</table>

Cash flow: \( CF(0) \) \( CF(1) \) \( CF(k) \) \( CF(n) \)

\[
PV = CF(0) + \frac{CF(1)}{1+i} + \frac{CF(2)}{(1+i)^2} + \ldots + \frac{CF(n)}{(1+i)^n}
\]

### Net Present Value

http://www.youtube.com/watch?v=IH1UUh2_XFbM

http://www.youtube.com/watch?v=PCrBvhTJiAw

NPV is just the PV of future cash flows minus the initial cost at time zero.

To make money in the future, often we have to spend money at time zero (to buy machines or build a factory for example). Then to calculate our total wealth at time zero, we need to subtract, from the PV of future cash flows, our initial cost at time zero. The result is called NPV.

\[
NPV = -\text{Cost} + PV
\]

**Example.**

If you invest $10 today, you’ll get $6 at the end of Year 1 and $8 at the end of Year 2. The interest rate is 12%. What’s your NPV?

\[
PV = -10 + \frac{6}{1.12} + \frac{8}{1.12^2} = 1.7347
\]

### Internal rate of return (IRR)

http://www.youtube.com/watch?v=B89vwItBFfk

IRR is the interest rate such that the net present value of future cash flows is equal to zero. To find the IRR, we need to solve the following equation:
\[ 0 = CF(0) + \frac{CF(1)}{1+i} + \frac{CF(2)}{(1+i)^2} + \ldots + \frac{CF(n)}{(1+i)^n} \]

You can use BA II Plus/BA II Plus Professional Cash Flow Worksheet to calculate IRR.

**Example.** Calculate IRR of the following cash flows:

<table>
<thead>
<tr>
<th>Time t (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>-100</td>
<td>24</td>
<td>35</td>
<td>20</td>
<td>16</td>
<td>13</td>
</tr>
</tbody>
</table>

**Solution**

Enter the following into BA II Plus/BA II Plus Professional Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>Cash flow</th>
<th>CF0</th>
<th>C01</th>
<th>C02</th>
<th>C03</th>
<th>C04</th>
<th>C05</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>F01</td>
<td>F02</td>
<td>F03</td>
<td>F04</td>
<td>F05</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Press “IRR” “CPT.” You should get: IRR=3.01%.

**Multiple IRRs.** The number of positive roots in

\[ 0 = CF(0) + \frac{CF(1)}{1+i} + \frac{CF(2)}{(1+i)^2} + \ldots + \frac{CF(n)}{(1+i)^n} \]

is equal to the number of sign changes. For example, if \( CF(0) \) is negative and \( CF(1), CF(2), \ldots, CF(n) \) are all positive, then there’s only one sign change. As a result, there is only one positive root in polynomial equation; there is a unique IRR.

If \( CF(0) \) is negative, \( CF(1) \) is positive, \( CF(2) \) is negative, and all the remaining cash flows are positive, then we have two sign changes. As a result, there are two IRRs.

Please note that BA II Plus/BA II Plus Professional Cash Flow Worksheet generates only one IRR value.
Example. Calculate IRR of the following cash flows:

<table>
<thead>
<tr>
<th>Time t (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>- $100</td>
<td>230</td>
<td>-132</td>
</tr>
</tbody>
</table>

Solution

We need to solve the following equation:

\[
0 = -100 + \frac{230}{1+r} - \frac{132}{(1+r)^2}
\]

\[
\frac{1.32}{(1+r)^2} - \frac{2.33}{1+r} + 1 = 0 \Rightarrow \left( \frac{1}{1+r} - \frac{1}{1.1} \right) \left( \frac{1}{1+r} - \frac{1}{1.2} \right) = 0 \Rightarrow r_1 = 10\%, \quad r_2 = 20\%
\]

Multiple IRR’s are hard to explain. This is one of the drawbacks of using IRR to determine profitability.

Asset and its price

Asset = a stream of cash flows

The price of an asset = PV of future cash flows

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>......</th>
<th>k</th>
<th>......</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset</td>
<td>(CF(0))</td>
<td>(CF(1))</td>
<td>(CF(k))</td>
<td>(CF(n))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
PV = CF(0) + \frac{CF(1)}{1+i} + \frac{CF(2)}{(1+i)^2} + \ldots + \frac{CF(n)}{(1+i)^n}
\]

The price of an asset = PV of future cash flows

This is a critical concept for Exam FM. This concept is used in pricing bonds and stocks.
Convert a cash flow from one point of time to another point of time

Time $t_0$ $1$ $\ldots$ $t_1$ $\ldots$ $t_2$

Cash flow $A$ $\rightarrow$ ???

Assume that the interest rate is $i$. The discount rate is $d = 1 - \nu = 1 - (1+i)^{-1}$.

We have a cash flow of $A$ at $t_1$ and we want to convert it to a cash flow at $t_2$, where $t_2 \geq t_1$.

$A$ @ $t_1$ $\rightarrow$ $A(1+i)^{t_2-t_1} = A(1-d)^{-(t_2-t_1)}$ @ $t_2$ (accumulating)

Alternatively, we have a cash flow $B$ at $t_2$ and we want to convert it to a cash flow at $t_1$ where $t_2 \geq t_1$.

Time $t_0$ $1$ $\ldots$ $t_1$ $\ldots$ $t_2$

Cash flow ??? $\rightarrow$ $B$

$B$ @ $t_2$ $\rightarrow$ $Bv^{t_2-t_1} = B(1-d)^{t_2-t_1}$ @ $t_1$ (discounting)

Example.

Assume the discount rate $d = 10\%$.

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$X$</td>
<td>$10$</td>
<td>$Y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$10$ @ $t = 3$ $\rightarrow$ $10(1-d)^{3-1} = 10(1-10\%)^2 = 8.1$ @ $t = 1$

$10$ @ $t = 3$ $\rightarrow$ $10(1-d)^{-7}$ $= 10(1-10\%)^{-4} = 15.24157903$ @ $t = 7$
Alternative calculation:
\[ v = 1 - d = 1 - 10\% = 0.9, \quad 1 + i = v^{-1} = 0.9^{-1} \]

\[ \$10 \quad \text{at} \; t = 3 \quad \rightarrow \quad \$10 \; v^{3-1} = \$10 \; v^2 = \$10 \left( 0.9^2 \right) = \$8.1 \quad \text{at} \; t = 1 \]

\[ \$10 \quad \text{at} \; t = 3 \quad \rightarrow \quad \$10 \left( 1 + i \right)^{7-3} = \$10 \left( 0.9^{-1} \right)^4 = \$15.24157903 \quad \text{at} \; t = 7 \]

**Collapse multiple cash flows into a single cash flow**

Many times we need to collapse (i.e. consolidate) multiple cash flows occurring at different times into a single cash flow occurring at a common point of time. To collapse a stream of cash flows into a single cash flow, we can NOT simply add up multiple cash flows (unless the interest rate is zero).

**Procedures to collapse multiple cash flows into a single cash flow:**

- Choose a common point of time.
- Convert, either by discounting or by accumulating, each cash flow to an equivalent cash flow occurring at this common point of time.
- Add up the converted values of all cash flows. This is the single cash flow into which multiple cash flows collapse.

**Example.**

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$1</td>
<td></td>
<td>$1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Assume the interest rate is 10%.

We have 2 cash flows: $1 at t=2 and $1 at t=6. We want to collapse these 2 cash flows into one cash flow, perhaps because we want to calculate the total value of these 2 cash flows.

**1) If we choose t=0 as the common time.**

\[ \$1 \quad \text{at} \; t = 2 \quad \rightarrow \quad \$1.1^2 = \$0.82644628 \quad \text{at} \; t = 0 \]

\[ \$1 \quad \text{at} \; t = 6 \quad \rightarrow \quad \$1.1^6 = \$0.56447393 \quad \text{at} \; t = 0 \]

The total value of the 2 cash flows \( @ \; t = 0 \):

\[ \$0.82644628 + \$0.56447393 = \$1.39092021 \]
(2) If we choose t=2 as the common time.

\[
\begin{align*}
$1 @ t = 2 & \rightarrow $1 @ t = 2 \\
$1 @ t = 6 & \rightarrow $1.1^{-4} = 0.68301346 @ t = 2
\end{align*}
\]

The total value of the 2 cash flows @ t=2:

\[1 + 0.68301346 = 1.68301346\]

(3) If we choose t=4 as the common time.

\[
\begin{align*}
$1 @ t = 2 & \rightarrow $1.1^2 = 1.21 @ t = 0 \\
$1 @ t = 6 & \rightarrow $1.1^2 = 0.82644628 @ t = 4
\end{align*}
\]

The total value of the 2 cash flows @ t=4:

\[1.21 + 0.82644628 = 2.03644628\]

(4) If we choose t=6 as the common time.

\[
\begin{align*}
$1 @ t = 2 & \rightarrow $1.1^4 = 1.4641 @ t = 6 \\
$1 @ t = 6 & \rightarrow $1 @ t = 6
\end{align*}
\]

The total value of the 2 cash flows @ t=6:

\[1.4641 + 1 = 2.4641\]

(5) If we choose t=7 as the common time.

\[
\begin{align*}
$1 @ t = 2 & \rightarrow $1.1^5 = 1.61050 @ t = 7 \\
$1 @ t = 6 & \rightarrow $1.1 @ t = 7
\end{align*}
\]

The total value of the 2 cash flows @ t=7:

\[1.61050 + 1.1 = 2.71051\]

So far, we have collapsed the two cash flows into five single cash flows occurring at t=0, 2, 4, 6, and 7 respectively. Since these five single cash flows each represent the total value of the identical cash flows of $1 @ t=2 and $1 @ t=6, they should convert to each other following the standard conversion rule.
Time $t$  0   1   2   3   4   5   6   7

Cash flow

$1.39092021$  $1.68301346$  $2.03644628$  $2.4641$  $2.71051$

For example:

$1.39092021$ @ $t = 0$ → $1.39092021(1.1^2) = 1.68301345$ @ $t = 2$

We did not get $1.68301346$ @ $t = 2$ due to rounding.

$2.03644628$ @ $t = 4$ → $2.03644628(1.1^3) = 2.71051$ @ $t = 7$ (OK)
Annuity – collapsing \( n \) parallel cash flows into a single cash flow

- \( n \) parallel evenly-spaced cash flows of $1 each

\[
\begin{array}{cccccccc}
\$1 & \$1 & \$1 & \$1 & \ldots & \$1 & \$1 \\
| & | & | & | & | & |
\hline
i & i & i & i & i & i & i \\
1 & 2 & 3 & 4 & n-1 & n
\end{array}
\]

- The total value of \( n \) parallel cash flows of $1 each at one step to the left of the 1\textsuperscript{st} cash flow is \( a_{\overline{m}|i} = \frac{1-v^n}{i} \). In other words, we can collapse \( n \) parallel cash flows of $1 each into a single cash flow \( a_{\overline{m}|i} \) at one step to the left of the 1\textsuperscript{st} cash flow.

- The total value of \( n \) parallel cash flows of $1 each at the 1\textsuperscript{st} cash flow time is \( \ddot{a}_{\overline{m}|i} = \frac{1-v^n}{d} \). We can collapse \( n \) parallel cash flows of $1 each into a single cash flow \( \ddot{a}_{\overline{m}|i} \) at the 1\textsuperscript{st} cash flow time.

- The total value of \( n \) parallel cash flows of $1 each at the final cash flow time is \( s_{\overline{m}|i} = \frac{(1+i)^n-1}{i} \). We can collapse \( n \) parallel cash flows of $1 each into a single cash flow \( s_{\overline{m}|i} \) at the final cash flow time.

- The total value of \( n \) parallel cash flows of $1 each at one step to right of the final cash flow is \( \ddot{s}_{\overline{m}|i} = \frac{(1+i)^n-1}{d} \). We can collapse \( n \) parallel cash flows of $1 each into a single cash flow \( \ddot{s}_{\overline{m}|i} \) at one step to the right of the final cash flow.
Please note that $i$ is the effective interest rate between two consecutive cash flows. For example, if two consecutive cash flows are 3 months apart, then $i$ is the effective interest rate for the 3 month period (i.e., 3 months = 1 unit time); if two consecutive cash flows are 3 years apart, then $i$ is the effective interest rate for the 3 year period (i.e., 3 years = 1 unit time).

Since the four single cash flows $a_{ni}$, $\ddot{a}_{ni}$, $s_{ni}$, and $\ddot{s}_{ni}$ each represent the total value of the identical $n$ parallel cash flows, they should convert to each other following the standard conversion rule. So we have:

$$a_{ni} = v \ddot{a}_{ni}, \quad \ddot{a}_{ni} = a_{ni} (1+i)$$
$$s_{ni} = v \ddot{s}_{ni}, \quad \ddot{s}_{ni} = s_{ni} (1+i)$$
$$a_{ni} = v^n s_{ni}, \quad s_{ni} = a_{ni} (1+i)^n$$
$$\ddot{a}_{ni} = v^n \ddot{s}_{ni}, \quad \ddot{s}_{ni} = \ddot{a}_{ni} (1+i)^n$$
$$a_{ni} = v^{n+1} \ddot{s}_{ni}, \quad \ddot{s}_{ni} = a_{ni} (1+i)^{n+1}$$
$$\ddot{a}_{ni} = v^{n+1} s_{ni}, \quad s_{ni} = \ddot{a}_{ni} (1+i)^{n+1}$$

**Avoid the common pitfall**

Question:
“What’s the difference between annuity due and annuity immediate?”

Answer:
In annuity due, the 1st payment is at $t=0$;
in annuity immediate, the 1st payment is at $t=1$.

If you agree with this answer, you are wrong!

This is a common mistake made by many. This common mistake originates from the fact that when textbooks derive the formula for annuity due and annuity immediate, they always draw the following diagram (making the annuity due have the 1st payment at $t=0$ and the annuity immediate have the 1st payment at $t=1$):
Common diagram for annuity due—potentially misleading

<table>
<thead>
<tr>
<th>Payment $1</th>
<th>$1</th>
<th>$1</th>
<th>…</th>
<th>$1</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Time $t = 0</th>
<th>$t = 1</th>
<th>$t = 2</th>
<th>…. $t = n - 1</th>
<th>$t = n</th>
</tr>
</thead>
</table>

\[ a_{\bar{n}|t} = 1 + v + v^2 + \ldots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d} \]

The truth, however, is that in an annuity due, the 1st payment can start at any time. For example, the 1st payment can be at $t = 2$:

<table>
<thead>
<tr>
<th>Payment $1</th>
<th>$1</th>
<th>$1</th>
<th>…</th>
<th>$1</th>
<th>$1</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Time $t = 0</th>
<th>$t = 1</th>
<th>$t = 2</th>
<th>$t = 3</th>
<th>…. $t = n</th>
<th>$t = n + 1</th>
<th>$t = n + 2</th>
</tr>
</thead>
</table>

\[ a_{\bar{n}|t} = 1 + v + v^2 + \ldots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d} \]

To avoid the faulty thinking that in an annuity due the 1st payment is at $t = 0$, you should use the following diagram(i.e. don’t include the time in your diagram):

<table>
<thead>
<tr>
<th>Payment $1</th>
<th>$1</th>
<th>$1</th>
<th>…</th>
<th>$1</th>
<th>$1</th>
</tr>
</thead>
</table>

\[ a_{\bar{n}|t} = 1 + v + v^2 + \ldots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d} \]

Rule to remember: If you discount $n$ evenly spaced cash flows back to the $1$st cash flow time, you’ll have an annuity due, regardless of when the $1$st payment is made.
Common diagram for annuity immediate—potentially misleading

<table>
<thead>
<tr>
<th>Payment</th>
<th>$1</th>
<th>$1</th>
<th>…</th>
<th>$1</th>
<th>$1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>$t = 0$</td>
<td>$t = 1$</td>
<td>$t = 2$</td>
<td>…</td>
<td>$t = n - 1$</td>
</tr>
</tbody>
</table>

$$a_{n|t} = v + v^2 + ... + v^{n-1} + v^n = \frac{v - v^{n+1}}{1 - v} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{v} = \frac{1}{i}$$

The truth, however, is that in an annuity immediate, the 1st payment can start at any time. For example, the 1st payment can be at $t = 2$:

<table>
<thead>
<tr>
<th>Payment</th>
<th>$1$</th>
<th>$1$</th>
<th>…</th>
<th>$1$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>$t = 0$</td>
<td>$t = 1$</td>
<td>$t = 2$</td>
<td>$t = 3$</td>
<td>…</td>
</tr>
</tbody>
</table>

$$a_{n|t} = v + v^2 + ... + v^{n-1} + v^n = \frac{v - v^{n+1}}{1 - v} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{v} = \frac{1}{i}$$

To avoid the faulty thinking that in an annuity immediate the 1st payment is at $t = 1$, you should use the following diagram (i.e. don’t include the time in your diagram):

<table>
<thead>
<tr>
<th>Payment</th>
<th>$1$</th>
<th>$1$</th>
<th>$1$</th>
<th>…</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>$1$st</td>
<td>$2$nd</td>
<td>$3$rd</td>
<td>…</td>
<td>$n$-th payment</td>
</tr>
</tbody>
</table>

$$a_{n|t} = v + v^2 + ... + v^{n-1} + v^n = \frac{v - v^{n+1}}{1 - v} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{v} = \frac{1}{i}$$

Rule to remember: If you discount $n$ evenly spaced cash flows back to one interval prior to 1st cash flow time, you’ll have an annuity immediate, regardless of when the 1st payment is made.
Similarly, when draw the diagram for $s_{n|i}$ and $\overline{s}_{n|i}$, don’t include the timeline:

<table>
<thead>
<tr>
<th>Payment</th>
<th>$1$</th>
<th>$1$</th>
<th>$1$</th>
<th>$\ldots$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st</td>
<td>2nd</td>
<td>3rd</td>
<td>$\ldots$</td>
<td>$n$-th</td>
</tr>
</tbody>
</table>

$$s_{n|i} = \frac{(1+i)^n - 1}{d}$$

$$\overline{s}_{n|i} = \frac{(1+i)^n - 1}{i}$$

If you accumulate $n$ evenly spaced cash flows the final cash flow time, use $s_{n|i}$, regardless of when the final payment is made. If you accumulate $n$ evenly spaced cash flows the final cash flow time plus one interval, use $\overline{s}_{n|i}$, regardless of when the final payment is made.

**Summary:** Just memorize the following diagram and you’ll do fine.

<table>
<thead>
<tr>
<th>Payment</th>
<th>$1$</th>
<th>$1$</th>
<th>$1$</th>
<th>$\ldots$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st</td>
<td>2nd</td>
<td>3rd</td>
<td>$\ldots$</td>
<td>$n$-th</td>
</tr>
</tbody>
</table>

One step before the first cash flow time, use $a_{n|b}$;

at the 1$^\text{st}$ cash flow time, use $\overline{a}_{n|i}$;

at the final cash flow time, use $s_{n|i}$;

one step after the final cash flow time, use $\overline{s}_{n|i}$.
Example 1.

You are given the following cash flows:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$5</td>
<td>$5</td>
<td>$5</td>
<td>$5</td>
<td>$5</td>
<td>$5</td>
<td>$5</td>
<td>$5</td>
<td></td>
</tr>
</tbody>
</table>

The interest rate is 10%.

Calculate, at t=4, the total value of these cash flows.

Solution

We need to collapse the cash flows into a single cash flow @ t=4. There are many ways to do so.

Method 1
First, we split the original cash flows into two streams of cash flows. The first stream consists of cash flows at t=2, 3, and 4. The second stream consists of cash flows at t=5, 6, 7, and 8. If we add up these two streams, we should get the original cash flows.

Next, we collapse each of the two streams into a single cash flow at t=4. For the 1st stream of cash flows, we need to collapse 3 parallel cash flows into a single cash flow at the final payment time. This gives us an equivalent single cash flow at t=4 of $5s\_3^{10\%}$.

For the 2nd stream of cash flows, we need to collapse 4 parallel cash flows into a single cash flow at one step to the left of the 1st cash flow. As the result, the equivalent single cash flow at t=4 is $5a\_4^{10\%}$.
Consequently, the total value of the original cash flows @ t=4 is:

\[ 5 \cdot d_{4|10\%} + 5 \cdot a_{4|10\%} = 5 \cdot \frac{1.1^3 - 1}{0.1} + 5 \cdot \frac{1 - 1.1^{-4}}{0.1} \approx 32.40 \]

**Method 2**

We split the original cash flows into two streams of cash flows. The 1st stream consists of cash flows at t=2 and 3. The 2nd stream consists of cash flows at t=4, 5, 6, 7, and 8.

Next, we collapse each of the two streams of cash flows into a single cash flow at t=4. For the 1st stream, we need to collapse 2 parallel cash flows to a single cash flow one step to the right of the final cash flow. This gives us an equivalent cash flow of \( 5 \cdot d_{4|10\%} \) at t=4.

For the 2nd stream, we need to collapse five parallel cash flows at the first cash flow time. This gives us an equivalent cash flow of \( 5 \cdot a_{4|10\%} \) at t=4.

Finally, we add up these two equivalent cash flows at t=4.

\[ 5 \cdot d_{4|10\%} + 5 \cdot a_{4|10\%} = 5 \cdot \frac{1.1^3 - 1}{1-1.1^{-3}} + 5 \cdot \frac{1 - 1.1^{-5}}{1-1.1^{-3}} \approx 32.40 \]
Method 3
This time, we don’t split the original cash flows.

First, we collapse the original cash flows to \( t=1 \). Since 7 parallel cash flows are collapsed into a single cash flow one step to the left of the 1\(^{st} \) cash flow, the equivalent single cash flow is \( 5a_{\bar{1}10\%} \).

\[
\begin{array}{cccccccccc}
\text{Time} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{Cash flow} & \uparrow & $5 & $5 & $5 & $5 & $5 & $5 & $5 & $5 \\
\end{array}
\]

Next, we’ll convert the single cash flow of \( 5a_{\bar{1}10\%} \) @ \( t=1 \) to a cash flow @ \( t=4 \).

\[5a_{\bar{1}10\%} \quad @ \quad t = 1 \rightarrow \quad 5a_{\bar{1}10\%} (1.1^{1-1}) = 5a_{\bar{1}10\%} (1.1^1) \quad @ \quad t = 4\]

Finally, the total value of the original cash flow @ \( t=4 \):

\[5a_{\bar{1}10\%} (1.1^1) = \frac{5(1-1.1^{-7})}{0.1} (1.1^1) \approx 32.40\]

Method 4
First, we collapse the original cash flows to \( t=2 \). Since 7 parallel cash flows are collapsed into a single cash flow at the 1\(^{st} \) cash flow time, the equivalent single cash flow is \( 5\bar{a}_{\bar{1}10} \).

\[
\begin{array}{cccccccccc}
\text{Time} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{Cash flow} & \uparrow & $5 & $5 & $5 & $5 & $5 & $5 & $5 & $5 \\
\end{array}
\]

Next, we’ll convert the single cash flow of \( 5\bar{a}_{\bar{1}10} \) @ \( t=2 \) to a cash flow @ \( t=4 \).

\[5\bar{a}_{\bar{1}10} \quad @ \quad t = 2 \rightarrow \quad 5\bar{a}_{\bar{1}10} (1.1^{2-2}) = 5\bar{a}_{\bar{1}10} (1.1^2) \quad @ \quad t = 4\]
Finally, the total value of the original cash flow @ $t = 4$:

$$5 \bar{a}_{10}\left(1.1^2\right) = 5 \frac{1-1.1^{-7}}{1-1.1^{-1}}(1.1^2) \approx 32.40$$

**Method 5**

First, we collapse the original cash flows to $t=8$. Since 7 parallel cash flows are collapsed into a single cash flow at the final cash flow time, the equivalent single cash flow is $5 s_{7\%}$.

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
</tr>
</tbody>
</table>

Next, we’ll convert the single cash flow of $5 s_{7\%}$ @ $t=8$ to a cash flow @ $t=4$.

$$5 s_{7\%} @ t = 8 \rightarrow 5 s_{7\%}(1.1^{4-8}) = 5 s_{7\%}(1.1^4) @ t = 4$$

Finally, the total value of the original cash flow @ $t = 4$:

$$5 s_{7\%}(1.1^4) = 5 \frac{1.1^7 - 1}{0.1}(1.1^4) \approx 32.40$$

**Method 6**

First, we collapse the original cash flows to $t=9$. Since 7 parallel cash flows are collapsed into a single cash flow at one step after the final cash flow time, the equivalent single cash flow is $5 \bar{x}_{7\%}$.

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
<td>$5$</td>
</tr>
</tbody>
</table>

Next, we’ll convert the single cash flow of $5 \bar{x}_{7\%}$ @ $t=9$ to a cash flow @ $t=4$. 
\[
\$5 \dd{\bar{a}}_{7\%,9} \at t = 9 \rightarrow \$5 \dd{\bar{a}}_{7\%,1-9} (1.1^{4-9}) = 5 \dd{\bar{a}}_{7\%,1-5} (1.1^{-5}) \at t = 4
\]

Finally, the total value of the original cash flow \at t = 4:

\[
5 \dd{\bar{a}}_{7\%,1-3} (1.1^{-3}) = 5 \frac{1.1^7 - 1}{1 - 1.1^{-1}} (1.1^{-3}) \approx 32.40
\]

The above 6 methods may seem an overkill, but they are good exercises on how to collapse an identical cash flow stream in various ways.

**Example 2**

Explain why \( \dd{\bar{a}}_{n,4} = \dd{\bar{a}}_{n,i} (\dd{\bar{a}}_{n,j}) = \dd{\bar{a}}_{n,i} (1 + v^n + v^{2n} + v^{3n}) \), where \( n \) is a positive integer and \( j = (1+i)^n - 1 \).

**Solution**

We can collapse \( 4n \) parallel cash flows of $1 into \( \dd{\bar{a}}_{4n,i} \):

![Diagram showing 4n parallel cash flows](http://actuary88.com)
Next, we split the $4n$ parallel cash flows into 4 sets of $n$ parallel cash flows. We collapse each set of $n$ parallel cash flows into a single cash flow of $\ddot{a}_{n|i}$, arriving at 4 sets of parallel cash flows of $\ddot{a}_{n|i}$ each. Finally, we collapse the 4 parallel cash flows of $\ddot{a}_{n|i}$ each into a single cash flow of

$$\left(\ddot{a}_{n|i}\right)\left(\ddot{a}_{n|j}\right).$$

![Diagram of cash flows](http://actuary88.com)

We use an annuity factor of $\ddot{a}_{n|j}$. $j$ represents the interest rate between two consecutive cash flows of $\ddot{a}_{n|i}$. Among the four parallel cash flows of $\ddot{a}_{n|i}$, two consecutive cash flows are $n$ time apart. Consequently,

$$j = (1+i)^n - 1.$$

Because $4n$ parallel cash flows can be collapsed into two single cash flows $\ddot{a}_{4n|i}$ and $\left(\ddot{a}_{n|i}\right)\left(\ddot{a}_{n|j}\right)$ that occur at the same time, it follows that

$$\ddot{a}_{4n|i} = \left(\ddot{a}_{n|i}\right)\left(\ddot{a}_{n|j}\right).$$

Because $\ddot{a}_{n|j} = 1 + (1+j)^{-1} + (1+j)^{-2} + (1+j)^{-3} = 1 + v^n + v^{2n} + v^{3n}$, we have:

$$\ddot{a}_{4n|i} = \left(\ddot{a}_{n|i}\right)\left(\ddot{a}_{n|k}\right) = \ddot{a}_{n|i} \left(1 + v^n + v^{2n} + v^{3n}\right)$$

Make sure you understand the above logic. This sharpens your skill to quickly collapse a complex stream of cash flows into a single cash flow.
We can also prove \( \ddot{a}_{\dot{4}n|i} = (\ddot{a}_{\dot{4}n|i})(\dddot{a}_{\dot{4}n}) \) using the standard annuity formula.

\[
\ddot{a}_{\dot{4}n|i} = \frac{1 - v^{4n}}{d}, \quad \dddot{a}_{\dot{4}n|i} = \frac{1 - v^n}{d}
\]

\[
\ddot{a}_{\dot{4}n} = \frac{1 - (1+i)^{-4n}}{1 - (1+i)^{-n}} = \frac{1 - v^{4n}}{1 - v^n} = 1 + v^n + v^{2n} + v^{3n}
\]

\[
(\ddot{a}_{\dot{4}n|i})(\dddot{a}_{\dot{4}n}) = \frac{1 - v^n}{d} \cdot \frac{1 - v^{4n}}{1 - v^n} = \frac{1 - v^{4n}}{d}
\]

\[\Rightarrow \ddot{a}_{\dot{4}n|i} = (\ddot{a}_{\dot{4}n|i})(\dddot{a}_{\dot{4}n}) = \dddot{a}_{\dot{4}n} (1 + v^n + v^{2n} + v^{3n})\]

Generally, for any positive integer \( k \):

\[
\ddot{a}_{\dot{k}n|i} = (\ddot{a}_{\dot{k}n|i})(\dddot{a}_{\dot{k}n}) = \dddot{a}_{\dot{k}n|i} \left[ 1 + v^n + v^{2n} + \ldots + v^{(k-1)n} \right]
\]

\[
a_{\ddot{k}n|i} = (a_{\ddot{k}n|i})(\dddot{a}_{\dot{k}n}) = a_{\ddot{k}n|i} \left[ 1 + v^n + v^{2n} + \ldots + v^{(k-1)n} \right]
\]

\[
s_{\ddot{k}n|i} = (s_{\ddot{k}n|i})(\dddot{a}_{\dot{k}n}) = s_{\ddot{k}n|i} \left[ 1 + (1+i)^n + (1+i)^{2n} + \ldots + (1+i)^{(k-1)n} \right]
\]

\[
\dddot{s}_{\ddot{k}n|i} = (\dddot{s}_{\ddot{k}n|i})(\dddot{a}_{\dot{k}n}) = \dddot{s}_{\ddot{k}n|i} \left[ 1 + (1+i)^n + (1+i)^{2n} + \ldots + (1+i)^{(k-1)n} \right]
\]

where \( j = (1+i)^n - 1 \)

You don't need to memorize the above formula. Just make sure that you can derive the above formulas by collapsing cash flows.
**Example 3 (SOA May 2002 EA-1 #11)**

S1 = The accumulated value as of 12/31/2002 of $500 invested at the end of each month during 2002 at a nominal interest rate of 8% per year, convertible quarterly.

A1 = The present value as of 1/1/2002 of S1, at a nominal discount rate of 6% per year, convertible semiannually.

S2 = The accumulated value as of 12/31/2002 of $1,500 invested at the end of each quarter during 2002 at a nominal discount rate of 6% per year, convertible monthly.

A2 = The present value as of 1/1/2002 of S2, at a nominal interest rate of P% per year, convertible once every two years.

In what range is P% such that A1 = A2?

[A] Less than 4.60%
[B] 4.60% but less than 4.70%
[C] 4.70% but less than 4.80%
[D] 4.80% but less than 4.90%
[E] 4.90% or more

**Solution**

To simplify the calculation, let's set $500 as one unit of money.

<table>
<thead>
<tr>
<th>Date</th>
<th>1/1/2002</th>
<th>12/31/2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t (months)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Payments</td>
<td>$1</td>
<td>1</td>
</tr>
</tbody>
</table>

The quarterly effective interest rate is \( i^{(4)}/4 = 8%/4 = 2\% \)

Remember 1 year = 4 quarters

The monthly effective interest \( i \) is calculated as follows:

\[
(1+i)^{1/3} = 1 + 2\% \quad \text{(remember that 1 quarter = 3 months)}
\]

\[
i = (1 + 2\%)^{1/3} - 1 = 0.66227096\% \quad S_1 = \ddot{s}_{\frac{1}{4}} = 12.44689341
\]
Because the nominal discount rate is 6% per year convertible semiannually, the discount rate during a 6-month period is 

\[ 1 - d^{(2)} = 1 - \frac{6\%}{2} = 1 - 3\% = 0.97 \]

We can find the annual discount rate \( d \) : 

\[ 1 - d = \left[ 1 - \frac{d^{(2)}}{2} \right]^2 = 0.97^2 \]

Here we use the formula: 

\[ 1 - d = \left[ 1 - \frac{d^{(m)}}{m} \right]^m \]

Let \( v \) represent the annual discounting factor for A1. Then 

\[ A1 = (S1)v \]. However, \( v = 1 - d \). So we have:

\[ A1 = (S1)v = (S1)(1 - d) = 12.44689341 \left[ 1 - \frac{6\%}{2} \right]^2 = 11.7112820 \]

Next, we calculate \( S2 \).

<table>
<thead>
<tr>
<th>Date</th>
<th>1/1/2002</th>
<th>12/31/2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time ( t ) (quarters)</td>
<td>0 1 2 3 4</td>
<td></td>
</tr>
<tr>
<td>Payments</td>
<td>$3 3 3 3</td>
<td>3 3 3 3</td>
</tr>
</tbody>
</table>

3 units of money=$1,500

\[ S_2 = 3 \times \bar{a}_{\overline{3|j}} \]

The nominal discount rate is 6% per year convertible monthly. We can find the annual rate of discount \( d \) and the annual discounting factor \( v \).

\[ v = \frac{1}{1+i} = 1 - d = \left[ 1 - \frac{d^{(12)}}{12} \right]^{12} \]

Once again the formula is 

\[ 1 - d = \left[ 1 - \frac{d^{(m)}}{m} \right]^m \]

Once we have the annual effective rate \( i \), we can find the quarterly effective interest rate \( j \).

\[ (1 + j)^4 = 1 + i = v^{-1} \quad \text{(1 year=4 quarters)} \]
To calculate $A_2$, we need to know how to deal with a bizarre nominal rate: “a nominal interest rate of $P\%$ per year, convertible once every two years.” Let’s start with something simple. If we have a nominal interest rate of $P\%$ per year convertible once every 6 months, then the 6-month-period interest rate is $\frac{P\%}{2}$. Here the denominator 2 is the # of 6-month-periods per year (so one year = two 6-months). Similarly, if we have a nominal interest rate of $P\%$ per year convertible monthly, then the monthly interest rate is $\frac{P\%}{12}$. Here the denominator 12 is the # of months per year (so one year = 12 months).

Now let’s apply this logic to a nominal interest rate convertible once every two years. One year is 0.5 of 2 years. Then the interest rate for a 2-year period is $\frac{P\%}{0.5} = 2P\%$. Then the discounting factor for a 2-year period is $\frac{1}{1+2P\%} = (1+2P\%)^{-1}$. Then the discounting factor for one year is $\left[(1+2P\%)^{-1}\right]^{\frac{1}{2}} = (1+2P\%)^{\frac{1}{2}}$.

$A_2 = \text{The present value as of 1/1/2002 of S}_2, \text{ at a nominal interest rate of } P\% \text{ per year, convertible once every two years.}$

Then $A_2 = S_2 \left(1+2P\%\right)^{\frac{1}{2}} = 12.27548783 \left(1+2P\%\right)^{\frac{1}{2}}$.

$A_1 = A_2 \Rightarrow 11.7112820 = 12.27548783 \left(1+2P\%\right)^{\frac{1}{2}}$.

$\Rightarrow P\% = 4.93367417\%$. So the answer is E.
Perpetuity

**collapsing** $+\infty$ parallel cash flows into a single cash flow

- $+\infty$ parallel evenly-spaced cash flows of $1$ each

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
$1$ & $1$ & $1$ & $1$ & ... & $1$ & $1$ & \\
1 & 2 & 3 & 4 & & & & \\
\end{array}
\]

\[a_{+\infty|i} = \frac{1}{i} \quad \ddot{a}_{+\infty|i} = \frac{1}{d}\]

We can collapse $+\infty$ parallel cash flows of $1$ each into a single cash flow $\frac{1}{i}$ at one step to the left of the 1st cash flow.

We can collapse $+\infty$ parallel cash flows of $1$ each into a single cash flow $\frac{1}{d} = 1 + \frac{1}{i}$ at the 1st cash flow time.
Example 1 (SOA May 2000 EA-1 #1)

Purchase date of a perpetuity-due: 1/1/2000
  Level payment amount: $100
  Frequency of payments: Annual
  Cost of perpetuity: $1,100
  Interest rate for perpetuity: $i\%$, compounded annually

Immediately following the payment on 1/1/2014, the remaining future payments are sold at a yield rate of $i\%$. The proceeds are used to purchase an annuity certain as follows:
  Term of annuity: 10 years
  1st payment of annuity: 1/1/2018
  Frequency of annuity payment: semi-annual on January 1 and July 1
  Interest rate for annuity: $\frac{i}{2}\%$ compounded annually

In what range is the semi-annual annuity payment?
- (A) Less than $75
- (B) $75 but less than $77
- (C) $77 but less than $79
- (D) $79 but less than $81
- (E) $81 or more

Solution

Perpetual annuity due:

| Time t (years) | 0   | 1   | 2   | 3   | ... | $\infty$
|----------------|-----|-----|-----|-----|-----|-------------
| Payment        | $100| $100| $100| $100| $100| $100        |

\[
P V = \frac{100}{d} = 1,100, \quad \Rightarrow \quad d = 1 - \frac{1}{1 + i\%} = \frac{1}{11}, \quad \Rightarrow \quad i\% = 10\%
\]

Perpetual annuity due sold on 1/1/2014:

| # of payments | 1     | 2     | 3     | 4     | ... | $\infty$
|---------------|-------|-------|-------|-------|-----|-------------
| Time t        | ...   | 1/1/2014 | 7/1/2014 | 1/1/2015 | 7/1/2015 | ... | $\infty$
| Payment       | ...   | $100$ | $100$ | $100$ | ... | $100$        |

\[
selling \ price = \frac{100}{i\%} = \frac{100}{10\%} = 1,000
\]
The sales proceeds are used to purchase a 10 year annuity certain.

<table>
<thead>
<tr>
<th>Time</th>
<th>1/1/2018</th>
<th>7/1/2018</th>
<th>1/1/2019</th>
<th>7/1/2019</th>
<th>…</th>
<th>1/1/2027</th>
<th>7/1/2027</th>
</tr>
</thead>
<tbody>
<tr>
<td># of payments</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>…</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>Payment</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

\[ PV = X \ddot{a}_{20|j} \]

Where \( j \) is the semi-annual effective interest rate.

\[
j = \left(1 + \frac{i\%}{2}\right)^{0.5} - 1 = \left(1 + \frac{10\%}{2}\right)^{0.5} - 1 = 2.47\%
\]

At 1/1/2018, the accumulated value of the sales proceeds should be equal to the PV of the 10 year annuity certain.

\[
\Rightarrow 1,000 \left(1 + \frac{i\%}{2}\right)^4 = X \ddot{a}_{20|j}, \quad \Rightarrow 1,000(1.05)^4 = X \ddot{a}_{20|2.47\%}, \quad X = 75.87
\]

So the answer is C.
Annuity – payable m-thly in advance

To derive the above formula, we first collapse $m$ cash flows of $\frac{1}{m}$ each that occur in each unit of time into an equivalent single cash flow. We should have $n$ equivalent single cash flows (because we have a total of $n$ units of time). Next, we collapse these $n$ single cash flows into one single cash flow of $\frac{d}{d^{(m)} a_{|m|}}$. 

\[
\ddot{a}_{\overline{n}|m|} = \frac{1-v^n}{d^{(m)}} = \frac{d}{d^{(m)}} \frac{1-v^n}{d} = \frac{d}{d^{(m)}} \ddot{a}_{\overline{n}|m|}, \quad \text{where} \quad \left[1 - \frac{d^{(m)}}{m}\right]^m = 1 - d = v
\]
$\frac{1}{m} \ddot{a}_{m|j}$ where $j = (1+i)^{\frac{1}{m}} - 1$

$\frac{1}{m} \ddot{a}_{m|j} = \frac{1}{m} \frac{1 - (1+j)^m}{1 - (1+i)^{\frac{1}{m}}} = \frac{1}{m} \frac{1 - (1+i)^{\frac{1}{m}}}{1 - (1+i)^{-\frac{1}{m}}} = \frac{d}{m \left(1 - d^\frac{1}{m}\right)} = \frac{d}{d^{(m)}}$

Time t

0 1 ...... n-1 n

$0 \quad \frac{1}{m} \quad \frac{2}{m} \quad ...... \quad \frac{m-1}{m} \quad \frac{m}{m}$

$\frac{1}{m} \quad \frac{1}{m} \quad \frac{1}{m} \quad ...... \quad \frac{1}{m}$

$\frac{d}{d^{(m)}} \quad \frac{d}{d^{(m)}} \quad ...... \quad \frac{d}{d^{(m)}}$

$\frac{d}{d^{(m)}} \ddot{a}_{m|i}$

1 unit of time

n-th unit of time

n cash flows

Guo FM, fall 2009
Annuity – payable $m$-thly in arrears

Time $t$  0                        1             …… $n-1$   $n$

$\frac{s}{m}$  $\frac{s}{m}$  $\frac{s}{m}$  …  $\frac{s}{m}$  ……  $\frac{s}{m}$  $\frac{s}{m}$  $\frac{s}{m}$  …  $\frac{s}{m}$

1 unit of time  $n$-th unit of time

$\ddot{a}_{m|i}^{(m)} = \frac{1-v^n}{d_{m|i}^{(m)}} = \frac{d}{a_{m|i}^{(m)}} \dot{a}_{m|i}$ and $a_{m|i}^{(m)} = \frac{1-v^n}{l_{m|i}^{(m)}} = \frac{i}{a_{m|i}^{(m)}} a_{m|i}$ are the only two formulas you need to memorize for annuities where multiple cash flows occur in a one unit time. If a problem asks you to find $s_{m|i}^{(m)}$ and $\ddot{s}_{m|i}^{(m)}$, you can calculate this way:

$s_{m|i}^{(m)} = a_{m|i}^{(m)} (1+i)^n, \quad \ddot{s}_{m|i}^{(m)} = \dot{a}_{m|i}^{(m)} (1+i)^n$

If the symbols and the related formulas of $a_{m|i}^{(m)}$, $\dot{a}_{m|i}^{(m)}$, $s_{m|i}^{(m)}$, and $\ddot{s}_{m|i}^{(m)}$ look too ugly and complex, you can always use the cash flow frequency $\frac{1}{m}$ as the unit time, thus forcing the cash flow frequency and the interest compounding frequency to be identical. This greatly simplifies the number of concepts and formulas you need to memorize.

If the unit time is of $\frac{1}{m}$ long, then the effective interest rate during the unit time is $j = (1+i)^{\frac{1}{m}} - 1$. 

Guo FM, fall 2009
Under this simplifying technique, the PV of an annuity payable m-thly in advance in each unit time is:

\[
\begin{array}{cccccc}
\text{Time } t & 0 & 1 & \ldots & n-1 & n \\
\hline
\frac{1}{m} & \frac{1}{m} & \frac{2}{m} & \ldots & \frac{m-1}{m} & \frac{m}{m} \\
\frac{s}{m} & \frac{s}{m} & \frac{s}{m} & \ldots & \frac{s}{m} & \frac{s}{m} \\
m \text{ cash flows} \\
\hline
\frac{1}{m} \cdot a_{mn|j} & \frac{1}{m} \cdot s_{mn|j} \\
\end{array}
\]

\[
\begin{array}{cccccc}
\text{Time } t & 0 & 1 & \ldots & n-1 & n \\
\hline
\frac{1}{m} & \frac{1}{m} & \frac{2}{m} & \ldots & \frac{m-1}{m} & \frac{m}{m} \\
\frac{s}{m} & \frac{s}{m} & \frac{s}{m} & \ldots & \frac{s}{m} & \frac{s}{m} \\
m \text{ cash flows} \\
\hline
\frac{1}{m} \cdot a_{mn|j} & \frac{1}{m} \cdot s_{mn|j} \\
\end{array}
\]
Example 1

A loan of $100,000 borrowed at 6% annual effective is repaid by level monthly payments in advance over the next 30 years. After 10 years, the outstanding balance of the loan is refinanced at 4% annual effective and is paid by level monthly payments in advance over 20 years.

Calculate:
- The monthly payment of the original loan.
- The principal portion and the interest portion of the 37th payment.
- The monthly payment of the refinanced loan.
- The accumulated value of the reduction in monthly payments invested at 4% annual effective.

Solution

Method 1 - use a year as the compounding period

Find the monthly payment of the original loan

Let \( X \) represent the monthly payment of the original loan.

\[
X \cdot \dd_{30|6\%}^{(12)} = 100,000, \quad \dd_{30|6\%}^{(12)} = \frac{1 - v^{30}}{d^{(12)}} = 1 - d = \left[ 1 - \frac{d^{(12)}}{12} \right]^{12}
\]
Find the principal portion and the interest portion of the 37th payment

The 37th payment is the 1st payment in the 4th year.

The # of compounding period remaining immediately after the 37th payment is 30 – 3 = 27.

\[ X \ddot{a}_{30\,6\%}^{(12)} = X \frac{1 - v^{30}}{d^{(12)}} = \left(7,038.186588\right) \frac{1 - 1.06^{-27}}{5.812767\%} = 95,973.09 \]

The interest portion of the 37th payment is:

\[ \left(95,973.09\right) \frac{d^{(12)}}{12} = \left(95,973.09\right) \frac{5.812767\%}{12} = 464.89 \]

The principal portion is:

\[ 586.52 - 464.89 = 121.63 \]
Calculate the monthly payment of the refinanced loan.

<table>
<thead>
<tr>
<th>Time t (Year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>……</th>
<th>10</th>
<th>……</th>
<th>27</th>
<th>28</th>
<th>29</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 years' payments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ X \ddot{a}_{20|6\%}^{(12)} \]

The outstanding balance of the original loan at \( t = 10 \):

\[
X \ddot{a}_{20|6\%}^{(12)} = X \left[ \frac{1-v^{20}}{d^{(12)}} \right]_{i=6\%}
\]

\[
\ddot{a}_{20|6\%}^{(12)} = \left[ \frac{1-v^{20}}{12} \right]_{i=6\%} = \frac{1-1.06^{-20}}{12(1-1.06^{-12})} = 11.83937535
\]

Let \( \frac{A}{12} \) represent the monthly payment of the refinanced loan.

\[
A \ddot{a}_{20|4\%}^{(12)} = X \ddot{a}_{20|6\%}^{(12)}
\]

\[
\ddot{a}_{20|4\%}^{(12)} = \left[ \frac{1-v^{20}}{12} \right]_{i=4\%} = \frac{1-1.04^{-20}}{12(1-1.04^{-12})} = 13.88301906
\]

\[
\Rightarrow A = X \frac{\ddot{a}_{20|6\%}^{(12)}}{\ddot{a}_{20|4\%}^{(12)}} = (7,038.186588) \frac{11.83937535}{13.88301906} = 6,002.1334
\]

So the monthly payment in advance of the refinanced loan is:

\[
\frac{6,002.1334}{12} = 500.1777833
\]
Find the accumulated value of the reduction in monthly payments invested at 4% annual effective.

Reduction of the monthly payment due to refinancing:

\[
\frac{1}{12} (X - A) = \frac{1}{12} (7,038.1866 - 6,002.1334) = 86.3377667
\]

The accumulated value of the reductions at 4%:

\[
(X - A) \overline{s}_{\infty}^{(12) 4%}
\]

\[
\overline{s}_{\infty}^{(12) 4%} = \frac{(1 + i)^{20} - 1}{d^{(12)}} = \frac{1.04^{20} - 1}{12 \left( 1 - \frac{1}{1.04^{12}} \right)} = 30.4194036
\]

\[
(X - A) \overline{s}_{\infty}^{(12) 4%} = 1,036.0532(30.4194036) = 31,516.12044
\]

Method 2 – use a month as the compounding period

Find the monthly payment of the original loan

Let \( Y \) represent the monthly payment in advance.
The number of compounding periods: \(30(12) = 360\)

The interest rate per period: \(j = 1.06^{\frac{1}{12}} - 1 = 0.48675506\%\)

\[
Y \ddot{a}_{360\mid j} = 100,000
\]

\[
\ddot{a}_{360\mid j} = \frac{1 - v^{360}}{d} = \frac{1 - 1.0048675506^{\frac{360}{-1}}}{1 - 1.0048675506^{-1}} = 170.4984712
\]

\[
Y = \frac{100,000}{\ddot{a}_{360\mid j}} = 586.5155230 \approx 586.52
\]

**Find the principal portion and the interest portion of the 37th payment**

<table>
<thead>
<tr>
<th>Time t (Month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>36</th>
<th>\ldots</th>
<th>357</th>
<th>358</th>
<th>359</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>\ldots</td>
<td>Y</td>
<td>\ldots</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td></td>
<td>\underline{324 monthly payments}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The # of compounding period remaining immediately before the 37th payment is \(27(12) = 324\).

The outstanding loan is:

\[
Y \ddot{a}_{324\mid j} = Y \frac{1 - v^{324}}{d} = \left(586.5155230\right) \frac{1 - 1.0048675506^{\frac{324}{-1}}}{1 - 1.0048675506^{-1}} = 95,973.09
\]

The interest portion of the 37th payment is:

\[
(95,973.09) d, \text{ where } d = 1 - v = 1 - 1.0048675506^{-1}
\]

The interest is: \(95,973.09 \left(1 - 1.0048675506^{-1}\right) = 464.8909\)

The principal portion is: \(586.52 - 464.89 = 121.63\)
Calculate the monthly payment of the refinanced loan.

<table>
<thead>
<tr>
<th>Time t (Month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>……</th>
<th>120</th>
<th>……</th>
<th>357</th>
<th>358</th>
<th>359</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>……</td>
<td>Y</td>
<td>……</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

The outstanding balance of the original loan at $t=120$ is:

$$Y \ddot{a}_{240|j} = Y \left( 1 - v^{240} \right) \frac{d}{j=0.48675506\%} = \left( 586.5155230 \right) \frac{1-1.0048675506^{-240}}{1-1.0048675506^{-1}} = 83,327.73367$$

Let $B$ represent the monthly payment of the refinanced loan.

$$B\ddot{a}_{240|k} = Y\ddot{a}_{240|j} \quad \text{where} \quad k = 1.04^{1/12} - 1 = 0.032737398\%$$

$$\ddot{a}_{240|k} = \left( 1 - v^{240} \right) \frac{1}{k} = \frac{1-1.0032737398^{-240}}{1-1.0032737398^{-1}} = 166.5962287$$

$$B = \frac{Y\ddot{a}_{240|j}}{\ddot{a}_{240|k}} = \frac{83,327.73367}{166.5962287} = 500.177791$$

Find the accumulated value of the reduction in monthly payments invested at 4% annual effective.

<table>
<thead>
<tr>
<th>Time t (Month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>……</th>
<th>120</th>
<th>……</th>
<th>357</th>
<th>358</th>
<th>359</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>……</td>
<td>Y</td>
<td>……</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

Reduction of the monthly payment due to refinancing:
\[ Y - B = 586.5155230 - 500.177791 = 86.337732 \]

The accumulated value of the reductions at 4%:

\[ (Y - B)^{\bar{x}_{240}}_k \]

\[ \bar{x}_{240k} = \frac{(1 + k)^{240} - 1}{1 - (1 + k)^{-1}} = \frac{(1 + 0.32737398\%)^{240} - 1}{1 - (1 + 0.32737398\%)^{-1}} = 365.0329421 \]

The accumulated value is:

\[ (Y - B)^{\bar{x}_{240}}_k = 86.337732 \times (365.0329421) = 31,516.11632 \]

**Example 2 (2002 May EA-1 #3)**

Given values:

\[ \overline{s}^{\text{ln}t}_{2^n} = 180.24943 \]

\[ d^{(m)} = 0.08 \]

In what range is \( \overline{s}^{\text{ln}t}_{2^n} \)?

[A] Less than 2,930

[B] 2,930 but less than 2,970

[C] 2,970 but less than 3,010

[D] 3,010 but less than 3,050

[E] 3,050 or more

**Solution B**

\[ \overline{s}^{(m)}_{2^n} = \frac{(1+i)^{2n} - 1}{d^{(m)}} = 180.24943 \]. Using \( d^{(m)} = 0.08 \), we find that

\[ (1+i)^{2n} = 180.24943(0.08) + 1 \approx 15.42 \]

\[ \overline{s}^{(m)}_{4^n} = \frac{(1+i)^{4n} - 1}{d^{(m)}} = \frac{15.42^4 - 1}{0.08} = 2,959.71 \]
Example 3 (2002 May EA-1 #4)
A 20-year immediate annuity certain is payable monthly. Immediately after the 43\textsuperscript{rd} payment has been made, the present value of the remaining annuity payments is calculated to be $X$.

$N$ is the number of the payment after which the present value of the remaining annuity payments is less than $\frac{X}{2}$ for the first time.

$d^{(4)} = 0.08$

What is $N$?

[A] 67
[B] 68
[C] 171
[D] 172
[E] 173

Solution

First, let's calculate the monthly interest rate $i$. We are given $d^{(4)} = 0.08$.

$$v^3 = (1+i)^{-3} = 1 - \frac{d^{(4)}}{4}, \quad \Rightarrow \quad v = 0.99328839 \text{ (monthly discount factor)}$$

Let $P$ represent the monthly payment.

<table>
<thead>
<tr>
<th>Time $t$ (months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>43</th>
<th>45</th>
<th>46</th>
<th>…</th>
<th>240</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$$P \ a_{240\text{–}N|i} = P \ a_{1\text{–}N|i} = X$$

We are asked to find $N$ such that $P \ a_{240\text{–}N|i} < \frac{X}{2}$. First, let's find $N$ such that $P \ a_{240\text{–}N|i} = \frac{X}{2}$.

$$\Rightarrow \quad \frac{P \ a_{240\text{–}N|i}}{P \ a_{1\text{–}N|i}} = \frac{X}{X} = \frac{1}{2}$$
We need to solve the equation:

$$0.99328839^{240-N} = 1 - \frac{1}{2}(1 - 0.99328839^{197}) = 0.63268311$$

$$(240 - N) \ln 0.99328839 = \ln 0.63268311$$

$$240 - N = \frac{\ln 0.63268311}{\ln 0.99328839} = 67.9788$$

$$N = 240 - 67.9788 \approx 172.02$$

As $N$ increases, the present value of the remaining annuity payments decreases. As the extreme, if $N = 240$, then there’s no payments left and present value of the remaining annuity payments is zero.

So $N = 173$ is the 1st time that remaining annuity payments is less than $\frac{X}{2}$. The answer is E.
Example 4 (2004 May EA-1 #8)
The present value of a 15-year monthly annuity-immediate is $20,600. Payments are as follows:

- Years 1 – 7: \(X\) per month
- Years 7 – 15: \(X + $300\) per month

Interest rate: 8%, compounded annually.

Calculate \(X\)

Solution

<table>
<thead>
<tr>
<th>Time (t) (year)</th>
<th>0</th>
<th>7</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (t) (month)</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Payment</td>
<td>(X)</td>
<td>(X)</td>
<td>(X)</td>
</tr>
</tbody>
</table>

We’ll break down the original cash flows into the following two streams:

**Stream #1**

\[
P V = X \cdot a_{180|1}^{1} \]

Where \(i = 1.08^{\frac{1}{12}} - 1 = 0.6430301\%\) (monthly effective interest rate)

**Stream #2**

\[
P V = (300 \cdot a_{96|7}^{1}) \cdot 1.08^{-7} = 300 \cdot a_{96|7}^{1} = 300 \cdot a_{96|7}^{1}
\]

So the total PV of the payments is:

\[
X \cdot a_{180|1}^{1} + (300 \cdot a_{96|7}^{1}) \cdot 1.08^{-7} = 20,600
\]

\[
a_{180|1} = 106.4275863, \quad a_{96|7} = 71.45305627
\]

Solving the equation, we get: \(X = 76.03619\)
Increasing annuity

\[
\begin{array}{cccccc}
$1$ & $2$ & $3$ & $4$ & \ldots & $n$
\end{array}
\]

\[n \text{ cash flows}\]

\[\frac{1}{i} n\]

Just remember one increasing annuity formula: 

\[(Ia)_{\overline{m}|i} = \frac{\ddot{a}_{\overline{m}|i} - n \nu^n}{i}\]

Then, calculate the remaining increasing annuity factors as follows:

\[(I\ddot{a})_{\overline{m}|i} = (1+i)(Ia)_{\overline{m}|i}, \quad (I\ddot{s})_{\overline{m}|i} = (1+i)(Ia)_{\overline{m}|i}, \quad (I\ddot{s})_{\overline{m}|i} = (1+i)^{n+1}(Ia)_{\overline{m}|i}\]

Continuously increasing annuity

\[(\overline{Ia})_{\overline{m}|i} = \int_0^a e^{\delta t} dt = \frac{\overline{a}_{\overline{m}} - n \nu^n}{\delta}\]
Decreasing annuity

\[
\begin{array}{cccccc}
\$n & \$\(n-1\) & \ldots & \$4 & \$3 & \$2 & \$1 \\
\hline
\end{array}
\]

\(n\) cash flows

\[
(Da)_{\overline{m}\mid i} \quad (Da\bar{a})_{\overline{m}\mid i} \quad (Ds)_{\overline{m}\mid i} \quad (Ds\bar{s})_{\overline{m}\mid i}
\]

\[
(Da)_{\overline{m}\mid i} = \frac{n - a_{\overline{m}\mid i}}{i}, \quad (Da\bar{a})_{\overline{m}\mid i} = \frac{n - a_{\overline{m}\mid i}}{d}
\]

\[
(Ds)_{\overline{m}\mid i} = \frac{n(1+i)^n - s_{\overline{m}\mid i}}{i}, \quad (Ds\bar{s})_{\overline{m}\mid i} = \frac{n(1+i)^n - s_{\overline{m}\mid i}}{d}
\]

Just remember one decreasing annuity formula: \((Da)_{\overline{m}\mid i} = \frac{n - a_{\overline{m}\mid i}}{i}\)

Then, calculate the remaining increasing annuity factors as follows:

\[
(Da\bar{a})_{\overline{m}\mid i} = (1+i)(Da)_{\overline{m}\mid i}, \quad (Ds)_{\overline{m}\mid i} = (1+i)^n (Da)_{\overline{m}\mid i}, \quad (Ds\bar{s})_{\overline{m}\mid i} = (1+i)^{n+1}(Da)_{\overline{m}\mid i}
\]
Example 1
A company is participating in a project. The cash flows of the project are as follows:

- The company will invest $10 million per year for the 1st three years of the project. The investment will be made continuously.
- The company will receive a cash flow at the end of each year starting from Year 4.
- At the end of Year 4, the company will receive the 1st cash flow of $9 million. This amount will be reduced by $0.5 million for each subsequent year, until the company receives $5 million in a year.
- Starting from that year, the cash flow received by the company will be reduced by $1 million each year, until the company receives zero cash flow.

Calculate the NPV of the project if the discount rate is 12%.

Solution

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$9.0</td>
<td>$8.5</td>
<td>$8.0</td>
<td>$7.5</td>
<td>$7.0</td>
<td>$6.5</td>
<td>$6.0</td>
<td>$5.5</td>
<td>$5.0</td>
<td>$4.0</td>
<td>$3.0</td>
<td>$2.0</td>
<td>$1.0</td>
</tr>
</tbody>
</table>

$-10\left(\ddot{a}_{3\mid 12\%}\right) = -10\left(\frac{1-\nu^3}{\delta}\right)_{t=12\%} = -10\frac{1-1.12^{-3}}{\ln 1.12} = -25.43219763$

The initial investment at t=0 is:

$-10\left(\ddot{a}_{3\mid 12\%}\right) = -10\frac{1-1.12^{-3}}{\ln 1.12} = -25.43219763$

Next, let’s calculate the PV of cash flows from t=4 to t=12.

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$9.0</td>
<td>$8.5</td>
<td>$8.0</td>
<td>$7.5</td>
<td>$7.0</td>
<td>$6.5</td>
<td>$6.0</td>
<td>$5.5</td>
<td>$5.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$9\ddot{a}_{7\mid 12\%} - 0.5(1\ddot{a}_{8\mid 5\%})$
\[ 9 \ddot{a}_{9|12\%} = 9 \frac{1-1.12^{-9}}{1-1.12^{-1}} = 53.708759 \]

\[ 0.5(\ddot{a}_{9|12\%}) = 0.5 \frac{\ddot{a}_{9|12\%} - 8(1.12^{-8})}{0.12} = 0.5 \frac{1-1.12^{-8}}{0.12} - 8(1.12^{-8}) \]

\[ = 0.5 \frac{5.56375654 - 8(1.12^{-8})}{0.12} = 9.71954465 \]

\[ \Rightarrow 9 \ddot{a}_{9|12\%} - 0.5(\ddot{a}_{9|12\%}) = 53.708759 - 9.71954465 = 43.98921325 \]

PV of cash flows from t=4 to t=12:

\[(1.12^{-4})43.98921325 = 27.95594028 \]

Next, let’s calculate the PV of cash flows from t=13 to t=16:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash</td>
<td>$4.0</td>
<td>$3.0</td>
<td>$2.0</td>
<td>$1.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[(Da)_{12\%} = \frac{n-a_{\ddot{a}}}{i} = 4 - a_{\ddot{a}} = 4 - 3.03734935 = 8.02208878 \]

PV is: \[(1.12^{-12})8.02208878 = 2.05907038 \]

So the PV of all cash inflows is:

\[= 27.95594028 + 2.05907038 = 30.01501066 \]

Finally, the NPV of the project is:

\[30.01501066 - 25.43219763 = 4.58281303 \]
**Example 2 (May 2004 SOA EA-1 #3)**

<table>
<thead>
<tr>
<th>Type of Annuity</th>
<th>Annuity immediate, with 19 annual payments.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual Payments</td>
<td>First payment is $1, increasing each year by $1 until payment reaches $10, then decreasing by $1 each year to the final payment of $1</td>
</tr>
<tr>
<td>Interest rate</td>
<td>5% annual effective</td>
</tr>
</tbody>
</table>

In what range is the present value of this annuity at the date of the purchase?

[A] Less than $57
[B] $57 but less than $60
[C] $60 but less than $63
[D] $63 but less than $66
[E] $66 or more

**Solution C**

The fastest solution is to use the cash flow worksheet in BA II Plus/BA II Plus Professional. Enter the following into Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash flow</th>
<th>Amt</th>
<th>Frequency</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>CF0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>C01</td>
<td>$1</td>
<td>F01</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>C02</td>
<td>$2</td>
<td>F02</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>C03</td>
<td>$3</td>
<td>F03</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>C04</td>
<td>$4</td>
<td>F04</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>C05</td>
<td>$5</td>
<td>F05</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>C06</td>
<td>$6</td>
<td>F06</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>C07</td>
<td>$7</td>
<td>F07</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>C08</td>
<td>$8</td>
<td>F08</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>C09</td>
<td>$9</td>
<td>F09</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>C10</td>
<td>$10</td>
<td>F10</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>C11</td>
<td>$9</td>
<td>F11</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>C12</td>
<td>$8</td>
<td>F12</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>C13</td>
<td>$7</td>
<td>F13</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>C14</td>
<td>$6</td>
<td>F14</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>C15</td>
<td>$5</td>
<td>F15</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>C16</td>
<td>$4</td>
<td>F16</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>C17</td>
<td>$3</td>
<td>F17</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>C18</td>
<td>$2</td>
<td>F18</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>C19</td>
<td>$1</td>
<td>F19</td>
<td>1</td>
</tr>
</tbody>
</table>
Though the above table lists the cash flow frequencies from F01 to F19, you really don’t need to enter any cash flow frequency. If a user doesn’t enter the # of cash flows, BA II Plus/BA II Plus Profession automatically sets the # of a cash flow to one. This should be fine because all our cash flow frequencies are one.

Next, set the interest rate to 5%. You should get:

\[
NPV = 62.60644983
\]

**Alternative method (a little slower, but not too bad)**

We break down the original cash flows into two streams:

Stream #1: Increasing annuity from t=1 to t=9
Stream #2: Decreasing annuity from t=10 to t=19

We then separately calculate the present value of each stream and find the total present value.

The present value of Stream #1: \((Ia)_{0\%5}\)

To find the present value of Stream #2, we first calculate the present value of this stream at t=9; the PV should be \((Da)_{0\%5}\). Then, we discount this PV to t=0.

The present value of Stream #2: \(v^9 (Da)_{0\%5}\)

The PV of the original cash flows is:

\[
(Ia)_{0\%5} + v^9 (Da)_{0\%5} = 33.23465027 + 29.37179956 = 62.60644983
\]

\(33.23465027 + 29.37179956 = 62.60644983\)
Example 3 (May 2001 SOA EA-1 #7)

Repayment schedule for a loan:

<table>
<thead>
<tr>
<th>End of each odd numbered year</th>
<th>Amount of repayment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$100</td>
</tr>
<tr>
<td>3</td>
<td>$300</td>
</tr>
<tr>
<td>5</td>
<td>$500</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>X</td>
<td>$100X</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>25</td>
<td>$2,500</td>
</tr>
</tbody>
</table>

Interest rate: 6% per year, compounded annually

$A$ is the total of the payments to be made after the 15th year.

$B$ is the present value of the remaining payments as of the beginning of the 16th year.

In what range is $A - B$?

(A) Less than $3,120
(B) $3,120 but less than $3,150
(C) $3,150 but less than $3,180
(D) $3,180 but less than $3,210
(E) $3,210 or more

Solution C
Let’s set $100 as one unit of money.
The interest rate per 2 years is: $i = (1.06)^2 - 1 = 12.36\%$

<table>
<thead>
<tr>
<th>Time $t$ (year)</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reset time $t$ (2 years)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Repayment</td>
<td>$17$</td>
<td>$19$</td>
<td>$21$</td>
<td>$23$</td>
<td>$25$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$A = 17 + 19 + 21 + 23 + 25 = 105$

Next, we’ll calculate $B$. Please note that $B$ is the PV as of the beginning of the 16th year (i.e. end of the 15th year). So $B$ is the PV at $t = 15$, not at $t = 16$. 
You can use the increasing annuity formula to calculate $B$. However, that calculation is overly complex. A simple approach is this:

$$B = 17v + 19v^2 + 21v^3 + 23v^4 + 25v^5$$

$$= \frac{17}{1.1236} + \frac{19}{1.1236^2} + \frac{21}{1.1236^3} + \frac{23}{1.1236^4} + \frac{25}{1.1236^5} = 73.37424439$$

$$A - B = 105 - 73.37 = 31.63 = $3,163$$

The fastest solution is to use BA II Plus Cash Flow Worksheet. Enter the following into Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>CF0</th>
<th>C01</th>
<th>C02</th>
<th>C03</th>
<th>C04</th>
<th>C05</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0</td>
<td>$17</td>
<td>$19</td>
<td>$21</td>
<td>$23</td>
<td>$25</td>
</tr>
</tbody>
</table>

Then set $I=12.36$ (so the interest rate is 12.36%). You should get:

$$NPV = 73.37424439. \quad \Rightarrow \quad B = 73.37424439.$$  

To calculate $B$, simply set $I=0$ (so the interest rate is zero). You should get:

$$NPV = 105. \quad \Rightarrow \quad A = 105 \Rightarrow A - B = 105 - 73.37 = 31.63 = $3,163$$

**Moral of this problem:**

Having an increasing annuity doesn’t mean you have to use the increasing annuity formula.

**Example 4 (May 2004 SOA EA-1 #25)**

Smith buys a 10-year decreasing annuity-immediate with annual payments of 10, 9, 8, ..., 2, 1.

On the same date, Smith buys a perpetuity-immediate with annual payments. For the first 11 years, payments are 1, 2, 3, ..., 11. After year 11, payments remain constant at 11.

At an annual effective interest rate of $i$, both annuities have a present value of $X$.

Calculate $X$. 

Guo FM, fall 2009
Solution
As usual, we draw a cash flow diagram:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>...</th>
<th>∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annuity #1</td>
<td>$10</td>
<td>$9</td>
<td>$8</td>
<td>$7</td>
<td>$6</td>
<td>$5</td>
<td>$4</td>
<td>$3</td>
<td>$2</td>
<td>$1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Annuity #2</td>
<td>$1</td>
<td>$2</td>
<td>$3</td>
<td>$4</td>
<td>$5</td>
<td>$6</td>
<td>$7</td>
<td>$8</td>
<td>$9</td>
<td>$10</td>
<td>$11</td>
<td>$11</td>
<td>$11</td>
<td>$11</td>
</tr>
</tbody>
</table>

Since we don't know the interest rate, we can't use BA II Plus Cash Flow Worksheet. We have to use the formulas for the increasing and decreasing annuity.

\[
(Da)_{\bar{n}|i} = \frac{10 - a_{\bar{n}|i}}{i}
\]

To calculate the PV of annuity 2, we'll break it down into 2 streams:

Stream #1

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>...</th>
<th>∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annuity #2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$11</td>
<td>$11</td>
<td>$11</td>
</tr>
</tbody>
</table>

At \( t=10 \), the PV of this stream is \( \frac{11}{i} \). At \( t=0 \), the PV of this stream is \( \frac{11}{i} \).

Stream #2

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annuity #2</td>
<td>$1</td>
<td>$2</td>
<td>$3</td>
<td>$4</td>
<td>$5</td>
<td>$6</td>
<td>$7</td>
<td>$8</td>
<td>$9</td>
<td>$10</td>
<td></td>
</tr>
</tbody>
</table>

\[
(Ia)_{\bar{n}|i} = \frac{\ddot{a}_{\bar{n}|i} - 10v^{10}}{i}
\]

So at \( t=0 \), the PV of Annuity #2 is \( \frac{11}{i}v^{10} + \frac{\ddot{a}_{\bar{n}|i} - 10v^{10}}{i} \).

We are told that:

\[
\frac{10 - a_{\bar{n}|i}}{i} = \frac{11}{i}v^{10} + \frac{\ddot{a}_{\bar{n}|i} - 10v^{10}}{i} = X
\]

\[
\Rightarrow 10 - a_{\bar{n}|i} = 11v^{10} + \ddot{a}_{\bar{n}|i} - 10v^{10}
\]
\[ \ddot{a}_m + a_{m|i} + v^{10} - 10 = 0 \]

To quickly solve this equation, we'll convert the equation into cash flows:

Convert \( \ddot{a}_m \) into a stream of cash flows:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td></td>
</tr>
</tbody>
</table>

Convert \( a_{m|i} \) into a stream of cash flows:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td></td>
</tr>
</tbody>
</table>

Convert \( v^{10} - 10 \) into a stream of cash flows:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>-$9 $2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td></td>
</tr>
</tbody>
</table>

Sum up the above 3 streams of cash flows:

\[ PV = \ddot{a}_m + a_{m|i} + v^{10} - 10 = 0 \]

To find the interest rate \( i \), we'll use Cash Flow Worksheet. Enter the following into Cash Flow Worksheet:

\[ CF0= -9, \; C01= 1, \; F01 = 10 \]

Press “IRR” “CPT.” You should get: IRR=17.96301385.

So \( i=17.96301385\% \)

\[ X = \frac{10 - a_{m|i}}{i} = \frac{10 - a_{m|i}^{17.96301385\%}}{17.96301385\%} = 30.62092178 \]
Example 5 (May 2000 SOA EA-1 #10 modified)
Term of a 20-year annuity-certain:
  Initial payment: $300 due 1/1/200

Payment patterns:
  • All payments are made January 1
  • Payments increase by $300 each year beginning 1/1/2001 through 1/1/2009
  • Payments decrease by $200 each year beginning 1/1/2001 through 1/1/2019

Interest rate: 7% per year, compounded annually for the 1st 9 years.
  6% per year, compounded annually thereafter.

Calculate the present value of the annuity.

Solution

First, let’s list all of the cash flows.

<table>
<thead>
<tr>
<th>Time t</th>
<th>Date</th>
<th>Payment (Use $100 as one unit of money)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/1/2000</td>
<td>$3</td>
</tr>
<tr>
<td>1</td>
<td>1/1/2001</td>
<td>$6</td>
</tr>
<tr>
<td>2</td>
<td>1/1/2002</td>
<td>$9</td>
</tr>
<tr>
<td>3</td>
<td>1/1/2003</td>
<td>$12</td>
</tr>
<tr>
<td>4</td>
<td>1/1/2004</td>
<td>$15</td>
</tr>
<tr>
<td>5</td>
<td>1/1/2005</td>
<td>$18</td>
</tr>
<tr>
<td>6</td>
<td>1/1/2006</td>
<td>$21</td>
</tr>
<tr>
<td>7</td>
<td>1/1/2007</td>
<td>$24</td>
</tr>
<tr>
<td>8</td>
<td>1/1/2008</td>
<td>$27</td>
</tr>
<tr>
<td>9</td>
<td>1/1/2009</td>
<td>$30</td>
</tr>
<tr>
<td>10</td>
<td>1/1/2010</td>
<td>$28</td>
</tr>
</tbody>
</table>
Here we have two interest rates: 7% from $t = 0$ to $t = 9$ and 6% from $t = 9$ to $t = 19$. As a result, we have to break down the cash flows into two streams. For each stream, we'll directly enter the cash flows into BA II Plus Cash Flow Worksheet – this is the fastest way.

**Stream #1** (Enter the following into Cash Flow Worksheet)

<table>
<thead>
<tr>
<th>Time t</th>
<th>Date</th>
<th>Payment (Use $100 as one unit of money)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>1/1/2009</td>
<td>CF0</td>
</tr>
<tr>
<td>10</td>
<td>1/1/2010</td>
<td>$28</td>
</tr>
<tr>
<td>11</td>
<td>1/1/2011</td>
<td>$26</td>
</tr>
<tr>
<td>12</td>
<td>1/1/2012</td>
<td>$24</td>
</tr>
<tr>
<td>13</td>
<td>1/1/2013</td>
<td>$22</td>
</tr>
<tr>
<td>14</td>
<td>1/1/2014</td>
<td>$20</td>
</tr>
<tr>
<td>15</td>
<td>1/1/2015</td>
<td>$18</td>
</tr>
<tr>
<td>16</td>
<td>1/1/2016</td>
<td>$16</td>
</tr>
<tr>
<td>17</td>
<td>1/1/2017</td>
<td>$14</td>
</tr>
<tr>
<td>18</td>
<td>1/1/2018</td>
<td>$12</td>
</tr>
<tr>
<td>19</td>
<td>1/1/2019</td>
<td>$10</td>
</tr>
</tbody>
</table>
Using the interest rate of 6%, the PV at $t = 9$ of Stream #1 is:
146.8777947

**Stream #2** At $t = 9$, we add the PV of Stream #1 to the cash flow of #30

<table>
<thead>
<tr>
<th>Time t</th>
<th>Date</th>
<th>Date</th>
<th>CF0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/1/2000</td>
<td>$3</td>
<td>CF0</td>
</tr>
<tr>
<td>1</td>
<td>1/1/2001</td>
<td>$6</td>
<td>C01</td>
</tr>
<tr>
<td>2</td>
<td>1/1/2002</td>
<td>$9</td>
<td>C02</td>
</tr>
<tr>
<td>3</td>
<td>1/1/2003</td>
<td>$12</td>
<td>C03</td>
</tr>
<tr>
<td>4</td>
<td>1/1/2004</td>
<td>$15</td>
<td>C04</td>
</tr>
<tr>
<td>5</td>
<td>1/1/2005</td>
<td>$18</td>
<td>C05</td>
</tr>
<tr>
<td>6</td>
<td>1/1/2006</td>
<td>$21</td>
<td>C06</td>
</tr>
<tr>
<td>7</td>
<td>1/1/2007</td>
<td>$24</td>
<td>C07</td>
</tr>
<tr>
<td>8</td>
<td>1/1/2008</td>
<td>$27</td>
<td>C08</td>
</tr>
<tr>
<td>9</td>
<td>1/1/2009</td>
<td>$30+146.8777947=176.8777947</td>
<td>C09</td>
</tr>
</tbody>
</table>

Using the interest rate of 7%, you should get:
$\text{NPV} = 191.4044063 = \$19,140.44$ (one unit=$\$100$)
If you prefer the formula-driven approach, this is how. We still use $100 as one unit of money. We break down the cash flows into two streams. One stream consists of increasing annuity payments from \( t = 0 \) to \( t = 9 \); the other consists of decreasing annuity payments from \( t = 10 \) to \( t = 19 \).

At \( t = 9 \), the PV of the decreasing annuity

\[
\begin{array}{cccccc}
\text{Time } t & 9 & 10 & 11 & 12 & \ldots & 18 & 19 \\
\text{Cash flow} & 30-2(1) & 30-2(2) & 30-2(3) & \ldots & 30-2(9) & 30-2(10) \\
\end{array}
\]

\[
30 a_{10|6\%} - 2(Ia)_{10|6\%} = 146.8777947
\]

At \( t = 0 \), the PV of the increasing annuity

\[
\begin{array}{cccccc}
\text{Time } t & 0 & 1 & 2 & 3 & \ldots & 8 & 9 \\
\text{Cash flow} & 3(1) & 3(2) & 30(3) & 3(4) & \ldots & 3(9) & 3(10) \\
\end{array}
\]

\[
3(Ia)_{10|7\%} = 111.51261773
\]

The total PV at \( t = 0 \) is:

\[
3(Ia)_{10|7\%} + \left[ 30 a_{10|6\%} - 2(Ia)_{10|6\%} \right] 1.07^{-9} = 111.51261773 + 146.8777947(1.07^{-9})
\]

\[
= 191.4044063 = $19,140.44
\]

**Example 6 (May 2005 SOA EA-1 #8)**

\[a_{2n|t} = 8.00407\]

\[-t \cdot a_{2n|t} = 8.63279\]

\(i = \)the annual effective interest, compounded annually.

Calculate \(i\).

**Solution**

\(a_{2n|t}\) and \(-t \cdot a_{2n|t}\) are symbols for deferred annuities. In a deferred annuity, all the cash flows are shifted rightwards. For example, this is the cash flow diagram for \(a_{2n|t}\):
To draw the diagram for $a_{2n\mid i}$, an $n$-year deferred annuity, we simply shift all the above cash flows rightwards by $n$ units of time (so the 1$^{\text{st}}$ cash flow starts at $n+1$, instead of $t = 1$):

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$2n-1$</th>
<th>$2n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$\uparrow$</td>
<td>$$1$</td>
<td>$$1$</td>
<td>...</td>
<td>$$1$</td>
<td>$$1$</td>
</tr>
</tbody>
</table>

$$a_{2n\mid i} = \left( a_{2n\mid i}\right) v^n$$

Similarly, we draw the diagram for $a_{2n-1\mid i}$, (an $n-1$ year deferred annuity) by shifting the cash flows in $a_{2n\mid i}$ to the right by $n-1$ units of time (so the 1$^{\text{st}}$ cash flow starts at $n$, instead of $t = 1$):

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$n$</th>
<th>$n+1$</th>
<th>$n+2$</th>
<th>...</th>
<th>$(2n+1)+(n-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$\uparrow$</td>
<td>$$1$</td>
<td>$$1$</td>
<td>...</td>
<td>$$1$</td>
<td>$$1$</td>
<td>$$1$</td>
<td>$$1$</td>
<td>$$1$</td>
</tr>
</tbody>
</table>

$$a_{2n-1\mid i} = \left( a_{2n\mid i}\right) v^{n-1}$$

Come back to the problem. Let’s compare $n\mid a_{2n\mid i}$ and $n-1\mid a_{2n\mid i}$:

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$n$</th>
<th>$n+1$</th>
<th>$n+2$</th>
<th>...</th>
<th>$(2n+1)+(n-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$\uparrow$</td>
<td>$$1$</td>
<td>$$1$</td>
<td>...</td>
<td>$$1$</td>
<td>$$1$</td>
<td>$$1$</td>
<td>$$1$</td>
<td>$$1$</td>
</tr>
</tbody>
</table>
$n-1[a_{2n+1}]$ has $2n+1$ cash flows starting from $t = n$ and ending at $t = (2n+1) + (n-1) = 3n$.

$n[a_{2n}]$ has $2n$ cash flows starting from $t = n+1$ and ending at $t = 2n + n = 3n$.

So $n-1[a_{2n+1}]$, has all the cash flows in $n[a_{2n}]$, except $n-1[a_{2n+1}]$ has one additional cash flow at $t = n$. So we have:

$$n-1[a_{2n+1}] = n[a_{2n}] + v^n$$

$$\Rightarrow \quad v^n = n-1[a_{2n+1}] - n[a_{2n}] = 8.63279 - 8.00407 = 0.62872$$

$$n[a_{2n}] = \left( a_{2n} \right) v^n = \frac{1 - v^{2n}}{i} v^n,$$

$$\Rightarrow \quad 8.00407 = \frac{1 - 0.62872^2}{i} (0.62872), \quad i = 4.75\%$$
Chapter 4  Calculator tips

Best calculators for Exam FM

SOA/CAS approved calculators:

BA-35, BA II Plus, BA II Plus Professional, TI-30X, TI-30Xa, TI-30X II (IIS solar or IIB battery).

Best calculators for Exam P: BA II Plus, BA II Plus Professional, TI-30X IIS.

You should bring two calculators to the exam room -- BA II Plus Professional and TI-30 IIS. BA II Plus Professional is good for general calculations and the time-value-of-money calculations. TI-30 IIS is good for general calculations.

Even if you have BA II Plus, you might want to buy a BA II Plus Professional.

New features added in BA II Plus Professional.

We are only concerned with features relevant to Exam FM:

<table>
<thead>
<tr>
<th>BA II Plus Professional</th>
<th>BA II Plus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net Future Value (nice feature)</td>
<td>Doesn’t have this.</td>
</tr>
<tr>
<td>Modified Duration</td>
<td>Doesn’t have this.</td>
</tr>
<tr>
<td>In TVM, the default values are P/Y =1 and C/Y=1 (nice improvement over BA II Plus)</td>
<td>In TVM, the default values are P/Y =12 and C/Y=12 (This is a pain)</td>
</tr>
</tbody>
</table>

How to reset calculators to their best conditions for FM

According to exam rules, when you use BA II Plus, BA II Plus Professional, and TI-30X IIS for an SOA or CAS exam, exam proctors on site will need to clear the memories of your BA II Plus, BA II Plus Professional, and TI-30X IIS. Typically, a proctor will clear your calculator’s memories by resetting the calculator to its default setting. This is done by pressing “2nd” “Reset” “Enter” for BA II Plus and by simultaneously pressing “On” and “Clear” for TI-30X IIS and TI-30X IIB. You will need to know how to adjust the settings of BA II Plus and TI-30X IIS to your best advantage for the exam.
Best settings for BA II Plus

<table>
<thead>
<tr>
<th>Default setting</th>
<th>Optimal setting</th>
<th>Keystrokes to change the default setting to the optimal setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Display 2 decimal places. Enter 0.12345×2, you’ll get 0.25.</td>
<td>Display 8 decimal places. Enter 0.12345×2, you’ll get 0.24690000.</td>
<td><code>2nd</code> [Format] 8 Enter</td>
</tr>
<tr>
<td>Use the chain method. The calculator calculates numbers in the order that you enter them. If you enter 2+3×10, you’ll get 50. The calculator first calculates 2+3=5. Then it calculates 5×10=50.</td>
<td>Use AOS (Algebraic operating system). The calculator follows the standard rules of algebraic hierarchy in its calculation. If you enter 2+3×10, you’ll get 32. The calculator first calculates 3×10. Then it calculates 30+2=32.</td>
<td><code>2nd</code> [FORMAT], keep pressing multiple times until you see “Chn.” Press <code>2nd</code> [ENTER] (if you see “AOS”, your calculator is already in AOS, in which case press <code>[CLR Work]</code>)</td>
</tr>
<tr>
<td>Set P/Y=12 and C/Y=12 in TVM (time value of money) Worksheet. P stands for payment. C stands for compounding. Y stands for year. C/Y=12 means 12 compounding periods per year. If you enter I/Y=6 (i.e., set 6% annual interest rate), the calculator interprets this as a nominal rate compounding monthly and uses an interest rate of 6%/12=0.5% per month in its calculation. P/Y=12 means 12 payment in a year. If you enter 30 <code>2nd</code> xP/Y, you’ll get 360. This means that you are paying off a loan through 360 monthly payments. The setting of P/Y=12 and C/Y=12 is useful occasionally and harmful in majority of the times. In the heat of the exam, you can easily forget to switch settings.</td>
<td>Set P/Y=1 and C/Y=1. If you enter 6% per period (per year, per month, per day, etc), you’ll get 6% per period (per year, per month, per day, etc). If you enter 30 payments, you’ll get 30 payments, not 360 payments. <strong>Always use the setting P/Y=1 and C/Y=1. Under no circumstance should you change this setting.</strong> What if you are paying off a loan through monthly payments? Simply enter the monthly interest rate and the # of monthly payments into BA II Plus TVM.</td>
<td><code>2nd</code> P/Y 1 Enter <code>2nd</code> C/Y 1 Enter</td>
</tr>
</tbody>
</table>

BA II Plus Professional – the default setting is 2 decimal display, the chain method, and P/Y=1 and C/Y=1. As a result, you just need to set BA II Plus Professional to display 8 decimal places and the AOS. You don’t need to set P/Y=1 and C/Y=1 because this is the default setting.
AOS is more powerful than the chain method. For example, if you want to find $1 + 2e^3 + 4\sqrt{5}$, under AOS, you need to enter

$$1 + 2 \times 3 \ 2^{nd} e^x + 4 \times 5 \sqrt{x} \quad \text{(the result is about 50.1153)}$$

Under the chain method, to find $1 + 2e^3 + 4\sqrt{5}$, you have to enter:

$$1 + (2 \times 3 \ 2^{nd} e^x) + (4 \times 5 \sqrt{x})$$

AOS is better because the calculation sequence under AOS is the same as the calculation sequence in the formula. In contrast, the calculation sequence in the chain method is cumbersome.

TI-30X IIS --- You need change only one item on your once the proctor resets it. In its default settings, TI-30X IIS displays two decimal places. You should set it to display 8 decimal places. Press 2^{nd} Fix. The choose “8.”

The power of the TI-30X IIS lies in its ability to display the data and formula entered by the user. This “what you type is what you see feature” allows you to double check the accuracy of your data entry and of your formula. It also allows you to redo calculations with modified data or modified formulas.

For example, if you want to calculate $2e^{-2.5} - 1$, as you enter the data in the calculator, you will see the display:

$$2e^{(-2.5)} - 1$$

If you want to find out the final result, press the “Enter” key and you will see:

$$2e^{(-2.5)} - 1$$

-0.835830003

So $2e^{(-2.5)} - 1 = -0.835830003$

After getting the result of -0.835830003, you realize that you made an error in your data entry. Instead of calculating $2e^{(-2.5)} - 1$, you really wanted to calculate $2e^{(-3.5)} - 1$. To correct the data entry error, you simply change “-2.5” to “-3.5” on your TI-30X IIS. Now you should see:
\[ 2e^{-(3.5)} - 1 \]
\[ -0.939605233 \]

With the online display feature, you can also reuse formulas. For example, a problem requires you to calculate \( y = 2e^{-x} - 1 \) for \( x_1 = 5, \ x_2 = 6, \) and \( x_3 = 7. \) There is no need to do three separate calculations from scratch. You enter \( 2e^{-(5)} - 1 \) into the calculator to calculate \( y \) when \( x = 5. \) Then you modify the formula to calculate \( y \) when \( x = 6 \) and \( x = 7. \)

**Compound interest**

**Problem 1**

Today Mary deposits $23.71 into a bank account and earns 6% annual effective. Calculate the balance of her bank account 2 years from today.

**Solution**

The formula for simple interest rate is:  \[ A(n) = A(0)(1+i)^n \]

We are given: \( A(0) = 23.71, \ i = 6\%, \ n = 2. \)

\[ \Rightarrow A(2) = 23.71(1+6\%)^2 \]

**Method 1  BA II Plus/BA II Plus Professional**

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystrokes</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assume you already set the calculator to display 8 decimal place and to use AOS.</td>
<td>Omitted</td>
<td>Omitted</td>
</tr>
<tr>
<td>Calculate ( 23.71(1+6%)^2 )</td>
<td>23.71×</td>
<td>23.710000000</td>
</tr>
<tr>
<td></td>
<td>( 6%</td>
<td>0.060000000</td>
</tr>
<tr>
<td></td>
<td>+1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>)</td>
<td>1.060000000</td>
</tr>
<tr>
<td></td>
<td>( x^2 )</td>
<td>1.123600000</td>
</tr>
<tr>
<td></td>
<td>( = )</td>
<td>26.640556000</td>
</tr>
</tbody>
</table>
Watch out the % operator. In BA II Plus/BA II Plus Professional, $a \pm b\%$ is calculated as $a(\pm b\%)$. For example, if you enter 100+5%, you'll get 105, not 100.05. If you enter 100-5%, you'll get 95, not 99.95%. This is not a problem if you want to calculate $(1+6\%)$. However, if you want to calculate $(2+6\%)$ and you enter $(2+6\%)$, you'll get 2.12. To calculate $2+6\%$, you can enter $6\%+2$, which gives you 2.06. So to avoid any mistakes, always enter $b\%+a$ if you want to calculate $a+b\%$.

Please note that if you are to calculate $23.71(1.06)^{5.6}$, you can enter $23.71\times 1.06^{5.6}$. You should get 32.85824772.

In BA II Plus/BA II Plus Professional, “)” is the same as “=”. You can solve the problem using the following calculator key strokes:

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystrokes</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate</td>
<td>$23.71(1+6%)^2$</td>
<td>$23.71\times$ $23.71000000$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1+$ $1.00000000$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$6%$ $0.06000000$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$)$ $1.06000000$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$^2$ $1.12360000$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$)$ $26.64055600$</td>
</tr>
</tbody>
</table>

**Method 2 --- TI-30 IIS**

- Set to display 8 decimal places
- Double check your data entry. Press ↑ →. Double check that you indeed entered $23.71(1+6\%)^2$. Yes, you did. So the result 26.64055600 is correct.

Please note that if you are to calculate $23.71(1+6\%)^{5.6}$, you can enter $23.71(1+6\%)^{5.6}$. You should get 32.85824772. Please also note that TI-30 IIS calculates $a+b\%$ as $a+b\%$. For example, if you enter 2+6%, you'll get 2.06.
Method 3 --- Use Δ% Worksheet of BA II Plus/BA II Plus Professional

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystroke</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Choose Δ% Worksheet</td>
<td>2nd [Δ%]</td>
<td>OLD= (old content)</td>
</tr>
<tr>
<td>Clear worksheet</td>
<td>2nd [CLR Work]</td>
<td>OLD=0.00000000</td>
</tr>
<tr>
<td>Enter principal.</td>
<td>23.71 Enter</td>
<td>OLD=23.71000000</td>
</tr>
<tr>
<td>Enter the # of compounding period</td>
<td>↑</td>
<td>#PD=1.00000000</td>
</tr>
<tr>
<td>(can be a integer or fraction)</td>
<td></td>
<td>(The default period is one)</td>
</tr>
<tr>
<td></td>
<td>2 Enter</td>
<td>#PD=2.00000000</td>
</tr>
<tr>
<td>Enter the interest rate</td>
<td>↑</td>
<td>% CH=0.00000000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(The default is zero)</td>
</tr>
<tr>
<td></td>
<td>6 Enter</td>
<td>% CH=6.00000000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Enter 6, not 6%)</td>
</tr>
<tr>
<td>Calculate the balance</td>
<td>↑</td>
<td>NEW=0.00000000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(The default is zero)</td>
</tr>
<tr>
<td></td>
<td>CPT</td>
<td>NEW=26.64055600</td>
</tr>
</tbody>
</table>

Note – The # of compounding periods can be a fraction. If you are to calculate $23.71(1+6%)^{5.6}$, you simply enter the following:

OLD=23.71, % CH=6, # PD=5.6.

You should get NEW=32.85824772
Method 4 --- Use TVM Worksheet of BA II Plus/BA II Plus Professional

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystroke</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set C/Y=1 and P/Y=1. We always set C/Y=1 and P/Y=1 before using TVM.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Clear TVM Worksheet</td>
<td>2nd [CLR TVM]</td>
<td>0.00000000</td>
</tr>
<tr>
<td>Enter principal.</td>
<td>23.71 PV Enter</td>
<td>PV=23.71000000</td>
</tr>
<tr>
<td>Enter interest rate.</td>
<td>6 I/Y Enter</td>
<td>I/Y=6.00000000</td>
</tr>
<tr>
<td>Enter the # of compounding period (can be an integer or fraction)</td>
<td>2 N Enter</td>
<td>N=2.00000000</td>
</tr>
<tr>
<td>Calculate the accumulated value.</td>
<td>CPT FV</td>
<td>FV= - 26.64055600</td>
</tr>
</tbody>
</table>

Note:
(1) The negative sign in FV= - 26.64055600 means that you are paying off a loan of $26.64055600. As a general rule, a positive sign in TVM means money coming from someone else’s pocket and going to your pocket; a negative sign in TVM means money leaving your pocket and going to someone else’s pocket.

Together PV=23.71, I/Y=6%,N=2, and FV= -26.64055600 mean this:

- at time zero $23.71 flows from someone’s pocket to your pocket; in other words, you borrow $23.71 from someone.

- you use the money for 2 years and are charged with 6% interest rate.

- at the end of Year 2, $26.64055600 comes from your pocket to someone else’s pocket. In other words, you pay the lender $26.64055600 at the end of Year 2. $26.64055600 is greater than $23.71 because it includes interest payment.

(2) You can also set PV= - 23.71, I/Y=6%,N=2. This will give you FV= 26.64055600. You lend $23.71 for 2 years with 6% interest. At the end of Year 2, you receive $26.64055600.

(3) Because there are no regular payments in this problem, it does not matter whether you select end-of-period payments by setting 2nd END or you select beginning-of-period payments by setting 2nd BGN.
(4) The # of compounding periods can be a fraction. If you are to calculate $23.71(1+6\%)^{5.6}$, you simply enter the following:

$$PV=23.71, \ I/Y=6\%, N=5.6$$

You should get $FV= - 32.85824772$

**Annuity**

**Problem 1**

Calculate $a_{\text{im}6\%}$

**Method 1 – Use TVM of BA II Plus/BA II Plus Professional**

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystroke</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set to display 8 decimal places.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Make sure that you set $C/Y=1$ and $P/Y=1$. <strong>This is the golden rule. Never break this rule under any circumstances.</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Clear TVM Worksheet</td>
<td>$2^{nd}$ [CLR TVM]</td>
<td>0.00000000</td>
</tr>
<tr>
<td>Use the immediate annuity function (rather than annuity due).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>If last time when TVM was used and TVM was in the annuity due mode, you need to set it to the immediate annuity.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Forgetting to do so gives you a wrong result without you knowing it.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Enter the level payment.</td>
<td>1 [Payment] [Enter]</td>
<td>PMT=1.00000000</td>
</tr>
<tr>
<td>(We enter a positive 1 to indicate that we want to receive a payment of $1$.)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Enter # of payments</td>
<td>10 [Enter]</td>
<td>N=10.00000000</td>
</tr>
</tbody>
</table>
Calculate PV when the interest rate is zero. This is an extra step to double check whether we entered the right cash flow amounts and the correct # of payments.

\[
\text{CPT PV} \quad \text{PV} = 10.00000000
\]

(The result is correct. The PV of 10 level payments of $1 @ i=0 is 10.)

The negative sign indicates cash outgo. In other words, if we spend $10 now to buy a 10 year annuity immediate @ \( i=0 \), we’ll receive $1 per year.

**Enter interest rate.**

\[6 \ \text{I/Y Enter} \quad \text{I/Y} = 6.00000000\]

**Enter the # of compounding period (can be an integer or fraction).**

\[10 \ \text{N Enter} \quad \text{N} = 10.00000000\]

**Calculate the present value.**

\[
\text{CPT PV} \quad \text{PV} = -7.36008705
\]

So \( a_{10|6\%} = 7.36008705 \)

**Method 2 – Use Cash Flow Worksheet of BA II Plus/BA II Plus Professional**

Before using CF Worksheet, let’s identify all of the cash flows:

<table>
<thead>
<tr>
<th>Time ( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$0</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
</tr>
</tbody>
</table>

\[ a_{10|6\%} = ? \]

<table>
<thead>
<tr>
<th><strong>Procedure</strong></th>
<th><strong>Keystroke</strong></th>
<th><strong>Display</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Set to display 8 decimal places.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use Cash Flow Worksheet</td>
<td>( \text{CF} )</td>
<td>CF0=(old content)</td>
</tr>
<tr>
<td>Clear Worksheet</td>
<td>( \text{2nd [CLR WORK]} )</td>
<td>CF0=0.00000000</td>
</tr>
<tr>
<td>Enter the cash flow at ( t=0 ). Because the cash flow is zero, we don’t need to enter anything. Just press the down arrow.</td>
<td>( \downarrow )</td>
<td>CF0=0.00000000</td>
</tr>
</tbody>
</table>
Enter the dollar amount of the level payments.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>C01</td>
<td>0.00000000</td>
</tr>
<tr>
<td>1 Enter</td>
<td>C01=1.00000000</td>
</tr>
</tbody>
</table>

Enter the # of level payments. We have 10 level payments from \( t=1 \) to \( t=10 \).

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>F0</td>
<td>1.00000000</td>
</tr>
<tr>
<td>10 Enter</td>
<td>F0= 10.00000000</td>
</tr>
</tbody>
</table>

Use NVP portion of Cash Flow Worksheet.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>NPV</td>
<td>I=0.00000000</td>
</tr>
<tr>
<td></td>
<td>(The default interest rate is zero.)</td>
</tr>
</tbody>
</table>

Calculate NPV when the interest rate is zero. This is an extra step to double check whether we entered the right cash flow amounts and the correct # of payments.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>NPV</td>
<td>0.00000000</td>
</tr>
<tr>
<td></td>
<td>(The default # is 0.)</td>
</tr>
<tr>
<td>CPT</td>
<td>NPV= 10.00000000</td>
</tr>
<tr>
<td></td>
<td>(The result is correct. The PV of 10 level payments of $1 @ ( i=0 ) is 10.)</td>
</tr>
</tbody>
</table>

Enter the interest rate.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>I=0.00000000</td>
<td></td>
</tr>
<tr>
<td>6 Enter</td>
<td>I=6.00000000</td>
</tr>
</tbody>
</table>

Calculate NPV when the interest rate is zero. This is an extra step to double check whether we entered the right cash flow amounts and the correct # of payments.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>NPV</td>
<td>10.00000000</td>
</tr>
<tr>
<td></td>
<td>(This is the previous NPV we calculated last time @ ( i=0 ).)</td>
</tr>
<tr>
<td>CPT</td>
<td>NPV= 7.36008705</td>
</tr>
</tbody>
</table>

So \( a_{\bar{m}|6\%} = 7.36008705 \)

**Problem 2**

Calculate \( \bar{d}_{\bar{m}|6\%} \)

**Solution**
Method 1 – Use TVM Worksheet. The calculation procedures are identical to the procedure for calculating $a_{\overline{10}|6\%}$, except that we need to use the annuity due mode.

You should get: $\ddot{a}_{\overline{10}|6\%} = 7.80169227$

Method 2 – Use CF Worksheet.

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

The calculation procedures are identical to the procedure for calculating $a_{\overline{10}|6\%}$, except that we need to enter the following cash flows:

CF0=1; C01=1; F01=9.
You should get: $\ddot{a}_{\overline{10}|6\%} = 7.80169227$

Problem 3 Calculate $s_{\overline{10}|6\%}$

Solution

Method 1 – Use TVM

- Display 8 decimal places.
- Set C/Y=1, P/Y=1.
- Use the immediate annuity mode.
- Set PMT=1, N=10.
- CPT FV. You should get FV= - 10 @ $i=0$. The result is correct.
- Set I/Y=6. Press: CPT, FV.

You should get FV= - 13.18079494. So $s_{\overline{10}|6\%} = 13.18079494$
Method 2 – use Cash Flow Worksheet.

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$0</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td></td>
</tr>
</tbody>
</table>

\[ s_{10|6\%} = ? \]

Use BA II Plus Professional

Keystrokes

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystroke</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set to display 8 decimal places.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use Cash Flow Worksheet</td>
<td>CF</td>
<td>CF0=(old content)</td>
</tr>
<tr>
<td>Clear Worksheet</td>
<td>[2nd] [CLR WORK]</td>
<td>CF0=0.00000000</td>
</tr>
<tr>
<td>Enter the cash flow at ( t=0 ).</td>
<td>↓</td>
<td>CF0=0.00000000</td>
</tr>
<tr>
<td>Enter the dollar amount of the level payments.</td>
<td>↓</td>
<td>C01 0.00000000</td>
</tr>
<tr>
<td></td>
<td>1 Enter</td>
<td>C01=1.00000000</td>
</tr>
<tr>
<td>Enter the # of level payments. We have 10 level payments from ( t=1 ) to ( t=10 ).</td>
<td>↓</td>
<td>F01= 1.00000000 (The default # is 1.)</td>
</tr>
<tr>
<td></td>
<td>10 Enter</td>
<td>F01= 10.00000000</td>
</tr>
<tr>
<td>Use FVP portion of Cash Flow Worksheet.</td>
<td>NPV</td>
<td>I=0.00000000 (The default interest rate is zero.)</td>
</tr>
<tr>
<td></td>
<td>↓</td>
<td>NPV= 0.00000000 (The default # is 0.)</td>
</tr>
<tr>
<td></td>
<td>↓</td>
<td>NFV= 0.00000000 (The default net future value is zero.)</td>
</tr>
<tr>
<td>Calculate NPV when the interest rate is zero. This is an extra step to double check whether we entered the right cash flow amounts and the correct # of payments.</td>
<td>CPT</td>
<td>NPV= 10.00000000 (The result is correct. The NFV of 10 level payments of $1 @ ( i=0 ) is 10.)</td>
</tr>
<tr>
<td>Enter the interest rate.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Guo FM, fall 2009
Calculate NPV when the interest rate is zero. This is an extra step to double check whether we entered the right cash flow amounts and the correct # of payments.

<table>
<thead>
<tr>
<th>NPV</th>
<th>I=0.00000000 (The default interest rate is zero.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 Enter</td>
<td>I=6.00000000</td>
</tr>
</tbody>
</table>

NPV= 10.00000000
(NPV is 10 @ i=0. Even though we entered i=6%, we didn’t press CPT. If CPT is not pressed, BA II Plus Professional does not update the NPV using the latest interest rate. It merely displays the old NPV value.)

NFV= 10.00000000
(NFV is 10 @ i=0. Even though we entered i=6%, we didn’t press CPT. If CPT is not pressed, BA II Plus Professional does not update the NFV using the latest interest rate. It merely displays the old NPV value.)

CPT
NFV= 13.18079494 (BA II Plus Professional calculates NFV using the latest interest rate.)

So \( s_{10\%} = 13.18079494 \)

Use BA II Plus

BA II Plus does not have NFV. However, we can still calculate \( s_{10\%} \) using the following relationship:

\[
s_{\overline{n|m}} = a_{\overline{n|m}} (1+i)^n
\]

We have \( a_{\overline{10|6\%}} = 7.36008705 \)

\[
\Rightarrow (Is)_{\overline{10\%}} = 7.36008705(1.06^{10}) = 13.18079494
\]

Problem 4

Calculate \( \hat{s}_{10\%} \)
Solution

Method 1 – Use TVM

- Display 8 decimal places.
- Set C/Y=1, P/Y=1.
- Use the annuity due mode.
- Set PMT=1, N=10.
- CPT FV. You should get FV= - 10 @ \( i \). The result is correct.
- Set I/Y=6. Press: CPT, FV.

You should get FV= - 13.97164264. So \( \ddot{s}_{6|10\%} = 13.97164264 \)

Method 2 – use Cash Flow Worksheet.

BA II Plus Professional

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$0</td>
<td></td>
</tr>
</tbody>
</table>

Please note BA II Plus Professional Cash Flow Worksheet always accumulates cash flows to the final payment time. In order to calculate \( \ddot{s}_{6|10\%} \) (which accumulates value to \( t=10 \)), we needed to add an additional cash flow of zero at \( t=10 \). This tells BA II Plus Professional Cash Flow Worksheet to use \( t=10 \) as the ending time to accumulate cash flows. If we don’t add this additional cash flow of zero to \( t=10 \), BA II Plus will calculate the accumulated value to \( t=9 \).

Keystrokes – BA II Plus Professional only (not for BA II Plus):

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystroke</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set to display 8 decimal places.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use Cash Flow Worksheet</td>
<td>CF</td>
<td>CF0=(old content)</td>
</tr>
<tr>
<td>Clear Worksheet</td>
<td>2nd [CLR WORK]</td>
<td>CF0=0.00000000</td>
</tr>
</tbody>
</table>
Enter the cash flow at \( t=0 \).

<table>
<thead>
<tr>
<th>Enter</th>
<th>CF0 = 1.00000000</th>
</tr>
</thead>
</table>

Enter the dollar amount of the payments.

<table>
<thead>
<tr>
<th>↓</th>
<th>C01 = 0.00000000</th>
</tr>
</thead>
</table>

Enter the # of level payments. We have 9 level payments from \( t=1 \) to \( t=9 \).

<table>
<thead>
<tr>
<th>↓</th>
<th>F0 = 1.00000000</th>
</tr>
</thead>
</table>

Enter zero cash flow at \( t=10 \), increasing the # of time periods by one.

<table>
<thead>
<tr>
<th>↓</th>
<th>C02 = 0.00000000</th>
</tr>
</thead>
</table>

If we omit this step, BA II Plus Professional will accumulate values to \( t=9 \) and gives us the final result of \( \text{NPV}=13.18079494 \). You can verify this yourself.

Use FVP portion of Cash Flow Worksheet.

<table>
<thead>
<tr>
<th>NPV</th>
<th>I = 0.00000000</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>↓</th>
<th>NPV = 0.00000000</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>↓</th>
<th>NFV = 0.00000000</th>
</tr>
</thead>
</table>

(The default interest rate is zero.)

(The default # is 0.)

(The default net future value is zero.)
Calculate NPV when the interest rate is zero. This is an extra step to double check whether we entered the right cash flow amounts and the correct # of payments.

<table>
<thead>
<tr>
<th>CPT</th>
<th>NPV= 10.00000000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(The result is correct. The NFV of 10 level payments of $1 @ i=0 is 10.)</td>
</tr>
</tbody>
</table>

Enter the interest rate.

<table>
<thead>
<tr>
<th>NPV</th>
<th>I=0.00000000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(The default interest rate is zero.)</td>
</tr>
</tbody>
</table>

6 Enter

<table>
<thead>
<tr>
<th>I=6.00000000</th>
</tr>
</thead>
</table>

Calculate NPV when the interest rate is zero. This is an extra step to double check whether we entered the right cash flow amounts and the correct # of payments.

<table>
<thead>
<tr>
<th>↓</th>
<th>NPV= 10.00000000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(NPV is 10 @ i=0. Even though we entered i=6%, we didn’t press CPT. If CPT is not pressed, BA II Plus Professional does not update the NPV using the latest interest rate. It merely displays the old NPV value.)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>↓</th>
<th>NFV= 10.00000000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(NFV is 10 @ i=0. Even though we entered i=6%, we didn’t press CPT. If CPT is not pressed, BA II Plus Professional does not update the NFV using the latest interest rate. It merely displays the old NPV value.)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CPT</th>
<th>NFV= 13.97164264</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(BA II Plus Professional calculates NFV using the latest interest rate.)</td>
</tr>
</tbody>
</table>

So \( i_{10\%} = 13.97164264 \)

If we omit the step of entering the cash flow of zero at \( t=10 \), BA II Plus Professional will give us NFV=13.18079494. We can verify that

\[
13.18079494 = 13.97164264 \times 1.06^{-1}
\]

You see that 13.18079494 is the accumulated value at \( t=9 \), while 13.97164264 is the accumulated value at \( t=10 \).
How can we calculate the NFV if we need to accumulate the same cash flows to $t=11$? Now the cash flows diagram is:

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

We simply set $C02=0$, $F02=2$. This will move the ending time from to $t=9$ to $t=11$. We'll get the result that $NPF = 14.80994120$.

We can verify that $\frac{14.80994120}{13.97164264} = 1.06$.

**Use BA II Plus**

BA II Plus does not have NFV. However, we can still calculate $\ddot{s}_{0|\%}$ using the following relationship:

$$\ddot{s}_{0|\%} = \ddot{a}_{0|\%} (1+i)^g$$

We have $\ddot{a}_{0|\%} = 7.80169227$

$$\Rightarrow \ddot{s}_{0|\%} = 7.80169227 (1.06)^{10} = 13.97164263$$

**Loan/bond amortization**

In BA II Plus/BA II Plus Professional, TVM Worksheet is automatically tied to Amortization Worksheet. For a given loan or bond, if you find $PV$ (which is the principal of a loan), $PMT$ (regular payment), $N$ (# of payments) and $I/Y$ (the interest rate) through TVM Worksheet, Amortization Worksheet can generate an amortization schedule for you, splitting each payment into principal and interest.
Problem 1 (#10, Sample FM)

A 10,000 par value 10-year bond with 8% annual coupons is bought at premium to yield an annual effective rate of 6%.

Calculate the interest portion of the 7\text{th} payment.

(A) 632    (B) 642    (C) 651    (D) 660    (E) 667

Solution

One key thing to remember is that a bond is a loan. When you buy this $10,000 par 10 year bond with 8% annual coupons yielding 6%, you lend your money to whoever issued the bond. The money your lend (principal) is the present value of all the future cash flows (10 coupons of $800 each from t=1 to t=10 plus a final cash flow of $10,000 at t=10) discounted at an annual effective 6%. And the borrower (bond issuer) pays back your loan through 10 coupons of $800 each from t=1 to t=10 plus a final cash flow of $10,000 at t=10.

We first draw a cash flow diagram:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$10,000</td>
</tr>
</tbody>
</table>

For BA II Plus/BA II Plus Professional Amortization Worksheet to generate an amortization schedule, we first need to calculate PV. Let’s calculate PV of the bond.

Procedures for using TVM:

- Display 8 decimal places.
- Set C/Y=1, P/Y=1.
- Use the annuity immediate mode.
- Set PMT=800, N=10, FV=10,000. By making PMT and FV positive, we are getting 10 level payments of $800 each year and a final payment of $10,000 at t=10. So we bought the bond and should receive cash flows in the future. This will generate a negative PV,
which is our purchase price of the bond; we pay now to get cash flows in the future.

- Alternatively, we can make PMT= - 800 and FV= - 10,000; we sold the bond. We'll get a positive PV, which is our selling price of the bond; we get cash now but pay cash flows in the future. Either way is fine as long as we make PMT and FV have the same signs.

- CPT PV. This calculates PV @ i =0. We should get PV= - 18,000. The result is correct. If the interest rate is zero, PV is just the sum of all cash flows. PV=800(10)+10,000=18,000.

- Set I/Y=6. Press: CPT, PV.

We should get PV= - 11,472.01741.

Now Amortization Worksheet is ready to generate an amortization schedule for us, splitting each payment into principal and interest.

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystroke</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use Amortization Worksheet.</td>
<td>2nd Amort</td>
<td>P1=old content</td>
</tr>
<tr>
<td>Tell the calculator that we are</td>
<td>7 Enter</td>
<td>P1=7.000000000</td>
</tr>
<tr>
<td>interested in the 7th payment.</td>
<td>↓</td>
<td>So the payment to be split begins at t =7.</td>
</tr>
<tr>
<td></td>
<td>7 Enter</td>
<td>P2=7.000000000</td>
</tr>
<tr>
<td>Find the outstanding balance after the</td>
<td>↓</td>
<td>BAL= - 10,534.60239</td>
</tr>
<tr>
<td>7th payment</td>
<td></td>
<td>BA II Plus/BA II Plus Professional always calculates the outstanding balance after the P2 level payment is made.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>So the payment to be split ends at t =7.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>So the outstanding balance AFTER the 7th payment and immediately before the 8th payment is $10,534.60239. Since PV is negative, the outstanding balance should also be negative.</td>
</tr>
</tbody>
</table>
Find the principal portion of the 7th level payment.

PRN= 158.4187327

BA II Plus/BA II Plus Professional splits the total payments starting from P1 and ending with P2 into the principal repayment and interest payment.

Because we set P1=P2=7, the calculator splits only the 7th payment into principal and interest.

Of the $800 payment at t=7, $158.42 is the repayment of the principal. A positive $158.42 means that we receive $158.42. This makes sense. Our PV is negative (we lent our money at t=0). As a result, we will receive repayment of our principal.

Find the interest portion of the 7th payment.

INT= 641.5812674

Of the $800 payment at t=7, $641.58 is the interest payment. A positive $641.58 means that we receive $641.58.

So the interest portion of the 7th payment is about $642.

We can verify that the results are correct:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$10,000</td>
</tr>
</tbody>
</table>

The outstanding principal immediate after the 6th payment

\[
800a_{\overline{4}|6\%} + 10,000(1.06^{-4}) = 10,693.02112
\]

To verify, in the amortization worksheet, If you set P1=P2=6, you should get BAL= - 10,693.02112.

The interest accrued from t=7 to t=8

\[
10,693.02112(6\%) = 641.58
\]

The principal portion of the 7th payment

\[
800 - 641.58 = 158.42
\]

Please note that Amortization Worksheet uses data you entered or calculated in TVM Worksheet. Whenever you update TVM Worksheet,
Amortization Worksheet is automatically updated. As a result, you don’t need to use 2nd CLR Work to clear Amortization Worksheet. Amortization Worksheet is always in sync with TMV Worksheet. As long as the data in TVM Worksheet is correct, Amortization Worksheet will generate the correct amortization schedule.

**Additional calculations on this problem:**

- **What’s the outstanding balance immediately BEFORE the 7th payment?** Remember that BA II Plus/BA II Plus Professional Amortization Worksheet always calculates the outstanding balance after the P2 level payment is made. So first we find the outstanding balance immediately after the 7th payment is made. So we set P2=7. What about P1? We can set P1=1,2,3,4,5, 6, or 7. In other words, P1 needs to be a positive integer equal to or smaller than P2. We should get BAL=-10,534,60239. Immediately after the 7th payment is made, the outstanding balance is 10,534,60239. So the outstanding balance immediately before the 7th payment is made is 10,534,60239+800=11,334,60239. (Here’s another approach. We already know that the outstanding loan balance immediately after the 6th payment is 10,693,02112. Accumulating this amount with interest for one year is $10,693.02112(1.06)=11,334.60239$. So the outstanding loan balance immediately before the 7th payment is 11,334,6023.)

- **What’s the outstanding balance after the final coupon is paid at t=10?** Without using any calculators, we know that the outstanding balance must be $10,000 after the 10th coupon of $800 is paid. Will Amortization Worksheet produce this result? Let’s check. Set P2=10; P1 can be any positive integer no greater than 10. We get BAL=-10,000.

- **What’s the total principal and interest paid by the bond issuer during the life of the bond?** Remember BA II Plus/BA II Plus Professional always splits the payments starting from P1 and ending with P2 into principal and interest. Let’s set P1=1 and P2=10. We’ll get PRN=1,472.017410 and INT=6,527.982590. Let’s check:

<table>
<thead>
<tr>
<th>The total repayment of the loan (simple sum of all the future cash flows)</th>
<th>10(800)+10,000=18,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>The principal of the loan (the initial price of the bond)</td>
<td>PV=11,472.01741=PRN (OK)</td>
</tr>
</tbody>
</table>
How to split into principal and interest the total payments made during the 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, and 5\textsuperscript{th} payments? Easy! Set $P_1=2$ and $P_2=5$. We get $\text{PRN} = 517.8656978$ and $\text{INT} = 2,682.134302$. So out of the total payments made during the 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, and 5\textsuperscript{th} payments, the principal portion is 517.8656978; the interest portion is 2,682.134302. Let’s check:

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$800$</td>
<td>$800$</td>
<td>$800$</td>
<td>$800$</td>
<td>$800$</td>
<td>$800$</td>
<td>$800$</td>
<td>$800$</td>
<td>$800$</td>
<td>$800$</td>
<td>$10,000$</td>
</tr>
</tbody>
</table>

The outstanding loan immediately AFTER the 1\textsuperscript{st} payment is made

$$800d_{\frac{1}{6}} + 10,000 \left(1 + \frac{1}{1.06}\right)^{-1} = -11,360.33845$$

To verify, in the amortization worksheet, set $P_1=1$ and $P_2=1$. You should get $\text{BAL} = -11,360.33845$.

The outstanding loan immediately AFTER the 5\textsuperscript{th} payment is made

$$800d_{\frac{1}{6}} + 10,000 \left(1 + \frac{1}{1.06}\right)^{-5} = -10,842.47276$$

To verify, in the amortization worksheet, set $P_1=5$ and $P_2=5$. You should get $\text{BAL} = -10,842.47276$.

Reduction of principal due to the 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, and 5\textsuperscript{th} payments

$$11,360.33845 - 10,842.47276 = 517.8656979$$

(The result matches $\text{PRN}=517.8656978$. The slight difference is due to rounding)

Total repayment made in the 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, and 5\textsuperscript{th} payments

$$800(4) = 3,200$$

Total interest paid in the 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, and 5\textsuperscript{th} payments

$$3,200 - 517.8656979 = 2,682.134302$$

The result matches $\text{INT} = 2,682.134302$.

The total interest paid during the life of the bond

$$18,000 - 11,470.01741 = 6,579.982590 \ (\text{OK})$$
**Compare Cash Flow Worksheet with TVM Worksheet**

Pros of using Cash Flow Worksheet over TVM:
- Avoid inadvertently using C/Y=12 and P/Y=12
- Avoid painful switching between the annuity due mode and the annuity immediate mode
- Handle level and non-level payments

Cons of using Cash Flow Worksheet over TVM:
- A candidate can forget that the 1st cash flow in Cash Flow Worksheet is the CF0 (which takes place at $t=0$), not C01 (which takes place at $t=1$).
- A candidate needs to carefully track down the timing of each cash flow.
- TVM is automatically tied to Amortization Worksheet and can generate an amortization schedule; Cash Flow Worksheet is NOT tied to Amortization Worksheet and can NOT generate an amortization schedule.

I recommend that you master both methods. For a non-amortization exam problem, you can use both methods for the same problem and double check your calculations. If you are good at using BA II Plus/BA II Plus Professional, each method takes you only about 10 seconds. You should have time to use both methods for the same problem.

**Increasing annuity**

**Problem 1**
Calculate $(Ia)_{\overline{10}|5\%}$

**Solution**

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
<td>$3$</td>
<td>$4$</td>
<td>$5$</td>
<td>$6$</td>
<td>$7$</td>
<td>$8$</td>
<td>$9$</td>
<td>$10$</td>
</tr>
</tbody>
</table>

We’ll use BA II Plus/BA II Plus Professional Cash Flow Worksheet.
Enter the following:

| CF0=0; C01=1; C02=2; C03=3; C04=4; C05=5;  |
| C06=6; C07=7; C08=8; C09=9; C010=10.          |

You don’t need to enter F01=1, F02=1, …, F10=1. BA II Plus/BA II Plus Professional will automatically set them to one.

This is how BA II Plus/BA II Plus Professional sets the # of level cash flows:

- If a cash flow is zero, the default # of level cash flows is zero.
- If a cash flow is none zero, the default # of level cash flows is 1.
- If, for any cash flow, a user does not specifically enter the # of level cash flows, Cash Flow Worksheet uses the default # of level cash flows.

Next, calculate NPV. You should get 55. This is NPV @ $i=0$. We can verify this is correct:

\[
1 + 2 + 3 + \ldots + 10 = \frac{1}{2} (10)(11) = 55
\]

We have correctly entered the cash flow amounts and the # of cash flows. Next, set $I=6$. We get NPV=$36.96240842$.

Let’s check. We’ll use the formula:

\[
(1a)_n = \frac{\ddot{a}_m - n v^n}{i}
\]

\[
\Rightarrow (1a)_{10\text{\%}} = \frac{\ddot{a}_{10\text{\%}} - 10 \left(1.06^{-10}\right)}{6\%} = \frac{7.80169227 - 5.58394777}{6\%} = 36.96240842
\]

Why bother using the calculator when a formula works fine? The calculator reduces a complex increasing annuity formula to simple keystrokes. As long as you enter the correct data, the calculator will generate the result 100% right. In contrast, if you use the increasing annuity formula, you might miscalculate.

The secret to doing error-free calculations for a complex problem in the heat of the exam is to reduce a complex problem into a simple mechanic solution. We may not always be able to do so. However, if some problems
have mechanic solutions, we prefer to give our brain a rest and use mechanic solutions to solve the complex problems.

Knowing that BA II Plus/BA II Plus Professional can calculate an increasing annuity for us, do we still need to memorize the increasing annuity? Yes. SOA can always set up a problem in such way that a mechanic solution is impossible and some amount of thinking is needed. As a result, we’ll still need to memorize the increasing annuity formula.

**Problem 2**

Calculate \((l \ddagger)_{m\%}\)

**Solution**

\[
\begin{array}{c|ccccccccc}
\text{Time} t & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
\text{Cash flow} & $1 & $2 & $3 & $4 & $5 & $6 & $7 & $8 & $9 & $10 \\
\end{array}
\]

We’ll use BA II Plus/BA II Plus Professional Cash Flow Worksheet. Enter the following:

\[
\begin{align*}
\text{CF0} &= 1; \ C01 = 2; \ C02 = 3; \ C03 = 4; \ C04 = 5; \ C05 = 6; \\
& \quad C06 = 7; \ C07 = 8; \ C08 = 9; \ C09 = 10.
\end{align*}
\]

Calculate NPV. We get NPV=55 @ \(i=0\) (OK).

Set \(I=6\). We get NPV=39.18015293.

Let’s check. We’ll use the formula:

\[
(l \ddagger)_{m\%} = \frac{l m - n v^n}{d} = \frac{i}{d} (l a)_{m}\]

\(\Rightarrow\) \((l \ddagger)_{0.06\%} = \frac{0.06}{1-1.06^{-1}} \times 36.96240842 = 39.18015293\)
Problem 3

Calculate \( (I_s)_{10\%} \)

Solution

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$0</td>
<td>$1</td>
<td>$2</td>
<td>$3</td>
<td>$4</td>
<td>$5</td>
<td>$6</td>
<td>$7</td>
<td>$8</td>
<td>$9</td>
<td>$10</td>
</tr>
</tbody>
</table>

BA II Plus Professional Cash Flow Worksheet:

Enter the following:

CF0=0; C01=1; C02=2; C03=3; C04=4; C05=5; C06=6; C07=7; C08=8; C09=9; C010=10.

Next, calculate NFV. You should get NFV=55 @ \( i=0 \) (OK).

Finally, set \( I=6 \). We get \( NFV=66.19404398 \).

Let’s check. We’ll use the formula:

\[
(I_s)_{\frac{1}{(1+i)^n}} = (Ia)_{\frac{1}{(1+i)^n}} (1+i)^n
\]

\[
\Rightarrow (I_s)_{10\%} = 36.96240842 (1.06^{10}) = 66.19404398
\]

Use BA II Plus Cash Flow Worksheet (which doesn’t have the FPV function)

We use the formula: \( (I_s)_{\frac{1}{(1+i)^n}} = (Ia)_{\frac{1}{(1+i)^n}} (1+i)^n \)

Using Cash Flow Worksheet, we get: \( NPV = (Ia)_{\frac{1}{(1+i)^n}} = 36.96240842 \)

\[
\Rightarrow (I_s)_{10\%} = 36.96240842 (1.06^{10}) = 66.19404398
\]
Problem 4

Calculate \( (I\ddot{s})_{10\%} \)

Solution

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$1$</td>
<td>$2$</td>
<td>$3$</td>
<td>$4$</td>
<td>$5$</td>
<td>$6$</td>
<td>$7$</td>
<td>$8$</td>
<td>$9$</td>
<td>$10$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Use BA II Plus Professional Cash Flow Worksheet.

Enter the following:

\[
\begin{array}{c}
CF0=1; C01=2; C02=3; C03=4; C04=5; C05=6; \\
C06=7; C07=8; C08=9; C09=10.
\end{array}
\]

C10=0; F10=1
(This tells BA II Plus Professional to accumulate value at to t=10.)

Calculate NPV. We get NFV=55 @ i=0 (OK).
Set I=6. We get NFV=70.1658662.

Let’s check:

\[
(I\ddot{s})_{10\%} = (I\ddot{a})_{10\%} \times (1+i)^n
\]

\[
\Rightarrow (I\ddot{s})_{10\%} = (I\ddot{a})_{10\%} \times (1.06)^{10} = 39.18015293 \times (1.06)^{10} = 70.1658662
\]

Use BA II Plus Cash Flow Worksheet.

First, we calculate \((I\ddot{a})_{10\%}\) = 39.18015293

\[
\Rightarrow (I\ddot{s})_{10\%} = (I\ddot{a})_{10\%} \times (1.06)^{10} = 39.18015293 \times (1.06)^{10} = 70.1658662
\]
Problem 5

Calculate \( d_{\frac{12}{6\%}} \)

Solution

Let's first draw a cash flow diagram:

\[
\begin{array}{cccccccccccc}
\text{Time t} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\text{Cash flow} & $0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\
\end{array}
\]

This is a case where level payments of \( \frac{1}{12} \) are made monthly yet the interest rate of 6% is compounding annually. When the payments frequency and the compounding frequency do not match, we always use the payment frequency as the compounding period for our calculations in BA II Plus/BA II Plus Professional. Remember this rule. Never deviate from this rule.

Let's convert the annual interest rate to the monthly interest rate:

\[
\left(1 + \frac{i}{12}\right)^{12} = 1.06 \quad \Rightarrow \quad \frac{i}{12} = 0.06^{\frac{1}{12}} - 1 = 0.48675506\%
\]

Though here I explicitly write the monthly interest rate as 0.48675506\%, when you solve this problem in the exam, you can calculate the monthly interest rate but store the result in one of the calculator’s memories (we talked about this before). This eliminates the error-prone process of transferring a long decimal number back and forth between your calculator and the scrap paper.

Next, we’ll change the complex annuity \( d_{\frac{12}{6\%}} \) to a standard annuity:

\[
d_{\frac{12}{6\%}} = \frac{1}{12} d_{\frac{12}{0.48675506\%}}
\]

Using TVM Worksheet or Cash Flow Worksheet, we can easily calculate:
\[ a_{12}^{(12)} = 0.48675506\% = 11.62880032 \]

\[ a_{12}^{(12)} = \frac{1}{12} a_{12}^{(12)} = 0.96906669 \]

You can use the same method and calculate \( a_{\pi 12}^{(m)}, a_{\pi 12}^{(m)} s_{\pi 12}^{(m)}, a_{\pi 12}^{(m)} s_{\pi 12}^{(m)} \).

\[
\begin{align*}
  a_{\pi 12}^{(m)} &= \frac{1}{m} a_{\pi 12}^{(m)},
  a_{\pi 12}^{(m)} &= \frac{1}{m} a_{\pi 12}^{(m)} s_{\pi 12}^{(m)},
  s_{\pi 12}^{(m)} &= \frac{1}{m} s_{\pi 12}^{(m)},
  s_{\pi 12}^{(m)} &= \frac{1}{m} s_{\pi 12}^{(m)}
\end{align*}
\]

Where \( j = (1 + i)^{m} - 1 \)

**Problem 6**

Calculate \( a_{\pi 12}^{(12)} \), where the interest rate is \( i^{(12)} = 6\% \)

**Solution**

Let’s first draw a cash flow diagram:

<table>
<thead>
<tr>
<th>Time (Months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow ($0)</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td>$ \frac{1}{12}$</td>
<td></td>
</tr>
</tbody>
</table>

In this problem, the payment frequency is monthly and the interest rate given is the nominal rate compounding monthly. When the frequency of level payments matches the frequency by which a nominal interest compounds, BA II Plus/BA II Plus Professional TVM Worksheet has a shortcut way to calculate the annuity value. However, never use this shortcut; it causes more troubles than good.

Let’s first go through the shortcut in TVM Worksheet:

\[ a_{\pi 12}^{(12)} \]
Dangerous procedure to calculate $a_{[n]}^{(12)}$:

- Display 8 decimal places.
- Set $C/Y=12$ and $P/Y=12$. ($C$=compound. $C/Y=12$ means that the interest rate compounds 12 times in a year; $P$=payment. $P/Y=12$ means paying 12 times in a year).
- Use the annuity immediate mode.
- Set $PMT=1/12$, $N=12$.
- CPT PV. This calculates PV @ $i=0$. We should get PV= - 1 (OK).
- Set $I/Y=6$ (here we enter the nominal interest rate instead of converting the nominal rate to the monthly interest rate).
- CPT PV. We should get: PV= - 0.96824434

Though its result is correct, this procedure changes the safe setting of $C/Y=1$ and $P/Y=1$ to a dangerous setting of $C/Y=12$ and $P/Y=12$. If you set $C/Y=12$ and $P/Y=12$ but forget to reset to the safe setting of $C/Y=1$ and $P/Y=1$, when you enter an interest rate in TMV, TVM will treat this interest rate as the nominal interest rate compounding monthly. You might say, “This is OK. I’ll remember to change $C/Y=12$ and $P/Y=12$ back to $C/Y=1$ and $P/Y=1.” However, in the heat of the exam, it’s very easy to forget to reset $C/Y=1$ and $P/Y=1$. If you forget to reset $C/Y=1$ and $P/Y=1$, all your annuity calculations where an annual effective interest rate is entered will be wrong. For this reason, always stick to the safe setting $C/Y=1$ and $P/Y=1$. Never set $C/Y=12$ and $P/Y=12$.

The safe procedure to calculate $a_{[n]}^{(12)}$:

- Display 8 decimal places.
- Set $C/Y=1$ and $P/Y=1$.
- Use the annuity immediate mode.
- Set $PMT=1/12$, $N=12$.
- CPT PV. This calculates PV @ $i=0$. We should get PV= - 1 (OK).
- Set $I/Y=6/12=0.5$ (we simply enter the monthly interest rate).
- CPT PV. We should get: PV= - 0.96824434
Comprehensive calculator exercise

Problem 1
A loan of $100,000 borrowed at 6% annual effective is repaid by level monthly payments in advance over the next 30 year. After 10 years, the outstanding balance of the loan is refinanced at 4% annual effective and is paid by level monthly payments in advance over 20 years.

Calculate:
- The monthly payment of the original loan.
- The principal portion and the interest portion of the 37th payment.
- The monthly payment of the refinanced loan.
- The accumulated value of the reduction in monthly payments invested at 4% annual effective.

Solution

Find the monthly payment of the original loan

Keystrokes for TVM:
- Display 8 decimal places.
- Set C/Y=1 and P/Y=1.
- Use the annuity due mode.
- Set N=360, PV=100,000.
CPT PMT. This calculates PMT @ \( i = 0 \). We should get PMT= -277.7777778. Check: 100,000/360277.7777778. OK.

- Set I/Y=100 \( \left[ \frac{1}{1.06^{12} - 1} \right] \). Remember to multiple the interest rate by 100 (ex. enter 6 if the interest rate is 6%).

- CPT PMT. We should get: PMT= -586.5155230

Find the principal portion and the interest portion of the 37th payment

The 37th payment is the 1st payment in the 4th year.

<table>
<thead>
<tr>
<th>Time t (Month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>......</th>
<th>357</th>
<th>358</th>
<th>359</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>360 Months</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>...... Y</td>
</tr>
</tbody>
</table>

Keystrokes for TVM:

If you don’t clear TVM, TVM remembers all of the values you entered last time and the values it calculated last time. So you don’t need to reenter anything. You can simply pick up where you left off with TVM.

Keystrokes for additional calculations:

Enter 2nd AMORT (this activates the amortization worksheet)
Enter P1=37, P2=37 (this tells the calculator to look at the 37th payment)
Enter ↓, you’ll see:
BAL=95,386.57701 (This is the outstanding loan balance AFTER the 37th payment is made. Because you borrowed the PV of future cash flows, the PV is a cash inflow to you. So it’s positive. )

Enter ↓, you’ll see:
PRN= -121.6245226 (This is the principal portion of the 37th payment. The negative sign indicates cash outflow; this amount is your repayment of the principal in your 37th payment of the loan.)
Enter ↓, you’ll see:
INT= - 464.8910004 (This is the interest portion of the 37th payment. The negative sign indicates cash outflow; this amount is your interest portion of the 37th payment.)

Calculate the monthly payment of the refinanced loan.

Keystrokes for TVM:

Last time you used Amortization Worksheet. TVM keeps track of all the values in TVM and Amortization Worksheet. So you don’t need to reenter what you entered. You simply pick up where you left off.

Keystrokes for additional calculations:

Enter CE/C to leave Amortization Worksheet.

Enter N=240.

CPT PV. You should get: PV = 83,327.72914 (the outstanding balance of the original loan at t=240).

Set I/Y=$100 \left[ \frac{1}{0.04^{\frac{1}{12}}} - 1 \right]$

CPT PMT. You should get: PMT = - 500.1777638

Calculate the reduction of PMT due to refinancing:

PMT @6% = 586.5155230 (ignored the negative sign). Assume you store this value in a memory.

PMT @4% = 500.1777638 (ignored the negative sign). Assume you store this value in another memory.
Reduction = 586.5155230 - 500.1777638 = 86.33775916

Set PMT = 86.33775916.

By now, you are probably lost as to what values are currently stored in TVM and what keystrokes to press next. **When you are lost, you can always clear the current TVM Worksheet and start a fresh TVM Worksheet.** Of course, you need to reenter many values you entered before.

Under this method, you start a new TVM Worksheet. Enter N=240, I/Y=\[
\frac{1}{12}\log(1.04) - 1
\] PMT=86.33775916. Then, Press CPT FV. You should get FV= - 31,526.11850. This is accumulated value of the reductions of monthly payments at 4% to \( t=240 \) (months).

**As an easier alternatively,** you keep using the current TVM, recall each input in the current TVM, and change any input as necessary. For example, to find out the value of N, you simply press “RCL N.” You should get N=240. Next, you recall I/Y. You should get 0.32737398. You can check that this is \[
\frac{1}{12}\log(1.04) - 1
\]. Or if checking the interest rate is too much pain, you simply reset I/Y=\[
\frac{1}{12}\log(1.04) - 1
\]. So TVM uses the 4% annual interest rate. Then you recall PV and should get PV = 83,327.72914. Finally, you recall FV and should get FV = 0.

**Summary of your recall:**

\[
N=240, \ \ I/Y=100\left(\frac{1}{12}\log(1.04) - 1\right), \ \ PV = 83,327.72914, \ \ FV=0, \ \ PMT =86.33775916
\]

How do you calculate the accumulated value? Simply set PV=0. Press CPT FV. You should get \( FV= - 31,526.11850 \).

**Without setting** PV=0, if you press CPT FV, you should get a garbage \( FV=-214,097.4342 \). This garbage FV includes not only the accumulated value of the reduction of payments to \( t=240 \), but also the accumulated value of 83,327.72914 to \( t=240 \). You can check that:

\[
214,097.4342 = 83,327.72914\left(1.04^{20}\right) + 31,526.11850
\]
I recommend that you work through this problem multiple times with TVM Worksheet. Try to develop a mental picture of how your keystrokes will change the internal setting TVM.

Comprehensive calculator exercise #2
A company is participating in a project. The cash flows of the project are as follows:

- The company will invest $10 million per year for the 1st three years of the project. The investment will be made continuously.

- The company will receive a cash flow at the end of each year starting from Year 4.

- At the end of Year 4, the company will receive the 1st cash flow of $9 million. This amount will be reduced by $0.5 million for each subsequent year, until the company receives $5 million in a year.

- Starting from that year, the cash flow received by the company will be reduced by $1 million each year, until the company receives zero cash flow.

Calculate
- the NPV of the project if the discount rate is 12%.
- the IRR of the project.

Solution

Use NPV Worksheet (quick and easy)

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash</td>
<td>$9.0</td>
<td>$8.5</td>
<td>$8.0</td>
<td>$7.5</td>
<td>$7.0</td>
<td>$6.5</td>
<td>$6.0</td>
<td>$5.5</td>
<td>$5.0</td>
<td>$4.0</td>
<td>$3.0</td>
<td>$2.0</td>
<td>$1.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The initial investment at t=0 is:

$$-10\left(a_{\overline{3}|12\%}\right) = -10\left(\frac{1-v^3}{\delta}\right)_{i=12\%} = -10\frac{1-1.12^{-3}}{\ln1.12} = -25.43219763$$
Use NPV Worksheet:

\[
\text{CF0}= -25.43219763; \\
\text{C01}=0, \text{F01}=3; \\
\text{C04}=9; \text{C05}=8.5; \text{C06}=8; \text{C07}=7.5; \text{C08}=7; \text{C09}=6.5; \text{C10}=6; \\
\text{C11}=5.5; \text{C12}=5; \text{C13}=4; \text{C14}=3; \text{C15}=2; \text{C16}=1.
\]

CPT NPV. You should get \( NPV=47.56780237 \) @ \( i=0 \).

Double check:

\[
-25.43219763+9+8.5+8+7.5+7+6.5+6+5.5+5+4+3+2+1=47.56780237 \\
(\text{OK})
\]

Set \( I=12 \) and calculate NPV. You should get \( NPV=4.58281303 \)

Next, press IRR CPT. You should get: \( \text{IRR}=14.60149476 \).

So the \( \text{IRR} \) is \( 14.60149476\% \).

To calculate IRR, we need to solve the following equation:

\[
\begin{array}{ccccccccccccccc}
\text{Time } t & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
\text{cash} & $9.0 & $8.5 & $8.0 & $7.5 & $7.0 & $6.5 & $6.0 & $5.5 & $5.0 & $4.0 & $3.0 & $2.0 & $1.0 \\
\end{array}
\]

\[
-10(a)_{3\%}^{12\%} = -25.43219763
\]

\[
-25.43219763 + 9v^1 + 8.5v^2 + 8v^3 + 7.5v^4 + 7v^5 + 6.5v^6 + 6v^7 + 5.5v^8 + 5v^9 + 4v^{10} + 3v^{11} + 2v^{12} + v^{13} + v^{14} = 0
\]

We can NOT solve this equation manually. We need to use the IRR function of BA II Plus/BA II Plus Professional. We get:

\[
\text{IRR} = 14.60149476\%
\]
Comprehensive calculator exercise #3 (May 2005 FM, #35)
A bank customer takes out a loan of 500 with a 16% nominal interest rate convertible quarterly. The customer makes payments of 20 at the end of each quarter.

Calculate the amount of principal in the fourth payment.

[A] 0.0  
[B] 0.9  
[C] 2.7  
[D] 5.2  
[E] Not enough information

Solution

We’ll solve this problem with our imaginary cash flow method. To use this method, we need to find \( n \), the # of level payments. Once we find \( n \), we'll set up an imaginary cash flow of $20 at \( t = n + 1 \). Next, we'll discount this imaginary cash flow from \( t = n + 1 \) to \( t = 4 \) to find the principal portion of the payment made at \( t = 4 \):

\[
\text{Principal of the 4th payment} = 20 v_{(n+1)-4}
\]

So we need to calculate \( n \) by solving the following equation:

\[
500 = 20 a_{\overline{n}\rvert 4\%}
\]

This equation says that the PV of the all the quarterly payments of $20 @ 4% quarterly interest rate is the $500 (the total principal). The quarterly effective interest rate is 4% because we are given that \( i^{(4)} = 16\% \).

We’ll use BA II Plus/BA II Plus Professional TVM to solve \( 500 = 20 a_{\overline{n}\rvert 4\%} \).

Set PV= - 500, PMT = 20, I/Y=4. CPT N.

We get an error message. Perhaps we’ve entered wrong numbers. Once again, we enter:

Set PV= - 500, PMT = 20, I/Y=4. CPT N.

We get an error message again. Not knowing where the problem is, let’s throw away the calculator and solve the equation manually.

\[
500 = 20 a_{\overline{n}\rvert 4\%} \Rightarrow a_{\overline{4}\rvert 4\%} = 25
\]
Now we see where the trouble is. The loan is repaid through a perpetual immediate annuity. BA II Plus or BA II Plus Professional TVM can’t handle perpetual annuity (immediate or due).

Now we have \( n = \infty \).

Please note that our imaginary cash flow works even when \( n = \infty \).

Principal of the 4th payment is:

\[
20v^{(n+1)-4} = 20v^{-n-4} = 0
\]

Moral of this problem:

SOA can purposely design a problem to make our calculators useless. In studying for FM, we need to learn how to solve a problem with a calculator and how to solve it without a calculator.
Chapter 5  Geometrically increasing annuity

Key points:

Understand and memorize the following geometric annuity shortcuts:

\[
\frac{1}{1+k} \cdot a_{\overline{n|}}^{i-k}_{1+k}
\]

\[
(1) \cdot a_{\overline{n|}}^{i-k}_{1+k}
\]

\[
(1+k)^{n-1} \cdot s_{\overline{n|}}^{i-k}_{1+k}
\]

\[
(1+k)^{n} \cdot s_{\overline{n|}}^{i-k}_{1+k}
\]

Interpretation of this diagram:

For a geometrically increasing annuity where

- \( n \) geometrically increasing payments are made at a regular interval;
- the 1\(^{st} \) payment is $1;
- the next payment is always \((1+k)\) times the previous payment.

Then

(1) The present value at one interval prior to the 1\(^{st} \) payment is

\[
\frac{1}{1+k} \cdot a_{\overline{n|}}^{i-k}_{1+k}
\]. This value has a factor of \( \frac{1}{1+k} \) because the geometric payment pattern one interval prior to the 1\(^{st} \) payment is \( \frac{1}{1+k} \).
(2) The present value immediately after the 1st payment is \( \hat{a}_{j=1}^{i-k} \). The present value has a factor of 1 because the 1st payment is 1.

(3) The accumulated value immediately after the final payment is \( (1+k)^{n-1} \). This value has a factor of \( (1+k)^{n-1} \) because the final payment is made is \( (1+k)^{n-1} \).

(4) The accumulated value at one interval after the final payment is \( (1+k)^{n-n} \). This value has a factor of \( (1+k)^{n-n} \) because the geometric payment pattern one interval after the final payment is \( (1+k)^{n-n} \).

Explanations

**Present value of geometric annuity due:**

<table>
<thead>
<tr>
<th>Amount</th>
<th>$1</th>
<th>$(1+k)$</th>
<th>$(1+k)^2$</th>
<th>$(1+k)^3$</th>
<th>...</th>
<th>$(1+k)^{n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>n-1</td>
</tr>
</tbody>
</table>

\[
(1) \hat{a}_{j=1}^{i-k} = \hat{a}_{j=1}^{i-k}
\]

**Proof.**

The present value is: \( 1+(1+k)v+(1+k)^2v^2+\ldots+(1+k)^{n-1}v^{n-1} \).

Let’s set \( V=(1+k)v \) where \( V \) is the new discount factor.

The new interest is \( j = \frac{1}{V} - 1 = \frac{1}{1+k} - 1 = \frac{i-k}{1+k} \).

The present value is \( 1+V+V^2+\ldots+V^{n-1} = \hat{a}_{j=1}^{i-k} \).
How to memorize this formula:

- Geometric annuity due is still annuity due and its present value at time 0 should have an annuity due factor $\ddot{a}_{\bar{m},j}$ where $j$ is the new interest rate.
- The payment at time zero is 1. We say that the payment factor=1.
- We then multiply $\ddot{a}_{\bar{m},j}$ with 1.

Accumulated value of geometric annuity due:

<table>
<thead>
<tr>
<th>Amount</th>
<th>$1$</th>
<th>$(1+k)$</th>
<th>$(1+k)^2$</th>
<th>$(1+k)^3$</th>
<th>...</th>
<th>$(1+k)^{n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>$n-1$</td>
</tr>
</tbody>
</table>

Proof.

The accumulated value is

$$(1+i)^n + (1+i)^{n-1} (1+k) + (1+i)^{n-2} (1+k)^2 + ... + (1+i)(1+k)^{n-1}$$

$$= (1 + k)^n \left[ \frac{1 + i}{1 + k} + \left( \frac{1 + i}{1 + k} \right)^{n-1} + \left( \frac{1 + i}{1 + k} \right)^{n-2} + ... + \left( \frac{1 + i}{1 + k} \right) \right]$$

Let's set $1 + j = \frac{1 + i}{1 + k}$ where $j = \frac{1 + i}{1 + k} - 1 = \frac{i - k}{1 + k}$ is the new interest rate.

Then the accumulated value is:

$$(1 + k)^n \left[ (1 + j)^n + (1 + j)^{n-1} + (1 + j)^{n-2} + ... + (1 + j) \right] = (1 + k)^n \ddot{s}_{\bar{m},j-i-k}$$
How to memorize this formula:

- Geometric annuity due is still annuity due and its accumulated value at time \( n \) has an annuity factor \( \ddot{s}_{mj} \) where \( j = \frac{i - k}{1 + k} \) is the new interest rate.

- If we extend the geometrically increasing pattern to time \( n \), then the payment at time \( n \) will be \((1 + k)^n\). We say that the payment factor is \((1 + k)^n\).

- Future value = (Payment Factor)(Annuity Factor) = \((1 + k)^n \ddot{s}_{mj} = \frac{i - k}{1 + k}\)

Present value of geometric annuity immediate:

<table>
<thead>
<tr>
<th>Amount</th>
<th>$1$</th>
<th>$(1+k)$</th>
<th>$(1+k)^2$</th>
<th>……</th>
<th>$(1+k)^{n-2}$</th>
<th>$(1+k)^{n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>(n-1)</td>
<td>(n)</td>
</tr>
</tbody>
</table>

\[ \frac{1}{1 + k} \ddot{a}_{ji} = \frac{i - k}{1 + k} \]

Proof.

The present value is:

\[ v + (1+k)v^2 + (1+k)^2v^3 + … + (1+k)^{n-1}v^n = \frac{1}{1+k} \left\{ v(1+k) + [v(1+k)]^2 + … + [v(1+k)]^{n-1} \right\} \]

Let’s set \( V = v(1+k) \) where \( V \) is the new discount factor.

The new interest rate is \( j = \frac{1}{V} - 1 = \frac{1}{(1+k)v} - 1 = \frac{1}{1+k} - 1 = \frac{1+i}{1+k} - 1 = \frac{i - k}{1 + k} \)

The present value of geometric annuity immediate is:
\[ \frac{1}{1+k} \left[ V + V^2 + \ldots + V^{n-1} \right] = \frac{1}{1+k} \cdot a_{m, \frac{i-k}{1+k}} \]

How to memorize this formula:

- Geometric annuity immediate is still annuity immediate and its present value at time 0 has an annuity immediate factor \( a_{m, j} \) where \( j = \frac{i-k}{1+k} \) is the new interest rate.

- If we extend the geometrically increasing pattern to time zero, then the payment at time zero will be \( \frac{1}{1+k} \). We say that the payment factor is \( \frac{1}{1+k} \).

- \( PV = (\text{Payment Factor}) (\text{Annuity Factor}) = \frac{1}{1+k} a_{m, \frac{i-k}{1+k}} \)

### Accumulated value of geometric annuity immediate:

<table>
<thead>
<tr>
<th>Amount</th>
<th>$1$</th>
<th>( (1+k) )</th>
<th>( (1+k)^2 )</th>
<th>( \ldots )</th>
<th>( (1+k)^{n-2} )</th>
<th>( (1+k)^{n-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>( n-1 )</td>
<td>( n )</td>
</tr>
</tbody>
</table>

Proof.

The accumulated value is

\[
(1+i)^{n-1} + (1+i)^{n-2} (1+k) + (1+i)^{n-3} (1+k)^2 + \ldots + (1+i) (1+k)^{n-2} + (1+k)^{n-1} \\
= (1+k)^{n-1} \left[ \left( \frac{1+i}{1+k} \right)^{n-1} + \left( \frac{1+i}{1+k} \right)^{n-2} + \left( \frac{1+i}{1+k} \right)^{n-3} + \ldots + \left( \frac{1+i}{1+k} \right) + 1 \right]
\]
Let’s set \( 1 + j = \frac{1 + i}{1 + k} \) where \( j = \frac{1 + i}{1 + k} - 1 = \frac{i - k}{1 + k} \) is the new interest rate.

Then the accumulated value is:

\[
(1 + k)^{n-1} \left[ (1 + j)^{n-1} + (1 + j)^{n-2} + \ldots + (1 + j) + 1 \right] = (1 + k)^{n-1} \frac{s_{n, j = \frac{i-k}{1+k}}}{s_{1+k}}
\]

How to memorize this formula:

- Geometric annuity due is still annuity due and its accumulated value at time \( n \) has an annuity factor \( s_{n, j} \) where \( j = \frac{i - k}{1 + k} \) is the new interest rate.

- The payment at time \( n \) is \( (1 + k)^{n-1} \). We say that the payment factor is \( (1 + k)^{n-1} \).

- \( FV = (\text{Payment Factor}) (\text{Annuity Factor}) = (1 + k)^{n-1} \frac{s_{n, j = \frac{i-k}{1+k}}}{s_{1+k}} \)

**Sample Problems and Solutions**

**Problem 1**

You are given the following cash flows

\[
\begin{align*}
&\text{A} \quad \text{B} \quad \text{C} \quad \text{D} \\
&1+k \quad (1+k) \quad (1+k)^2 \quad (1+k)^3 \quad \ldots \quad (1+k)^n \\
&\text{n payments}
\end{align*}
\]
Calculate the value of the annuity at four different points:
- one step before the 1st payment (point A)
- 1st payment time (point B)
- final payment time (point C), and
- one step after the final payment (point D).

Solution

\[
\text{Value of a geometric annuity} = \text{payment factor} \times \text{annuity factor}
\]

The payment factor at one step before the 1st payment is $1. The 1st payment is \(1+k\). If we extend the geometric payment pattern to one step before the 1st payment time, we get $1. So the value of the annuity one step before the 1st payment time is \(\frac{a}{1+k}\).

The payment factor at the 1st payment is \(1+k\). The value of the annuity at the 1st payment time is \(\frac{\ddot{a}}{1+k}\).

The payment factor at the final payment is \((1+k)^n\). The value of the annuity at the final payment time is \(\frac{s}{1+k}\).
The payment factor at one step after the final payment is \((1+k)^{n+1}\). If we extend the geometric pattern to one step after the final payment time, we get \((1+k)^{n+1}\). So the value of the annuity at the 1\text{st} payment time is 

\[(1+k)^{n+1} \cdot S_{\overline{n+j-k}|i+k}\,.
\]

**Problem 2**

You are given the following cash flows

\[
\begin{array}{cccc}
(1+k)^2 & (1+k)^3 & (1+k)^4 & \ldots & (1+k)^{n+1} \\
\end{array}
\]

Calculate the value of the annuity at four different points:

- one step before the 1\text{st} payment (point A)
- 1\text{st} payment time (point B)
- final payment time (point C), and
- one step after the final payment (point D).
Problem 3

An annuity immediate has 15 payments. The 1st payment is $100. Each following payment is 8% larger than the previous payment. The annual effective interest rate is 14%. Calculate the present value of this annuity.

Solution

We are asked to find the present value of this geometric annuity at one interval prior to the 1st payment time.
From the shortcut, we know

\[ PV = (\text{Payment Factor}) \cdot (\text{Annuity Factor}) \]

To find the payment factor, we extend the geometric annuity payment pattern to one interval prior to the 1st payment time. Because the 1st payment is $100 and each payment is 8% larger than the previous payment, the payment that would have been made one interval prior to the 1st payment is: \( \frac{100}{1.08} \). So \( (\text{Payment Factor}) = \frac{100}{1.08} \)

To find the annuity factor, we simply change the original interest of 14% to the new interest rate:

\[
j = \frac{i - k}{1 + k} = \frac{14\% - 8\%}{1 + 8\%} = 5.5556\%
\]

\[ \Rightarrow (\text{Annuity Factor}) = a_{\overline{15}|5.5556} \]

\[ \Rightarrow PV = (\text{Payment Factor}) \cdot (\text{Annuity Factor}) = \frac{100}{1.08} a_{\overline{15}|5.5556} = 925.98 \]

Please note that $925.98 is calculated using BA II Plus or BA II Plus Professional. There is no need for you to manually calculate the annuity using the formula:

\[
a_{\overline{n}|i} = \frac{1 - v^n}{i}
\]

Please also note that you don’t need to draw the time line or to spell out each of the 15 payments; doing so is time-consuming and error-prone. So don’t draw the following diagram:

<table>
<thead>
<tr>
<th>Amount</th>
<th>$1</th>
<th>$(1 + 8%)</th>
<th>$(1 + 8%)^2</th>
<th>\ldots</th>
<th>$(1 + 8%)^{13}</th>
<th>$(1 + 8%)^{14}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

This diagram is good for you to initially prove the geometric annuity formula. Once you have proven the formula, don’t draw this diagram any more.
All you need to know that the present value of an annuity immediate is the present value one interval prior to the 1st payment.

Alternative method:

We first calculate the present value of the geometric annuity due. Then we discount this present value one interval prior to the 1st payment.

\[ PV \text{ geometric annuity due} = (\text{Payment Factor}) (\text{Annuity Factor}) \]

\[
\text{(Payment Factor)} = 100 \\
\text{(Annuity Factor)} = \bar{a}_{j} = \bar{a}_{[5.5556]} \\
\Rightarrow PV \text{ geometric annuity due} = 100\bar{a}_{[5.5556]} \\
\]

Next, we discount this value using the discount factor \( \frac{1}{1+14\%} \).

The present value of the geometric annuity immediate is:

\[ 100\bar{a}_{[5.5556]} \left( \frac{1}{1+14\%} \right) = 925.98 \]

**Problem 4**

An annuity immediate has 15 payments. The 1st payment is $100. Each following payment is 8% larger than the previous payment. The annual effective interest rate is 14%. Immediately after the 1st payment is made, this annuity is sold at a price of \( X \). Calculate \( X \).
Solution

Immediately after the 1st payment, there are 14 geometric payments left. We are asked to find the present value of these 14 geometric payments one interval prior to the 1st payment of $100(1+8\%). This should be the sales price $X$.

Once again, we use the shortcut:

$$PV = (\text{Payment Factor})(\text{Annuity Factor})$$

Please note that among the remaining 14 geometric payments, the 1st payment is $100(1+8\%)$. As a result, if we extend the geometric payment pattern one interval prior to this 1st payment, we'll get $100$.

$$\Rightarrow (\text{Payment Factor}) = 100, \quad (\text{Annuity Factor}) = a_{\overline{14}|5.5556\%}$$

$$PV = 100 a_{\overline{14}|5.5556\%} = 955.62$$

Problem 5

An annuity immediate has 15 payments. The 1st payment is $100. Each following payment is 8% larger than the previous payment. The annual effective interest rate is 14%. Calculate the accumulated value of this annuity immediately after the 15th payment.

Solution

$$FV = ?$$
\[ FV = (\text{Payment Factor})(\text{Annuity Factor}) \]

(Payment Factor) = \(100(1 + 8\%)^{14}\)

(Annuity Factor) = \(\frac{j}{i} = \frac{14\% - 8\%}{1 + 8\%} = 5.556\%\)

\[ FV = 100(1 + 8\%)^{14} \cdot s_{\overline{15.5556}\%} = 6,609.64 \]

**Problem 6**

In a perpetual annuity immediate, the 1st payment is $100 and each following payment is 8% larger than the previous payment. The annual effective interest rate is 14%. Calculate the present value of this annuity.

**Solution**

\[ PV = (\text{Payment Factor})(\text{Annuity Factor}) \]

(Payment Factor) = \(\frac{100}{1.08}\), (Annuity Factor) = \(a_{\overline{\infty}\mid j} = \frac{1}{j}\)

\[ j = \frac{14\% - 8\%}{1 + 8\%} = \frac{6\%}{1.08} \]

\[ \Rightarrow PV = \left(\frac{100}{1.08}\right) \cdot \frac{1}{j} = \frac{100}{6\%} \cdot \frac{6\%}{1.08} = 1,666.67 \]

Generally, the present value of a perpetual geometric annuity immediate is \(\frac{1}{i - k}\).
Problem 7

In a perpetual annuity immediate, the 1st payment is $100 and each following payment is 8% larger than the previous payment. The annual effective interest rate is 14%. Calculate the present value of this annuity immediately before the 1st payment is made.

Solution

\[
PV = (\text{Payment Factor}) \cdot (\text{Annuity Factor})
\]

(Payment Factor) = 100

(Annuity Factor) = \( \bar{a}_{\overline{n-1}_{i+k}} = \frac{1}{d} \)
\[ d = 1 - \frac{1}{1 + j} = 1 - \frac{1}{i - k} = \frac{i - k}{1 + i} \]

\[ \Rightarrow \quad (\text{Annuity Factor}) = \hat{a}_{\overline{j-k}|i} = \frac{1}{d} = \frac{1+i}{i-k} \]

\[ PV = 100 \frac{1+i}{i-k} = 100 \frac{1+14\%}{14\% - 8\%} = 1,900 \]

Generally, the present value of a perpetual geometric annuity due is \( \frac{1+i}{i-k} \).

<table>
<thead>
<tr>
<th>Amount</th>
<th>( (1+k) )</th>
<th>( (1+k)^2 )</th>
<th>( (1+k)^3 )</th>
<th>...</th>
<th>( (1+k)^{n-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>( n-1 )</td>
</tr>
</tbody>
</table>

\[ \hat{a}_{\overline{j-k}|i} = \frac{1+i}{i-k} \]

**Problem 8**

You plan to pay off your $150,000 mortgage by monthly installments for 15 years. Your 1st payment is one month from now. You plan to increase your monthly payments by 10% each year. You pay a nominal interest rate of 12% compounding quarterly.

Calculate your 1st monthly payment.

**Solution**

To simplify our calculation, let’s find the equivalent annual payments to replace each year’s 12 monthly payments.

Let \( X \) = your 1st monthly payment. Let \( Y \) = the present value of the annuity immediate of your 1st year payment.
Let's find the monthly interest rate.

\[(1+i)^3 = 1 + \frac{i(4)}{4} \quad \Rightarrow \quad i = \left[1 + \frac{i(4)}{4}\right]^3 - 1 = \left[1 + \frac{12\%}{4}\right]^\frac{1}{3} - 1 = 0.99016\%
\]

\[Y = X a_{11.2621} = 11.2621X\]

Because your monthly payments increase by 10% per year, the 12 monthly payments in the 2nd year can be replaced by a single cash flow of \(1.1Y\) at the beginning of the 2nd year. Similarly, the 12 payments in the 3rd year can be replaced by a single cash flow of \((1.1)^2 Y\) at the beginning of the 3rd year. As a result, you'll have a geometric annuity immediate for 15 payments (years).

\[PV = 150,000 = Y \ddot{a}_{15.2.319\%}^{j-r-k} = 11,683.3307\]

where \(r\) is the annual effective interest rate and \(k = 10\%\). The interest rate per 3-month is \(12\%/4 = 3\%\). So

\[r = 1.03^4 - 1 = 12.5509\%
\]

\[j = \frac{r-k}{1+k} = \frac{12.5509\% - 10\%}{1+10\%} = 2.319\%
\]

\[\Rightarrow \quad 150,000 = Y \ddot{a}_{15.2.319\%}^{j-r-k} = 12.8388Y, \quad Y = 11,683.3307
\]

\[X = \frac{Y}{1.12621} = \frac{11,683.3307}{11.2621} = 1,037.40\]
**Alternative method:**

We can use a single cash flow $Z$ at the end of Year 1 to replace the 1st year’s 12 monthly payments. Then, the 12 monthly payments in the 2nd year can be replaced by a single cash flow of $1.1Z$ at the end of Year 2. And the 12 monthly payments in the 3rd year can be replaced by a single cash flow of $(1.1)^2 Z$ at the end of Year 3. And so on.

\[
\begin{array}{ccc}
\text{15 payments} & \hline \\
Z & 1.1Z & (1.1)^2 Z & \ldots \\
\end{array}
\]

\[
PV = \frac{1}{1.1} Z a_{\frac{r}{12} \mid j=1} = 150,000, \text{ where } \frac{1}{1.1} \text{ is the payment factor.}
\]

\[
\Rightarrow 150,000 = \frac{1}{1.1} Z a_{\frac{r}{12} \mid j=1} = \frac{1}{1.1} Z a_{\frac{r}{12} \mid j=2.319\%}, \quad Z = 13,149.6670
\]

Next, we can calculate the 1st monthly payment $X$.

\[
\begin{array}{ccc}
\text{12 payments} & \hline \\
X & X & X & \ldots & X \\
\end{array}
\]

\[
FV = Z = X a_{\frac{r}{12} \mid j=1} = X a_{\frac{0.99016\%}{12}}
\]

\[
\Rightarrow X = \frac{Z}{a_{\frac{0.99016\%}{12}}} = \frac{13,149.6670}{a_{\frac{0.99016\%}{12}}} = 1,037.40
\]
Problem 9

An annuity immediate has 15 payments. The 1st payment is $100. Each following payment is 20% larger than the previous payment. The annual effective interest rate is 14%. Calculate the present value of this annuity.

Solution

To find the payment factor, we extend the geometric annuity payment pattern to one interval prior to the 1st payment time. Because the 1st payment is $100 and each payment is 20% larger than the previous payment, the payment that would have been made one interval prior to the 1st payment is: \( \frac{100}{1.2} \). So (Payment Factor) = \( \frac{100}{1.2} \)

To find the annuity factor, we simply change the original interest of 14% to the new interest rate:

\[
j = \frac{i - k}{1 + k} = \frac{14\% - 20\%}{1 + 20\%} = -5\%
\]

(OK if the new interest rate is negative)

\[
\Rightarrow\quad (\text{Annuity Factor}) = a_{\overline{15}|.-5}\%
\]

\[
\Rightarrow\quad PV = (\text{Payment Factor})(\text{Annuity Factor}) = \frac{100}{1.2} a_{\overline{15}|.-5\%} = $1,930.78
\]

where \( a_{\overline{15}|.-5\%} = \frac{1 - \left(\frac{1}{1-5\%}\right)^{15}}{-5\%} = 23.1694 \)
**Problem 10**

An annuity immediate has 15 payments. The 1st payment is $100. Each following payment is 20% larger than the previous payment. The annual effective interest rate is 14%. Calculate the accumulated value of this annuity immediately after the 15th payment.

**Solution**

\[
\begin{array}{cccc}
\text{15 payments} & \$100 & \$100(1.2) & \$100(1.2)^2 & \ldots & \$100(1.2)^{14} \\
\end{array}
\]

\[FV=?\]

\[FV = (\text{Payment Factor})(\text{Annuity Factor})\]

\[\text{(Payment Factor)} = 100(1.2)^{14}\]

\[\text{(Annuity Factor)} = s_{\overline{15}|j}, \quad j = \frac{14\%-20\%}{1+20\%} = -5\%\]

\[FV = 100(1.2)^{14}s_{\overline{15}|5\%} = 13,781.81\]

Where \[s_{\overline{15}|5\%} = \frac{0.95^{15} - 1}{-5\%} = 10.7342\]

**Problem 11**

In a perpetual annuity immediate, the 1st payment is $100. Each following payment is 20% larger than the previous payment. The annual effective interest rate is 14%. Calculate the present value of this annuity.
Solution

\[ \frac{1}{i-k} = \frac{1}{14\% - 20\%} = -20 \]

We got a nonsense number of negative 20. This really means that the present value is not defined.

We can easily check that the present value is not defined. Payments increase by 20\%, which is faster than the discount rate 14\%. As a result, the present value of this perpetual annuity becomes \( +\infty \).

Moral of this problem: If \( j = \frac{i-k}{1+k} < 0 \), then the present value of the perpetual geometric annuity is undefined.

Problem 12

Calculate \( 1+4+4^2+4^3+4^4+4^5 \)

Solution

This is a geometric annuity problem with \( i=0 \) and \( k = 3 \).

\[ PV = \ddot{a}_{\overline{6}|j} \]

\[ j = \frac{i-k}{1+k} = \frac{0-3}{1+3} = -75\% \]

\[ PV = \ddot{a}_{\overline{6}|j=-75\%} = 1,365 \]
We can check that the result is correct:

\[ 1 + 4 + 4^2 + 4^3 + 4^4 + 4^5 = \frac{1 - 4^6}{1 - 4} = 1,365 \]

Generally, \( 1 + q + q^2 + \ldots + q^n = \dot{a}_{m|j} \) where \( j = \frac{1}{q} - 1 \) (for \( q \neq 1 \)).

Alternatively, we can use the following formula:

\[ 1 + (1+r) + (1+r)^2 + (1+r)^3 + \ldots + (1+r)^{n-1} = s_{\overline{n|r}} \]

\[ \Rightarrow 1 + 4 + 4^2 + 4^3 + 4^4 + 4^5 = s_{\overline{6|0.03}} = \frac{(1+i)^n - 1}{i} = \frac{4^6 - 1}{3} = 1,365 \]

If you think it’s a joke to use annuity to calculate the sum of a geometric progression, think again. In the heat of the exam, it can be faster to calculate annuity to calculate the sum of a power series.

**Problem 13**

Starting from the current year, each year John will take out a level percentage of his salary and deposit it into a retirement fund. His goal is to accumulate $250,000 immediately after he retires.

Facts:

<table>
<thead>
<tr>
<th>John’s current age</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Retirement age</td>
<td>65</td>
</tr>
<tr>
<td>Current salary</td>
<td>$50,000</td>
</tr>
<tr>
<td>Annual growth of salary</td>
<td>3%</td>
</tr>
<tr>
<td>Interest rate earned by the retirement fund</td>
<td>7% annual effective</td>
</tr>
</tbody>
</table>

Alternative deposit plan #1

At the end of each year deposit \( X \% \) of the salary into the retirement fund.

Alternative deposit plan #2

At the beginning of each year, deposit \( Y \% \) of the salary into the retirement fund.

Calculate \( X - Y \).
Solution

We'll use $1,000 as the unit money to simplify our calculation.

<table>
<thead>
<tr>
<th>Age</th>
<th>40</th>
<th>41</th>
<th>42</th>
<th>43</th>
<th>...</th>
<th>64</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>24</td>
<td>25</td>
</tr>
</tbody>
</table>

\[
X \% (50) \quad X \% (50)1.03 \quad X \% (50)1.03^2 \quad \ldots \quad X \% (50)1.03^{23} \quad X \% (50)1.03^{24}
\]

At \( t=25 \)
- the payment factor = \( X \% (50)1.03^{24} \)
- the annuity factor = \( \ddot{s}_{25|} \) where \( i = \frac{7\% - 3\%}{1+3\%} = 3.8835\% \)

\[
\Rightarrow \quad X \% (50)1.03^{24} \ddot{s}_{25|} = 250
\]

\[
\Rightarrow \quad X \% = \frac{250}{(50)1.03^{24} \ddot{s}_{25|}} = 6\%
\]

<table>
<thead>
<tr>
<th>Age</th>
<th>40</th>
<th>41</th>
<th>42</th>
<th>43</th>
<th>...</th>
<th>64</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>24</td>
<td>25</td>
</tr>
</tbody>
</table>

\[
Y \% (50) \quad Y \% (50)1.03 \quad Y \% (50)1.03^2 \quad \ldots \quad Y \% (50)1.03^{23} \quad Y \% (50)1.03^{24}
\]

At \( t=25 \)
- the payment factor = \( Y \% (50)1.03^{25} \)
- the annuity factor = \( \ddot{s}_{25|} \) where \( i = \frac{7\% - 3\%}{1+3\%} = 3.8835\% \)

\[
\Rightarrow \quad Y \% (50)1.03^{25} \ddot{s}_{25|} = 250
\]

\[
\Rightarrow \quad Y \% = \frac{250}{(50)1.03^{25} \ddot{s}_{25|}} = 5.607\%
\]

\[
\Rightarrow \quad X - Y = 6 - 5.607 = 0.393
\]
Problem 12  (SOA May 2001 EA-1 #4)
Date of loan:    1/1/2001
Amount of loan:    $100,000
Frequency of payments:  Quarterly
Date of 1st payment:  3/31/2001
# of payments:   120

Amount of each of the 1st 110 repayments:   $3,100

Amount of each of the last 10 repayments:
Initial repayment of $ X , then doubling every quarter thereafter

Interest rate:   12% per year, compounding quarterly

Calculate the amount of the final payment.

Solution
Let’s use $1,000 as one unit of money and a quarter as one unit of time.

\[ P V = \$100 \]

The interest rate per quarter is \( i = \frac{12\%}{4} = 3\% \).

Let’s break down the cash flows into two streams:

Stream #1

\[ P V = 3.1a_{110} \]

Stream #2

\[ P V = X \ddot{a}_{110} \]

Where \( j = \frac{i-k}{1+k} = \frac{3\%-1}{1+1} = -48.5\% \) (here \( k = 1 \))
The present value of these two streams of cash flows should be $100 at $t = 0$.

$$100 = 3.1a_{110\%} + 1.03^{-111}X\bar{a}_{110\%}$$

$$a_{110\%} = 32.04275602$$

$$\bar{a}_{110\%} = 808.0213752$$

$$X = \frac{100 - 3.1a_{110\%}}{1.03^{-111}\bar{a}_{110\%}} = \frac{100 - 3.1(32.04275602)}{1.03^{-111}(808.0213752)} = 0.02197516$$

Alternatively, we can calculate the present value of the 2nd stream of the cash flows at $t = 110$:

**Stream #2**

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>……</th>
<th>110</th>
<th>111</th>
<th>112</th>
<th>113</th>
<th>……</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$X$</td>
<td>$2X$</td>
<td>$2^2X$</td>
<td>……</td>
</tr>
</tbody>
</table>

$$PV = \frac{X}{2}a_{10\%j}$$

$$100 = 3.1a_{110\%} + 1.03^{-110}\frac{X}{2}a_{10\%j}$$

$$a_{10\%j} = a_{10\%} = 1,568.973544$$

$$\frac{X}{1.03^{-110} \times \frac{a_{10\%j}}{2}} = \frac{100 - 3.1(32.04275602)}{1.03^{-110} \times \frac{1,568.973544}{2}} = 0.02197516$$

The final repayment (i.e. the 120th repayment) is:

$$2^9X = 2^9(0.02197516) = 11.25127973$$

Since one unit of money represents $1,000, the final repayment is $11,151.28. So the answer is B.
Problem 13  (SOA May 2005 EA-1 #17)
Smith retires on 1/1/2005 and receives his retirement benefit as monthly annuity payable at the end of each month for a period certain of 20 years.

The benefit for the 1st year is $2,000 per month. This monthly benefit is increased at the beginning of each year to be 5% larger than the monthly payment in the prior year.

$X$ is the present value on 1/1/2005 of the retirement benefit at a nominal interest rate of 6%, convertible monthly.

In what range is $X$?

(A) Less than $405,000  
(B) $405,000 but less than $410,000  
(C) $410,000 but less than $415,000  
(D) $415,000 but less than $420,000  
(E) $420,000 or more

Solution
We’ll do two things to simplify our calculation. First, we’ll use $1,000 as one unit of money. Second, we’ll convert the 12 monthly payments at Year 1 into one equivalent cash flow.

Convert the 12 monthly payments in Year 1 into one cash flow at $t = 0$:

<table>
<thead>
<tr>
<th>Time t (months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>......</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td></td>
<td>$2$</td>
<td>$2$</td>
<td>$2$</td>
<td>$2$</td>
<td>$2$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

$P = 2a_{12|i} = 2(11.61893207) = 23.2379$ where $i = \frac{6\%}{12} = 0.5\%$

We’ll draw a cash flow diagram for 20 years’ payments. The next year’s payment is 5% larger than the previous year’s payment.

<table>
<thead>
<tr>
<th>Time t (years)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>......</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>$P$</td>
<td>$1.05P$</td>
<td>$1.05^2P$</td>
<td>$1.05^3P$</td>
<td>......</td>
<td>$1.05^{19}P$</td>
</tr>
</tbody>
</table>

The present value of 20 years’ payments is:

$P \tilde{\alpha}_{20|j}$, where $j = \frac{r-k}{1+k} = \frac{r-5\%}{1+5\%}$
Because the unit time is one year, $r$ is the annual effective return.

\[
r = \left(1 + \frac{6\%}{12}\right)^{12} - 1 = 6.16778\%
\]

\[
\Rightarrow j = \frac{r - k}{1+k} = \frac{6.16778\% - 5\%}{1+5\%} = 1.1217\%
\]

So the present value is:

\[
P \ddot{a}_{\overline{20}|j} = 23.2379 \ddot{a}_{\overline{20}|1.11217\%} = 23.2379(18.0418) = 419.2538
\]

Because one unit of money is really $1,000, so the present value is $419,253.8.

The answer is D.

**Problem 14 (SOA May 2002 EA-1 #2)**

Annual payments into a fund: $1,000 at the end of year one, increasing by $500 per year in the 2\textsuperscript{nd} through the 10\textsuperscript{th} years. After the 10\textsuperscript{th} year, each payment increases by 3.5\% over the prior payment.

Interest rate: 7\% compounded annually.

Calculate the accumulated value of the fund at the end of year 20.

**Solution**

We'll use $1000 as one unit of money to keep our calculation simple.

First, let's use a table to keep track of cash flows:
We are asked to calculate the fund’s value at $t = 20$. There are many ways to find the answer. One way is to break the cash flows into two streams:
Stream #1

<table>
<thead>
<tr>
<th>Time t</th>
<th>payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$100</td>
</tr>
<tr>
<td>2</td>
<td>105</td>
</tr>
<tr>
<td>3</td>
<td>110</td>
</tr>
<tr>
<td>4</td>
<td>115</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
</tr>
<tr>
<td>6</td>
<td>125</td>
</tr>
<tr>
<td>7</td>
<td>130</td>
</tr>
<tr>
<td>8</td>
<td>135</td>
</tr>
<tr>
<td>9</td>
<td>140</td>
</tr>
<tr>
<td>10</td>
<td>145</td>
</tr>
</tbody>
</table>

Stream #2

<table>
<thead>
<tr>
<th></th>
<th>145*1.035</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>145*1.035^2</td>
</tr>
<tr>
<td>12</td>
<td>145*1.035^3</td>
</tr>
<tr>
<td>13</td>
<td>145*1.035^4</td>
</tr>
<tr>
<td>14</td>
<td>145*1.035^5</td>
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<tr>
<td>15</td>
<td>145*1.035^6</td>
</tr>
<tr>
<td>16</td>
<td>145*1.035^7</td>
</tr>
<tr>
<td>17</td>
<td>145*1.035^8</td>
</tr>
<tr>
<td>18</td>
<td>145*1.035^9</td>
</tr>
<tr>
<td>19</td>
<td>145*1.035^10</td>
</tr>
<tr>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

A fast way to find the accumulated value of stream #1 is to use BA II Plus Cash Flow Worksheet. Enter the following into Cash Flow Worksheet:
<table>
<thead>
<tr>
<th>Time t</th>
<th>payment</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0</td>
<td>CF0</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>C01</td>
</tr>
<tr>
<td>2</td>
<td>105</td>
<td>C02</td>
</tr>
<tr>
<td>3</td>
<td>110</td>
<td>C03</td>
</tr>
<tr>
<td>4</td>
<td>115</td>
<td>C04</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>C05</td>
</tr>
<tr>
<td>6</td>
<td>125</td>
<td>C06</td>
</tr>
<tr>
<td>7</td>
<td>130</td>
<td>C07</td>
</tr>
<tr>
<td>8</td>
<td>135</td>
<td>C08</td>
</tr>
<tr>
<td>9</td>
<td>140</td>
<td>C09</td>
</tr>
<tr>
<td>10</td>
<td>145</td>
<td>C10</td>
</tr>
</tbody>
</table>

Set $I=7$ (so the interest rate is 7%). You should get $NPV=840.9359126$. This is the PV of Stream #1 at $t=0$. The accumulated value of Stream #1 at $t=20$ is:

$$840.9359126 \times 1.07^{20} = 3,254.156635$$

The accumulated value of Stream #2 at $t=20$ is:

$$145 \left( 1.035^{10} \right) s_{\overline{10}|j}, \text{ where } j = \frac{7\% - 3.5\%}{1 + 3.5\%} = 3.38164251\%$$

Using BA II Plus TVM, you should get:

$$145 \left( 1.035^{10} \right) s_{\overline{10}|j} = 2,386.418027$$

So at $t=20$ the accumulated value of the entire fund is:

$$3,254.156635 + 2,386.418027 = 5,640.574662$$

Because one unit is equal to $100, the accumulated value is $564,000.57.
Chapter 6  Real vs. nominal interest rate

Key points:

1. Nominal interest rate
   - the growth rate of your money
   - interest rate quoted in the market

2. Real interest rate
   - the growth rate of your purchasing power

3. Formulas

\[
1 + \text{real interest rate} = \frac{1 + \text{nominal interest rate}}{1 + \text{inflation rate}}
\]

\[\Rightarrow \quad \text{real interest rate} = \frac{\text{nominal interest rate} - \text{inflation rate}}{1 + \text{inflation rate}}\]

\[\Rightarrow \quad \text{real interest rate} \approx \text{nominal interest rate} - \text{inflation rate}\]

Real Cash Flow = \(\frac{\text{Nominal Cash Flow}}{1 + \text{inflation rate}}\)

4. Constant dollar vs. real dollar

   - Current dollars – money received. If you get $100 on your part time job last week, you get $100 current dollars. Your $100 current dollars have not been adjusted for inflation.

   - Constant or real dollars -- dollars reported in terms of the value they had on a previous date. The $100 you got last week could buy 90 Big Macs from MacDonald’s. However, several years ago, $50 could buy 90 Big Macs. So your current $100 is worth only $50 in constant or real dollars. Constant or real dollars are current dollars adjusted for inflation.
5. Discounting rule

- Discount nominal dollar cash flows by the nominal interest rates
- Discount real dollar cash flows by the real interest rates

Sample Problems

Problem 1

Starting from the current year, each year John will take out money from his salary and deposit it into a retirement fund. The retirement fund earns the market interest rate. His goal is to accumulate $100,000 constant dollars immediately after he retires.

Facts:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>John’s current age</td>
<td>40</td>
</tr>
<tr>
<td>John’s Retirement age</td>
<td>65</td>
</tr>
<tr>
<td>Market interest rate</td>
<td>7% annual effective</td>
</tr>
<tr>
<td>Inflation</td>
<td>3% per year</td>
</tr>
<tr>
<td>Alternative deposit Plan #1</td>
<td>Deposit $X constant dollars at the end of each year into the retirement fund.</td>
</tr>
<tr>
<td>Alternative deposit Plan #2</td>
<td>Deposit $Y nominal dollars at the end of Year 1, each subsequent annual deposit is 5% larger than the previous one.</td>
</tr>
</tbody>
</table>

[1] Use the nominal dollar method, calculate the annual deposit at the end of Year 1 for Plan #1 and Plan #2

[2] Use the constant dollar method, calculate the annual deposit at the end of Year 1 for Plan #1 and Plan #2

Solution

[1] Nominal dollar method Plan #1
- Use nominal dollar cash flows
- Discount all nominal cash flows at the nominal interest rate

The nominal interest rate is 7%. The inflation is 3%.
John wants to accumulate, from $t=0$ to $t=25$, $\$100,000$ in constant dollars. $\$100,000$ constant dollars are equivalent to the following nominal dollars:

$$100,000 (1 + \text{inflation})^{25} = 100,000 (1.03)^{25} = 209,377.79$$

<table>
<thead>
<tr>
<th>Age</th>
<th>40</th>
<th>41</th>
<th>42</th>
<th>43</th>
<th>…</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time $t$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>…</td>
<td>25</td>
</tr>
<tr>
<td>Constant dollars</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>…</td>
<td>$X$</td>
<td></td>
</tr>
<tr>
<td>Nominal dollars</td>
<td>$1.03X$</td>
<td>$1.03^2X$</td>
<td>$1.03^3X$</td>
<td>…</td>
<td>$1.03^{25}X$</td>
<td></td>
</tr>
</tbody>
</table>

Nominal dollars:

$$100,000 (1.03)^{25} = 209,377.79$$

We have a geometric annuity. We’ll apply the 3 minute script.

$$\frac{X_{1.03^{25}}}{s_{25|j}} = 100,000 (1.03)^{25}, \text{ where } j = \frac{7\% - 3\%}{1 + 3\%} = 3.8835\%$$

$$\Rightarrow X_{s_{25|j}} = 100,000, \quad X = 2,439.12 \text{ (constant dollars)}$$

The actual annual deposit at the end of Year 1 is:

$$1.03X = 2,512.29 \text{ (nominal dollars)}$$

**Constant dollar method Plan #1**

- Use constant dollar cash flows
- Discount constant dollar cash flows at the real interest rate

The real interest rate is $j = \frac{7\% - 3\%}{1 + 3\%} = 3.8835\%$. 

<table>
<thead>
<tr>
<th>Age</th>
<th>40</th>
<th>41</th>
<th>42</th>
<th>43</th>
<th>…</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time $t$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>…</td>
<td>25</td>
</tr>
<tr>
<td>Constant dollars</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>…</td>
<td>$X$</td>
<td></td>
</tr>
<tr>
<td>Constant dollars</td>
<td>100,000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
This time, we have a simple annuity.

\[ X_{35|} = 100,000, \quad X = 2,439.12 \text{ (constant dollars)} \]

This matches the result in the nominal dollar method.

In this problem, the constant dollar method is simpler. This is because the problem gives us two figures, the level annual deposit and the accumulate value of the fund, in constant dollar to begin with.

**[2] Nominal dollar method Plan #2**

The fund’s accumulated value \( t=25 \):

\[ $100,000(1 + \text{inflation})^{25} = 100,000(1.03)^{25} = 209,377.79 \text{ (nominal dollars)} \]

<table>
<thead>
<tr>
<th>Age</th>
<th>40</th>
<th>41</th>
<th>42</th>
<th>43</th>
<th>…</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time ( t )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>…</td>
<td>25</td>
</tr>
<tr>
<td>Nominal dollars</td>
<td>( Y )</td>
<td>( 1.05Y )</td>
<td>( 1.05^2Y )</td>
<td>…</td>
<td>( 1.05^{24}Y )</td>
<td></td>
</tr>
<tr>
<td>Nominal dollars</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( 100,000(1.03)^{25} = 209,377.79 )</td>
</tr>
</tbody>
</table>

Once again, we use the 3 minute script for the geometric annuity:

\[ \frac{Y_{1.05^{24}}}{\text{payment factor at } t=25} \cdot s_{75|} = 100,000(1.03)^{25}, \text{ where } k = \frac{7\% - 5\%}{1 + 5\%} = 3.8835\% \]

\[ \Rightarrow Y = 2,051.64 \text{ (nominal dollars)} \]

So the actual annual deposit at the end of Year 1 is $2,051.64
Constant dollar method Plan #2

<table>
<thead>
<tr>
<th>Age</th>
<th>40</th>
<th>41</th>
<th>42</th>
<th>43</th>
<th>...</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>25</td>
</tr>
</tbody>
</table>

Nominal dollars: $Y, 1.05Y, 1.05^2Y, ... , 1.05^{24}Y$

Constant dollars: $(1.03^{-1})Y, (1.03^{-2})1.05Y, (1.03^{-3})1.05^2Y, ... , (1.03^{-25})1.05^{24}Y$

Constant dollars: 100,000

Cash flows increase by the following rate:

$$r = \frac{1.05}{1.03} - 1 = 1.9417\%$$

We need to accumulate the cash flows at the real interest rate:

$$j = \frac{7\% - 3\%}{1 + 3\%} = 3.8835\%$$

Once again, we have a geometric annuity.

$$Y(1.03^{-25})1.05^{24} \frac{s_{25}|}{R} = 100,000,$$ where

$$R = \frac{j - r}{1 + r} = \frac{3.8835\% - 1.9417\%}{1 + 1.9417\%} = 1.9048\%$$

Make sure you are not lost in the above equation.

$$\Rightarrow Y = 2,051.63 \text{ (nominal dollars)}$$

Problem 2

You invested in a foreign country and earned an annual effective return of 2% for 5 years. The country experienced deflation (negative inflation) of 3% per year effective during the period.

Calculate the real rate of return per year over the 5 year period.

Solution
The equation is:

\[ 1 + \text{real interest rate} = \frac{1 + \text{nominal interest rate}}{1 + \text{inflation rate}} \]

We are given:

The nominal interest rate = 2%
The inflation rate = -3%

So the real interest rate is:

\[ \text{real interest rate} = \frac{1 + \text{nominal interest rate}}{1 + \text{inflation rate}} - 1 = \frac{1 + 2\%}{1 + (-3\%)} - 1 = 5.15\% \]
Chapter 7 Loan repayment and amortization

- The borrower borrows a loan at $t=0$ and pays back the loan through $n$ installments

- Loan principal borrowed = PV of future payments

- At any time $t$ where $0 \leq t \leq n$ and $t$ is an integer before the loan is fully paid off, the outstanding balance of the loan can be calculated using one of the two methods

  1. **Prospective method.** The outstanding balance is equal to the PV of remaining payments to be paid in the future.

  2. **Retrospective method.** The outstanding balance is equal to the original loan principal accumulated to $t$ less than the previous payments accumulated to $t$.

- (A special case you need to remember) A loan is paid off by level payments $X$ payable at the end of each period for $n$ periods. For any time $t$ where $0 \leq t \leq n$ and $t$ is an integer, then

  1. PV of the loan at time zero = $X \, a_{n|}$

  2. The outstanding balance immediately after the $t$-th payment is $X \, a_{n-t|}$

  3. The principal portion of the 1-st, 2-nd, 3-rd, ..., $t$-th, .. $n$-th payment is $Xv^n, \, Xv^{n-1}, \, Xv^{n-2}, \ldots, Xv^{n-t+1}, \ldots, Xv$. (As time passes, more and more goes to the principal). Notice that the principal repayments nicely form a geometric progression. (However, the principal repayments nicely form a geometric progression only when the repayments are level.)

  4. The interest portion of the 1-st, 2-nd, 3-rd, ..., $k$-th, .. $n$-th payment is $X(1-v^n), \, X(1-v^{n-1}), \, X(1-v^{n-2}), \ldots, X(1-v^{n-t+1}), \ldots, X(1-v)$. (As time passes, less and less goes to the principal)
**Example 1**
A loan of $100,000 is repaid through monthly level payments over the next 20 years, the 1st payment due one month from today. The monthly payments are calculated using a nominal interest rate of 12% compounding monthly.

Calculate
- The monthly payments
- The principal outstanding immediately after 15th payment
- The principal and interest split of the 16th payment

**Solution**
Use a month as the compounding period

<table>
<thead>
<tr>
<th>Time t (Month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>......</th>
<th>237</th>
<th>238</th>
<th>239</th>
<th>240</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>......</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

\[ Y \, a_{\overline{240}|} = 100,000 \]

The monthly interest rate is:

\[ i = \frac{i^{(12)}}{12} = \frac{12\%}{12} = 1\% \]

\[ Y \, a_{\overline{240}|1\%} = 100,000 \quad \Rightarrow \quad Y = \frac{100,000 \, a_{\overline{240}|1\%}}{a_{\overline{240}|1\%}} = 1,101.086 \]

<table>
<thead>
<tr>
<th>Time t (Month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>......</th>
<th>15</th>
<th>16</th>
<th>......</th>
<th>237</th>
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</tbody>
</table>

\[ B_{15}^{p} = Y a_{\overline{240-15}|} = Y a_{\overline{225}|} \]
Next, we'll find the outstanding balance immediately after the 15th payment.

**Prospective method:**

\[ B_{15}^p = Ya_{240-155} = Ya_{225} = 1,101.086a_{225}|^{1%} = 98,372.815 \]

**Retrospective method:**

<table>
<thead>
<tr>
<th>Time t (Month)</th>
<th>Y</th>
<th>Y</th>
<th>Y</th>
<th>…</th>
<th>Y</th>
<th>Y</th>
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</table>

The value of the principal accumulated to \( t = 15 \):

\[ 100,000(1.01^{15}) = 116,096.896 \]

Previous payments from \( t = 1 \) to \( t = 15 \) accumulated to \( t = 15 \):

\[ Y_{15} = 1,101.086a_{225}|^{1%} = 17,724.066 \]

So the outstanding balance immediately after the 15th payment is:

\[ 116,096.896 - 17,724.066 = 98,372.83 \]

The slight difference between the results generated by the prospective method and the retrospective method is due to rounding.

The interest portion of the 16th payment:

\[ 98,372.8(1%) = 983.73 \]

The principal portion of the 16th payment:

\[ 1,101.09 - 983.73 = 117.36 \]

Alternatively, we can use the memorized rule on how to split a payment into principal and interest.

The principal portion:

\[ Yv^{n-t+1} = 1,101.09(1.01^{-1})^{240-16+1} = 117.36 \]

The interest portion:

\[ Y(1-v^{n-t+1}) = 1.101.09[1-(1.01^{-1})^{240-16+1}] = 983.73 \]
Example 2 (SOA 2005 May EA-1 #3)
Terms of a $1,000 loan issued by Smith:
Length of loan: 20 years
Payments: Level annual payments at the end of each year
Interest: 5% nominal, convertible semi-annually

When Smith receives each payment, it is immediately reinvested at 6%, compounded annually.

$R$ is the effective annual rate of interest earned by Smith on his combined investments over the 20 year period.

In what range is $R$?
(A) Less than 5.57%  
(B) 5.57%, but less than 5.62%  
(C) 5.62%, but less than 5.67%  
(D) 5.67%, but less than 5.72%  
(E) 5.72% or more

Solution
At $t = 0$ Smith invests (i.e. giving out) $1,000 cash. In return, he gets 20 annual payments. To find Smith’s annual rate of return, we need to find Smith’s wealth at $t = 20$. Then, we can calculate Smith’s return by solving the following equation:

$$1,000(1 + R)^{20} = \text{Smith's wealth @ } t = 20$$

Let $X$ represent the annual payment.

$$1,000 = Xa_{20\%, \text{semi-annual}}$$

where $i = \left(1 + \frac{5\%}{2}\right)^{2} - 1 = 5.0625\%$

Using BA II TVM or the annuity immediate formula, we get $X = 80.668$.  
Smith’s wealth @ $t = 20$ is $Xs_{20\%, \text{semi-annual}} = 2,967.4338$, using TVM or a formula.

$$\Rightarrow 1,000(1 + R)^{20} = 2,967.4338, \quad R = 5.59\%$$

So the answer is B.
Example 3 (SOA 2002 May EA-1 #1)
Loan repayment period: 5 years
Beginning loan amount: $75,000

Repayment Plan #1: Level annual payments at the beginning of each year
Repayment Plan #2: Level semi-annual payments at the end of each 6 month period

\[ A = \text{Annual payment under Repayment Plan #1} \]
\[ B = \text{Total payments in a year under Repayment Plan #2} \]
\[ 1,000d^{(i)} = 76.225 \]

In what range is the absolute value of \( |A - B| \)?
(A) Less than $1,000
(B) $1,000 but less than $1,025
(C) $1,025 but less than $1,050
(D) $1,050 but less than $1,075
(E) $1,075 or more

Solution
Let
\[ X = \text{the level annual payment under Repayment Plan #1} \]
\[ Y = \text{the level semi-annual payment under Repayment Plan #2} \]

Repayment Plan #1

<table>
<thead>
<tr>
<th>time ( t ) (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>payment</td>
<td>( X )</td>
<td>( X )</td>
<td>( X )</td>
<td>( X )</td>
<td>( X )</td>
</tr>
</tbody>
</table>

\[ 75,000 = X \, a_{3i} \]

where \( i \) is the annual effective rate

Repayment Plan #2

<table>
<thead>
<tr>
<th>time ( t ) (6 month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>payment</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
</tr>
</tbody>
</table>

\[ 75,000 = Y \, a_{\frac{3}{2}} \]
Where $j$ is the semi-annual effective rate

Next, we need to calculate $i$ and $j$.

\[
v = \frac{1}{1+i} = \left[1 - \frac{d(i)}{4}\right]^4 = \left[1 - \frac{0.076225}{4}\right]^4, \quad \Rightarrow \quad i = 8\%
\]

\[
(1 + j)^2 = 1 + i = 1.08, \quad \Rightarrow \quad j = 3.923\%
\]

Then using either BA II Plus TVM or annuity formulas, we can solve for $X$ and $Y$:

\[
75,000 = X \bar{a}_{3|}, \quad \Rightarrow \quad X = 17,392.81
\]

\[
75,000 = Y \bar{d}_{10|}, \quad \Rightarrow \quad Y = 9,211.43
\]

Then $A = X$, $B = 2Y$

\[
\Rightarrow |A - B| = |X - 2Y| = |17,392.81 - 2(9,211.43)| = 1,030.06
\]

The answer is C.

**Example 4 (SOA May 2002 EA-1 #5)**

Two $10,000 loans have the following repayment characteristics:

Loan 1: Level quarterly payments at the end of each quarter for five years.

Loan 2: Monthly interest payments on the original loan amount at the end of each month for 48 months plus a balloon repayment of principal at the end of the fourth year. The balloon repayment will be made using the accumulated value of a sinking fund created by level annual deposits at the beginning of each of the four years.

Effective annual interest rate on the loan: 8%.

Effective annual interest rate on the sinking fund: 9%.

$A = \text{Sum of repayments under Loan 1}.$

$B = \text{Sum of interest payments on Loan 2 plus sum of sinking fund payments}.$

In what range is the absolute value of $|A - B|$?
[A] Less than $875  
[B] $875 but less than $950  
[C] $950 but less than $1,025  
[D] $1,025 but less than $1,100  
[E] $1,100 or more

**Solution**

Loan #1  

<table>
<thead>
<tr>
<th>time $t$ (quarter)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>......</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>payment</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[10,000 = X \ a_{\frac{1}{3}^{|1/4|}}, \quad i = 1.08^{\frac{1}{3}} - 1 = 1.94265\%\]

\[\Rightarrow \quad X = 608.19 \text{ (using BA II Plus TVM or annuity formula)}\]

Loan #2 – interest payment  

<table>
<thead>
<tr>
<th>time $t$ (month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>......</th>
<th>47</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest payment</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[Y = 10,000 \left(1.08^{\frac{11}{12}} - 1\right) = 64.34\]

Loan #2 – installment in the sinking fund  

<table>
<thead>
<tr>
<th>time $t$ (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>installment</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[Z \ a_{\frac{1}{4}} = 10,000 \quad Z = 2,006.13\]

\[\Rightarrow \quad A = 20X, \quad B = 48Y + 4Z\]

\[\Rightarrow \quad |A - B| = |20X - B| = |20(608.19) - 48(64.34) + 4(2,006.13)| = 1,050.96\]

So the answer is D.
Example 5 (SOA May 2002 EA-1 #6)
Smith obtains a loan for $10,000 with 40 annual payments at an effective annual interest rate of 7%. The first payment is due one year from now.

A = Sum of interest paid in the even-numbered payments.
B = Sum of principal paid in the odd-numbered payments.

In what range is $A + B$?
[A] Less than $13,800
[B] $13,800 but less than $14,200
[C] $14,200 but less than $14,600
[D] $14,600 but less than $15,000
[E] $15,000 or more

Solution  D
We’ll use the imaginary cash flow method introduced in Chapter 1 “How to build a 3 minute solution script.”

<table>
<thead>
<tr>
<th>time $t$ (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>…</th>
<th>39</th>
<th>40</th>
<th>41</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Payment</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$\downarrow X$</td>
<td></td>
</tr>
<tr>
<td>Principal portion</td>
<td>$Xv^{40}$</td>
<td>$Xv^{39}$</td>
<td>$Xv^{38}$</td>
<td>$Xv^{37}$</td>
<td>$Xv^{2}$</td>
<td>$Xv$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interest portion</td>
<td>$X(1-v^{40})$</td>
<td>$X(1-v^{39})$</td>
<td>$X(1-v^{38})$</td>
<td>$X(1-v^{37})$</td>
<td>$X(1-v^{2})$</td>
<td>$X(1-v)$</td>
<td>$X(1-v^{39})$</td>
<td>$X(1-v^{37})$</td>
<td>$X(1-v)$</td>
</tr>
<tr>
<td>$A$ (even #)</td>
<td>$Xv^{40}$</td>
<td>$Xv^{38}$</td>
<td>$Xv^{2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B$ (odd #)</td>
<td>$Xv^{40}$</td>
<td>$Xv^{38}$</td>
<td>$Xv^{2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
10,000 = X \cdot \frac{a_{40|7\%}}{7\%}, \quad \Rightarrow \quad X = 750.09
\]

\[
A = X \left(1-v^{39}\right) + X \left(1-v^{37}\right) + \ldots + X \left(1-v\right)
\]

\[
= 20X - X \left(v^{39} + v^{37} + v^{35} + \ldots + v\right)
= X \left(20 - \frac{v^{41} - v^{2}}{1-v}\right)
\]

\[
= 750.09 \left(20 - \frac{1.07^{-1} - 1.07^{-31}}{1-1.07^{-2}}\right) = 9,832.75
\]

\[
B = X \left(v^{40} + v^{38} + \ldots + v^{2}\right)
= X \cdot \frac{v^{2} - v^{42}}{1-v^{2}}
= 750.09 \left(\frac{1.07^{-2} - 1.07^{-42}}{1-1.07^{-2}}\right) = 4,830.92
\]

\[
\Rightarrow \quad A + B = 9,832.75 + 4,830.92 = 14,663.37
\]
Example 6 (SOA May 2002 EA-1 #7)
Smith purchases a house for $120,000 and agrees to put 20% down. He takes out a 30-year mortgage, with monthly payments, with the first payment one month after the date of the mortgage. The interest rate is 8% compounded monthly.

Immediately following the 180th payment, Smith refinances the outstanding balance with a new 10-year mortgage, also with monthly payments, with the first payment one month after the date of the new mortgage. The new interest rate is 7.5% compounded monthly.

A = Amount of interest paid in the 100th payment of the first mortgage.
B = Amount of principal paid in the 100th payment of the refinanced mortgage.

In what range is \([A + B]\)?

- [A] Less than $1,300
- [B] $1,300 but less than $1,325
- [C] $1,325 but less than $1,350
- [D] $1,350 but less than $1,375
- [E] $1,375 or more

Solution

\[
120,000(80\%) = X a_{360|i}, \quad \text{where} \quad i = \frac{8\%}{12} = 0.66667\%
\]

\[
\Rightarrow \quad X = 704.41, \quad A = 704.41 \left(1 - v^{360-100+1}\right) = 580 \quad @0.66667\%
\]

The outstanding balance immediately before the refinancing

\[
P = X a_{180|i} = 73.710.30
\]

The monthly payment in the refinanced mortgage

\[
Y a_{180|j} = 73.710.30
\]
Where \( j = \frac{7.5\%}{12} = 0.625\% \)

The new monthly payment is \( \frac{73,710.3}{a_{(100)}^{0.625\%}} = 875 \)

\[ B = 875v^{120-100-1} = 768 \quad @0.625\% \]

\[ A + B = 1,348 \]

**Example 7 (SOA May 2001 EA-1 #3)**

<table>
<thead>
<tr>
<th>Amount of the loan</th>
<th>$100,000</th>
</tr>
</thead>
<tbody>
<tr>
<td># of originally scheduled level annual repayments</td>
<td>30</td>
</tr>
<tr>
<td>Time of 1(^{st}) repayment</td>
<td>1 year from the date of the loan</td>
</tr>
<tr>
<td>Additional payments made with the 5(^{th}) and 10(^{th}) scheduled repayments</td>
<td>$5,000 each</td>
</tr>
<tr>
<td>Effective annual interest rate</td>
<td>6%</td>
</tr>
</tbody>
</table>

Subsequent to the two additional payments, the loan continues to be repaid by annual repayments of the original size, plus a small final repayment one year after the last full repayment.

In what range is the total amount of interest saved due to the two additional payments?

- (A) Less than $23,500
- (B) $23,500 but less than $23,600
- (C) $23,600 but less than $23,700
- (D) $27,000 but less than $27,800
- (E) $23,800 or more

**Solution**

This problem is difficult. To solve it, you’ll need to find out the originally scheduled payments and the actual payments. To simplify our calculation, we’ll use $1,000 as one unit of money.

Let \( X \) represent the originally scheduled level annual repayment.
Scheduled repayments:

<table>
<thead>
<tr>
<th>Time $t$ (years)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>5</th>
<th>...</th>
<th>10</th>
<th>...</th>
<th>...</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repayments</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
</tr>
</tbody>
</table>


\[ 100 = X \ |

\[ \Rightarrow X = 7.26489115 \]

Actual repayments:

<table>
<thead>
<tr>
<th>Time $t$ (years)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>5</th>
<th>...</th>
<th>10</th>
<th>...</th>
<th>$m$</th>
<th>$m+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repayments</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X+5$</td>
<td>$X$</td>
<td>$X+5$</td>
<td>$X$</td>
<td>$X$</td>
<td>$Y$</td>
</tr>
</tbody>
</table>

To find $m$ and $Y$, we can calculate the outstanding loan balance at $t=10$ immediately after borrower pays $X+5$. First, though, we'll calculate the outstanding loan balance at $t=10$ immediately after the borrower pays $X$ assuming no extra repayments at $t=5$ or $t=10$:

<table>
<thead>
<tr>
<th>Time $t$ (years)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>5</th>
<th>...</th>
<th>10</th>
<th>...</th>
<th>...</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repayments</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
</tr>
<tr>
<td>Extra</td>
<td></td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

\[ P^{\text{scheduled}} = X \ |

\[ a_{\overline{30}|6\%} = 83.32772914 \]

To find the actual loan balance at $t=10$, we'll need to account for the two extra repayments of 5 each at $t=5$ or $t=10$:

\[ P^{\text{actual}} = P^{\text{scheduled}} - \left( 5 \times 1.06^5 + 5 \right) \]

\[ = 71.63660125 \]

Next, we are ready to calculate the # of actual full repayments after $t=10$:

\[ P^{\text{actual}} = X \ |

\[ a_{\overline{n}|6\%} \]

\[ \Rightarrow 71.63660125 = 7.26489115 a_{\overline{n}|6\%} \]

\[ \Rightarrow n = 15.37019964 \ (\text{using BA II Plus TVM}) \]
So there are 15 full repayments after $t = 10$.

Actual repayments:

<table>
<thead>
<tr>
<th>Time $t$ (years)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>5</th>
<th>...</th>
<th>10</th>
<th>11</th>
<th>...</th>
<th>25</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repayments</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X + 5$</td>
<td>$X$</td>
<td>$X + 5$</td>
<td>$X$</td>
<td>$X$</td>
<td>$Y$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$P_{\text{actual}} = 71.63660125$

At $t = 10$, the PV of the 15 full repayments from $t = 11$ to $t = 25$ and the PV of $Y$ must add up to $P_{\text{actual}}$:

$$Y \left(1.06^{-16}\right) + X a_{15|6\%} = P_{\text{actual}}$$

$$Y = \left(P_{\text{actual}} - X a_{15|6\%}\right) 1.06^{16} = \left(71.63660125 - 7.26489115 a_{15|6\%}\right) 1.06^{16} = 2.73892982$$

The total scheduled repayments: $30X = 30(7.26489115) = \$217.94673447$

Total interest paid if scheduled repayments are made:

$30X - 100 = 217.94673447 - 100 = 117.94673447$

The total actual repayments:

$25X + 5 + 5 + Y = 25(7.26489115) + 5 + 5 + 2.73892982 = 194.361208544$

Total actual interest paid:

$194.361208544 - 100 = 94.361208544$

Interest saved:

$117.94673447 - 94.361208544 = 23.585525926 = \$23,585.525926$

So the answer is B.

The following is the scheduled vs. actual payments (rounded to 2 decimals). In the exam, you don’t really need to write out the above table. Here I wrote the table just to make things clear.
<table>
<thead>
<tr>
<th>time t</th>
<th>scheduled repayments</th>
<th>actual repayments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>2</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>3</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>4</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>5</td>
<td>$7.26</td>
<td>$12.26</td>
</tr>
<tr>
<td>6</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>7</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>8</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>9</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>10</td>
<td>$7.26</td>
<td>$12.26</td>
</tr>
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<td>11</td>
<td>$7.26</td>
<td>$7.26</td>
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<td>12</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>13</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
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<td>14</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>15</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>16</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>17</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>18</td>
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</tr>
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</tr>
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<td>20</td>
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<td>$7.26</td>
</tr>
<tr>
<td>21</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>22</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>23</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>24</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>25</td>
<td>$7.26</td>
<td>$7.26</td>
</tr>
<tr>
<td>26</td>
<td>$7.26</td>
<td>$2.74</td>
</tr>
<tr>
<td>27</td>
<td>$7.26</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>$7.26</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>$7.26</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>$7.26</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$217.95</td>
<td>$194.36</td>
</tr>
</tbody>
</table>

The interest saved: $217.95 - $194.36 = 23.59 = $23,590.
Example 8 (SOA May EA-1 2001 #6)

Amount of the loan: $25,000
Term of loan: 8 years
Loan repayments: Quarterly, at the end of each quarter
Interest rate: 8% per year, compounded semiannually

The 11th and 12th scheduled payments are not made.

The loan is renegotiated immediately after the due date of the 12th (2nd missed) scheduled repayment with the following provisions:

13th (1st renegotiated) scheduled repayment: $X
14th through 32nd repayments:

- Each even-numbered repayment is $200 greater than the immediately proceeding (odd-numbered) repayment.
- Each odd-numbered repayment is equal to the immediately preceding even-numbered repayment.

The loan is to be completely repaid over the original term.

In what range is $X$?

- (A) Less than $250
- (B) $250 but less than $255
- (C) $255 but less than $260
- (D) $260 but less than $265
- (E) $265 or more

Solution C

The difficulty of this problem is to neatly keep track of the complex repayments in the renegotiated loan. One simple approach is to exhaustively list all of the repayments:

<table>
<thead>
<tr>
<th>Time (quarters)</th>
<th>loan repayments in the original plan</th>
<th>loan repayments in the revised plan</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(borrow 250)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$P$</td>
<td>$P$</td>
</tr>
<tr>
<td>2</td>
<td>$P$</td>
<td>$P$</td>
</tr>
<tr>
<td>3</td>
<td>$P$</td>
<td>$P$</td>
</tr>
<tr>
<td>4</td>
<td>$P$</td>
<td>$P$</td>
</tr>
<tr>
<td>5</td>
<td>$P$</td>
<td>$P$</td>
</tr>
<tr>
<td>6</td>
<td>$P$</td>
<td>$P$</td>
</tr>
<tr>
<td>7</td>
<td>$P$</td>
<td>$P$</td>
</tr>
<tr>
<td>8</td>
<td>$P$</td>
<td>$P$</td>
</tr>
</tbody>
</table>
In the above table, \( P \) is the level quarterly repayment originally scheduled. Please note that I used $100 as one unit of money. This way, the incremental repayment of $200 in the revised loan becomes 2; the loan amount changes from 25,000 to 250. This makes it easier for us to keep track of the loan repayments in the revised repayment plan.

\[
250 = P \cdot a_{\overline{3}|i}, \quad i = \left(1 + \frac{8\%}{2}\right)^{0.5} - 1 = 1.98\% \quad \Rightarrow \quad P = 10.6223
\]

The outstanding loan balance at \( t=10 \) immediately after the 10th repayment is made is: \( P \cdot a_{\overline{3}|i} = 10.6223 \cdot a_{\overline{3|1.98\%}} = 187.9560 \). This amount accumulates to \( 187.9560 \cdot (1+i)^{3} = 199.3454 \) at \( t=13 \). This outstanding principal must be paid in the renegotiated loan.

The repayments in the renegotiated loan consist of the following two cash flow streams:
The PV of the 1\textsuperscript{st} stream \( @ t = 13 \): 
\[ X \ddot{a}_{13\mid 1.98\%} = 16.7069X \]

To calculate the PV of the 2\textsuperscript{nd} stream \( @ t = 13 \), we simply enter the following cash flows into BA II Plus Cash Flow Worksheet:
Set I=1.98 (the interest rate). The NPV=156.1209. The PV of the two streams should be the outstanding loan balance at $t=13$:

$$199.3454 = 156.1209 + 16.7069X, \quad \Rightarrow \quad X = 2.5872 = \$258.72$$

**Example 9 (SOA May 2001 EA-1 #5)**

On 1/1/2002, Smith contributes $2,000 into a new saving account that earns 5% interest, compounded annually. On each January 1 thereafter, he makes another deposit that is 97% of the prior deposit. This continues until he has 20 deposits in all.

On each January 1 beginning on 1/1/2025, Smith makes annual withdrawals. There is to be a total of 25 withdrawals, with each withdrawal 4% more than the prior withdrawal, and the 25th withdrawal exactly depletes the account.

In what range is the sum of the withdrawals made on 1/1/2025 and 1/1/2026?

(A) Less than $5,410  
(B) $5,410 but less than $5,560  
(C) $5,560 but less than $5,710  
(D) $5,710 but less than $5,860  
(E) $5,860 or more

**Solution**

First, let’s organize the info into a table. We’ll use $1,000 as one unit of money. In the following table, the red numbers are deposits; the blue numbers are withdrawals. Let $X$ represent the 1st withdrawal.

<table>
<thead>
<tr>
<th>Time t (years)</th>
<th>1/1/2002</th>
<th>1/1/2003</th>
<th>...</th>
<th>1/1/2021</th>
<th>...</th>
<th>1/1/2025</th>
<th>1/1/2026</th>
<th>...</th>
<th>1/1/2049</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deposits (withdrawals)</td>
<td>2</td>
<td>2(0.97)</td>
<td>...</td>
<td>2(0.97)$^{19}$</td>
<td>...</td>
<td>$X$</td>
<td>1.04$X$</td>
<td>...</td>
<td>1.04$^{24}X$</td>
</tr>
</tbody>
</table>

Using the geometric annuity shortcut, we know that at $t=0$, the PV of the deposits is:

$$PV_1 = 2\bar{a}_{20|i}$$

$$PV_2 = X \bar{a}_{25|j}$$
\[ PV_i = 2\dot{a}_{20}, \text{ where } i = \frac{5\% - (-3\%)}{1 + (-3\%)} = 8.24742268\% \]

Please note that the 97% ratio is equivalent to -3% increase rate.

\[ \Rightarrow PV_i = 2\dot{a}_{20}8.24742268\% = 20.87005282 \]

At \( t = 23 \), the PV of the total withdrawals is:

\[ PV_2 = X \dot{a}_{23}, \text{ where } j = \frac{5\% - 4\%}{1 + 4\%} = 0.96153824\% \]

\[ \Rightarrow PV_2 = X \dot{a}_{23} = 22.34096995X \]

Because the 25th withdrawal depletes the saving account, we have:

\[
\frac{PV_1 \left(1.05^{23}\right)}{\text{accumulate deposits to } t = 23} = \frac{PV_2}{\text{PV of withdrawals at } t = 23}
\]

\[ 20.87005282 \left(1.05^{23}\right) = 22.34096995X \Rightarrow X = 2.8692963269 \]

So the sum of the withdrawals made on 1/1/2025 (1st withdrawal) and 1/1/2026 (2nd withdrawal) is:

\[ X + 1.04X = 2.04X = 2.04(2.8692963269) = 5.853364507 = \$5,853.364507 \]

The sum is \$5,853. The answer is D.
Example 10 (SOA May 2001 EA-1 #8)

<table>
<thead>
<tr>
<th>Date of the loan</th>
<th>1/1/2001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Date of 1st repayment</td>
<td>12/31/2001</td>
</tr>
<tr>
<td>Frequency of repayments</td>
<td>Annual</td>
</tr>
<tr>
<td>Term of loan</td>
<td>4 years</td>
</tr>
<tr>
<td>Amount of each repayment</td>
<td>$1,000</td>
</tr>
</tbody>
</table>

\[ v = \frac{1}{1+i} \]

The sum of the principal repayments in years one and two is equal to 10\(v^2\) times the sum of the interest repayments in years three and four.

Calculate \(v\).

Solution

We’ll use the imaginary cash flow method. In addition, we use $1,000 as one unit of money.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>time (t) (year)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>repayment</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
<td>▲ $1</td>
</tr>
<tr>
<td>principal portion</td>
<td>(v^{5-1} = v^4)</td>
<td>(v^{5-2} = v^3)</td>
<td>(v^{5-3} = v^2)</td>
<td>(v^{5-4} = v)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>interest portion</td>
<td>1(-v^4)</td>
<td>1(-v^4)</td>
<td>1(-v^4)</td>
<td>1(-v^4)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Imaginary cash flow

We are told: \(v^4 + v^3 = 10v^2(1 - v^2 + 1 - v)\)

\[ \Rightarrow v^2 + v - \frac{20}{11} = 0, \]

\[ \Rightarrow v = 0.938 \quad \text{(choose the positive root because } v > 0) \]
Example 11 (SOA May 2001 EA-1 #9)

<table>
<thead>
<tr>
<th>Amount of a loan</th>
<th>$1,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Date of loan</td>
<td>1/1/2001</td>
</tr>
<tr>
<td>Term of loan</td>
<td>30 years</td>
</tr>
<tr>
<td>Date of 1st repayment</td>
<td>1/1/2004</td>
</tr>
<tr>
<td>Frequency of repayments</td>
<td>Every 3 years</td>
</tr>
<tr>
<td>Interest rate</td>
<td>4% per year, compounded annually</td>
</tr>
</tbody>
</table>

Calculate the principal repaid in the 5th repayment.

Solution

We'll use $1,000 as one unit of money.

<table>
<thead>
<tr>
<th>Date</th>
<th>1/1/2001</th>
<th>1/1/2004</th>
<th>1/1/2007</th>
<th>...</th>
<th>...</th>
<th>...</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>time t t</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>5</td>
<td>..</td>
<td>10</td>
</tr>
<tr>
<td>repayment</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>principal repayment</td>
<td></td>
<td></td>
<td>X $v^6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The interest rate per 3 years is $i = 1.04^3 - 1 = 12.4864\%$

$1 = X \cdot a_{\overline{3}|t} \Rightarrow X = 0.18052244$

$X \cdot v^6 = 0.18052244 \cdot (1 + 12.4864\%)^6 = 0.08911095 = 89.11$

So the principal repaid in the 5th repayment is $89.11$. 
Example 12 (SOA May 2001 EA-1 #12)
A $200,000, 30-year variable rate mortgage loan is obtained. The 1st monthly payment is due one month from the date of the loan. At the time the loan is obtained, the interest rate is 7%, compounded monthly.

On the 2nd anniversary of the loan, the interest rate is increased to 7.5%, compounded monthly.

On the 4th anniversary of the loan, the interest rate is increased to 8%, compounded monthly, and remains fixed for the remainder of the mortgage repayment period.

Calculate the total interest paid on the loan.

Solution
We’ll use $1,000 as one unit of money.

Step 1 Calculate the monthly payment when the interest rate is 7%

\[
\begin{array}{c|cccccccc}
\text{Time t (month)} & 0 & 1 & 2 & \ldots & 24 & \ldots & 48 & \ldots & 360 \\
\text{payment} & X & X & X & X & X & X & X & X & X
\end{array}
\]

\[
200 = X a_{360|i}, \text{ where } i = \frac{7\%}{12}.
\]

In BA II Plus TVM, set PV= -200, N=360, I/Y=7/12. CPT “PMT.”

\[X = 1.33060499\]

After finding \(X\), don’t erase the data inputs in TVM. You’ll need to reuse these inputs.

Step 2 Calculate the outstanding loan balance when the interest rate changes to 7.5%

\[
\begin{array}{c|cccccccc}
\text{Time t (month)} & 0 & 1 & 2 & \ldots & 24 & \ldots & 48 & \ldots & 360 \\
\text{payment} & X & X & X & X & X & X & X & X & X
\end{array}
\]

\[
P_{24} = X a_{360-24|i} = X a_{336|i}.
\]

In TVM, simply reset N=336. CPT “PV.” You should get: \(P_{24} = 195.7898947\)
### Step 3  
Recalculate the level monthly payment under the new interest rate of 7.5%

<table>
<thead>
<tr>
<th>Time t (month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>24</th>
<th>…</th>
<th>48</th>
<th>…</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>payment</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$Y$</td>
<td>$Y$</td>
<td>$Y$</td>
<td>$Y$</td>
<td></td>
</tr>
</tbody>
</table>

\[ P_{24} = Y a_{\frac{360 - 24}{12},j} = Y a_{\frac{336}{12},j}, \text{ where } j = \frac{7.5\%}{12} \]

In TVM, simply reset I/Y=7.5/12. CPT “PMT.” You should get: 

\[ Y = 1.39572270 \]

### Step 4  
Calculate the outstanding loan balance when the interest rate changes to 8%

<table>
<thead>
<tr>
<th>Time t (month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>24</th>
<th>…</th>
<th>48</th>
<th>…</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>payment</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$Y$</td>
<td>$Y$</td>
<td>$Y$</td>
<td>$Y$</td>
<td></td>
</tr>
</tbody>
</table>

\[ P_{48} = Y a_{\frac{360 - 48}{12},j} = Y a_{\frac{312}{12},j} \]

In TVM, simply reset N=312. CPT “PV.” You should get: 

\[ P_{48} = 191.3502131 \]

### Step 5  
Recalculate the level monthly payment under the new interest rate of 8%

<table>
<thead>
<tr>
<th>Time t (month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>24</th>
<th>…</th>
<th>48</th>
<th>…</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>payment</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$Y$</td>
<td>$Y$</td>
<td>$Y$</td>
<td>$Y$</td>
<td></td>
</tr>
</tbody>
</table>

\[ P_{48} = Z a_{\frac{360 - 48}{12},k} = Z a_{\frac{312}{12},k}, \text{ where } k = \frac{8\%}{12} \]

In TVM, simply reset I/Y=8/12. CPT “PMT.” You should get: 

\[ Z = 1.45923312 \]

The total loan repayments: \[ 24X + 24Y + 312Z \]

The total interest paid on the loan: 

\[ 24X + 24Y + 312Z - 200 = 320.7125968 = \$320,712.60 \]
Example 13 (SOA May 2004 EA-1 #27)
A loan is made on 1/1/2004.

Loan repayments: 120 level monthly payments of interest and principal with 1\textsuperscript{st} payment at 2/1/2004.

Interest is charged on the loan at a rate of $i^{(12)} = 7.5\%$.

Amount of interest paid in the 54\textsuperscript{th} payment of loan = $100

$P$ = Principal outstanding on the loan after the 90\textsuperscript{th} payment.

Calculate $P$.

Solution

<table>
<thead>
<tr>
<th>time $t$ (month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>54</th>
<th>…</th>
<th>90</th>
<th>…</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>payments</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
</tr>
</tbody>
</table>

Using the imaginary cash flow method, we know that the interest portion of the 54\textsuperscript{th} payment is:

$$X \left( 1 - v^{121-54} \right) = X \left( 1 - v^{67} \right), \text{ where } X \text{ is the level monthly payment}$$

Next, we need to find the monthly effective interest rate:

$$i^{(12)} = \frac{7.5\%}{12} = 0.625\%$$

$$\Rightarrow \ X \left( 1 - v^{67} \right) = X \left[ 1 - (1 + 0.625\%)^{-67} \right] = 100 \ , \ X = 293.02$$

<table>
<thead>
<tr>
<th>time $t$ (month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>54</th>
<th>…</th>
<th>90</th>
<th>…</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>payments</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
</tr>
</tbody>
</table>

$$P = X \ a_{\overline{90}|i} = X \ a_{\overline{120}|i}$$

$$\Rightarrow \ P = X \ a_{\overline{90}|i} = 293.02 \ a_{\overline{120}|i} = 7,992.98$$
Example 14 (SOA May 2004 EA-1 #7)
Details of a loan made on 1/1/2004:

- # of payments: 10
- Amount of each payment: $5,000
- Date of 1st payment: 12/31/2004
- Interest rate: 8% compounded annually

Immediately after the 6th payment, an additional $10,000 payment is made. The loan is re-amortized over a longer term to provide for annual payment of $1,000 and a final smaller payment of \( X \) one year after the last $1,000 payment.

Calculate \( X \).

Solution

We'll use $1,000 as one unit of money.

The outstanding loan balance immediately after the 6th payment is made:

\[
\begin{array}{ccccccc}
\text{Time } t \text{ (year)} & 0 & 1 & 2 & \ldots & 6 & 7 & \ldots & 10 \\
\text{payment} & & 1 & 2 & \ldots & 5 & 5 & \ldots & 5 \\
\end{array}
\]

\[
5 \ a_{\bar{10}|8\%} = 16.5606342
\]

At \( t = 6 \), the borrowers pays additional $10 and immediately re-amortize the remaining loan balance. After that, he pays $1 per year for \( n \) years:

\[
16.5606342 - 10 = a_{\bar{n}|8}\%
\]

Enter the following into BA II Plus TVM:

- PV = -6.5606342, PMT=1, I/Y=8, FV=0. Press “CPT” “N.”

You should get: \( N = 9.66886977 \)

So after the re-amortization, the borrower pays 9 level payments of $1 each and pays a final payment of \( X \) after the 9th payment

\[
\Rightarrow \quad 16.5606342 - 10 = a_{\bar{9}|8}\% + X v^{10}
\]

Solving the above equation, we get:

\[
X = 0.67735471 = $677.35471
\]
Example 15 (SOA May 2000 EA-1 #24)

Amount of a loan: $250,000
Frequency of repayments: quarterly, at the end of each quarter
# of repayments: 100
Interest rate: 8% per year, compounded continuously

In which repayment does the principal component first exceed the interest component?

Solution
Let $X$ represent the level quarterly repayment.

<table>
<thead>
<tr>
<th>Time $t$ (quarter)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$k$</th>
<th>....</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
</tr>
</tbody>
</table>

Assume that in the $k$-th repayment the principal component first exceeds the interest component.

The principal component of the $k$-th repayment is $Xv^{101-k}$; the interest component is $X(1-v^{101-k})$.

We need to solve the following equation:

$$Xv^{101-k} > X(1-v^{101-k})$$

$$\Rightarrow v^{101-k} > 1 - v^{101-k} \quad 2v^{101-k} > 1, \quad v^{101-k} > 0.5$$

We are given the force of interest $\delta = 8\%$. So the quarterly discounting factor is:

$$v = e^{-\frac{\delta}{4}} = e^{-\frac{8\%}{4}} = e^{-0.02}$$

$$v^{101-k} > 0.5 \quad \Rightarrow e^{-0.02(101-k)} > 0.5$$

$$-0.02(101-k) > \ln 0.5 \quad 101-k < \frac{\ln 0.5}{-0.02} = 34.657 \quad k > 66.3426$$

So the 67th repayment is the 1st time the principal component exceeds the interest component.
Example 16 (SOA May 2000 EA-1 #13)
Date of a loan: 1/1/1990
Amount of loan: $100,000
Interest rate: 12% per year, compounded monthly
Term of loan: 360 level monthly repayments
First repayment date: 2/1/1990

Immediately after making the 120th repayment, the borrower decides to add $Q$ to each monthly repayment so that the loan will be repaid after having made a total of 160 monthly repayments.

Calculate $Q$.

Solution

Original repayment schedule where $X$ is the monthly repayment

<table>
<thead>
<tr>
<th>Time $t$ (month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>120</th>
<th>121</th>
<th>…</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td></td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
</tr>
</tbody>
</table>

$100,000 = X \ a_{120|12}$, where $i = \frac{12\%}{12} = 1\%$.

$\Rightarrow \ X = 1.028.612597$

$P = X \ a_{360|12} = 93,417.9957$

So the outstanding balance immediately after the 120th payment made is 93,417.9957.

Actual repayment

<table>
<thead>
<tr>
<th>Time $t$ (month)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>120</th>
<th>121</th>
<th>…</th>
<th>160</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td></td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X + Q$</td>
<td>$X + Q$</td>
<td>$X + Q$</td>
<td></td>
</tr>
</tbody>
</table>

$93,417.9957 = (X + Q) a_{160|12}$

$\Rightarrow \ X + Q = 2,845.100921 \Rightarrow Q = 1,816.488324$
Example 17
Payments are made at the end of the year for 30 years, with the payment equal to $12 for each of the first 20 payments and $9 for each of the last 10 payments. The interest portion of the 11th payment is twice the interest portion of the 21st payment.

Calculate the interest portion of the 21st payment.

Solution

First define some symbols:

- $OB_t$ is the outstanding loan balance immediately after the $t$th payment is made
- $K_t$ is the payment made at the $t$th payment. $K_t = I_t + P_t$
- $I_t$ is the interest portion of the $t$th payment
- $P_t$ is the principal portion of the $t$th payment
- $i$ is the effective interest rate per payment period (1 year in this problem)

Then $I_t = OB_{t-1} \times i$ and $P_t = K_t - I_t = OB_t - OB_{t+1}$

<table>
<thead>
<tr>
<th>Time t (Yr)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>10</th>
<th>11</th>
<th>12</th>
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<th>20</th>
<th>21</th>
<th>22</th>
<th>...</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>$12</td>
<td>$12</td>
<td>$12</td>
<td>$12</td>
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<td>$12</td>
<td>$9</td>
<td>$9</td>
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<td>$9</td>
</tr>
</tbody>
</table>

$OB_{10} = 12a_{\overline{10}|} + 9v^{10}a_{\overline{10}|}$

$OB_{20} = 9a_{\overline{10}|}$

The interest portion of the 11th payment is $I_{11} = OB_{10} \times i = \left(12a_{\overline{10}|} + 9v^{10}a_{\overline{10}|}\right)i$

The interest portion of the 21st payment is $I_{21} = OB_{20} \times i = \left(9a_{\overline{10}|}\right)i$

$\left(12a_{\overline{10}|} + 9v^{10}a_{\overline{10}|}\right)i = 2\left(9a_{\overline{10}|}\right)i$
\[12 + 9v^{10} = 29\]
\[v^{10} = \frac{6}{9}\]

\[\Rightarrow I_{21} = OB_{20} \times i = (9a_{\overline{10}|})i = 9 \times \frac{1-v^{10}}{i} \times i = 9 \times (1-v^{10}) = 9 \times \left(1 - \frac{6}{9}\right) = 3\]

Alternative method to calculate \(I_{21}\) after we find that \(v^{10} = \frac{6}{9}\).

Prospectively, the payments at \(t=21, 22, \ldots, 30\) are level. So we can use the imaginary cash flow method to calculate the interest portion of the payments made at \(t=21, 22, \ldots, 30\).

So we add an imaginary cash flow of $9 at \(t=31\).

<table>
<thead>
<tr>
<th>Time (t) (Yr)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>…</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>…</th>
<th>30</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$9</td>
</tr>
</tbody>
</table>

The principal portion of the 21\(^{st}\) payment is

\[P_{21} = 9v^{31-21} = 9v^{10} = 9 \times \frac{6}{9} = $6\]

The interest portion of the 21\(^{st}\) payment is

\[I_{21} = K_{21} - P_{21} = 9 - 6 = $3\]

**Example 18**

A loan of $100 at a nominal interest rate of 12\% convertible monthly is to be repaid by 6 monthly payments. The 1\(^{st}\) monthly payment starts one month from today. The first 3 monthly payments are \(x\) each; the last 3 monthly payments are \(3x\) each.

\(A = \)the principal portion of the 3\(^{rd}\) payment
\(B = \)the interest portion of the 5\(^{th}\) payment

Calculate \(A + B\)
Solution

\[
100 = xa_{\bar{n}} + 3xv^3a_{\bar{n}} \quad \text{where } i = 1%
\]

\[
x = \frac{100}{(1 + 3v^3)a_{\bar{n}}} = \frac{100}{(1 + 3\times 1.01^{-3})} = 8.692
\]

The interest portion of the 3rd payment: 

\[
I_3 = OB_3 \times i = 84.536 \times 1% = 0.845
\]

The principal portion of the 3rd payment: 

\[
P_3 = K_3 - I_3 = 8.692 - 0.845 = 7.847
\]

The interest portion of the 5th payment is: 

\[
I_5 = OB_5 \times i = 51.38 \times 1% = 0.514
\]

Alternative method to calculate $I_5$. Since prospectively the payments are level at $t = 5,6$, we can use the imaginary cash flow method. So we set a fake cash flow at $t = 7$.

\[
P_3 = 3x \times v^{7-5} = 3 \times 8.692 \times 1.01^{-2} = 25.562
\]

\[
I_5 = K_5 - P_3 = 3x \times v^{7-5} = 3 \times 8.692 - 25.562 = 0.514
\]

\[
\Rightarrow P_3 + I_5 = 7.847 + 0.514 = 8.361
\]
Example 19
A loan is repaid by annual installments of $X$ at the end of each year for 20 years. The annual interest rate is 6%.

Let $A=$ the total principal repaid in the first 5 years
Let $B=$ the total principal repaid in the last 5 years

Calculate $A/B$.

Solution
$$A = X \left( v^{20} + v^{19} + v^{18} + v^{17} + v^{16} \right)$$
$$B = X \left( v^5 + v^4 + v^3 + v^2 + v \right)$$
$$A/B = v^{15} = 1.06^{15} = 0.4173$$

Example 20
Payments are made at the end of the year for 30 years, with payment equal to 120 for each of the first 20 years and 90 for each of the last 10 years. The interest portion of the 11th payment is twice the interest portion of the 21st payment. Calculate the interest portion of the 21st payment.

Solution

<table>
<thead>
<tr>
<th>Time t (Yr)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>…</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>…</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>120</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
</tr>
</tbody>
</table>

The outstanding balance immediately after the 10th payment is made:
$$OB_{10} = 120a_{\overline{10}|10} + v^{10} \left( 9a_{\overline{10}|10} \right)$$

The interest portion of the 11th payment is
$$I_{11} = OB_{10} \times i = \left( 120a_{\overline{10}|10} + 90a_{\overline{10}|10}v^{10} \right)i$$

Similarly, the interest portion of the 21st payment is:
$$I_{21} = OB_{20} \times i = \left( 90a_{\overline{10}|10} \right)i$$

$$\left( 120a_{\overline{10}|10} + 90a_{\overline{10}|10}v^{10} \right)i = 2 \left( 90a_{\overline{10}|10} \right)i \quad \Rightarrow 120 + 90v^{10} = 2 \left( 90 \right) \quad \Rightarrow v^{10} = \frac{6}{9}$$

$$i = v^{-1} - 1 = \left( \frac{6}{9} \right)^{-1/10} - 1 = 4.1380\% \quad a_{\overline{10}|10} = \frac{1-v^{10}}{i} = \frac{1-6/9}{4.1380\%} = 8.0555$$

$$\Rightarrow I_{21} = \left( 90a_{\overline{10}|10} \right)i = 90 \times 8.0555 \times 4.1380\% = 30$$
Chapter 8   Sinking fund

Key points

- At $t = 0$, the borrower borrows the principal.

- At each year for $n$ years, the borrower pays only the interest accumulated during that year, stopping the outstanding principal from growing or declining. Thus, the outstanding principal immediately after the annual interest payment is always equal to the original principal at $t = 0$. (The borrower is only dealing with the interest due on year by year basis. He is not worrying about paying the principal at this stage.)

- At the end of the term of the loan, the borrower pays a lump sum equal to the borrowed principal, terminating the loan. (Eventually, the borrower has to pay the principal.)

- To make sure he can eventually pay off the principal, the borrower periodically deposits money into a fund. This fund accumulates with interest (can be different from the interest rate used to calculate the annual interest payment). At the end of the term of the loan, this fund accumulates enough money to pay off the principal.

Sample problems

Problem 1 (#12 SOA May 2003 EA-1)

A ten-year loan of $10,000 at an 8% annual effective rate can be repaid using any of the following methods:

I. Amortization method, with level annual payments at the end of each year.

II. Repay the principal at the end of the years while paying 8% annual effective interest on the loan at the end of each year. The principal is repaid by making equal annual deposits at the end of each year into a sinking fund earning interest at 6% annual effective so that the sinking fund accumulates to $10,000 at the end of the 10th year.

III. Same as II, except the sinking fund earns 8% annual effective.

IV. Same as II, except the sinking fund earns 12% annual effective.
Rank the annual payment amounts of each method.

[A] I < II < III < IV
[B] II < I = III < IV
[C] I < IV < III < II
[D] IV < I < III < II
[E] The correct answer is not given by [A], [B], [C], or [D] above.

**Solution**

Without doing any math, you should know

I = III
IV < III < II

**Reason for the 1st equation:**

The loan and sinking fund have the same interest rate. Consequently, our annual payments under I and III should be same. If we have the same interest rate, it doesn’t matter whether we separately pay the annual interest and the principal (the sinking fund method) or we combine the annual interest and the annual principal into one annual payment (loan amortization). We have the same pie, no matter how we slice it.

**Reason for the 2nd equation:**

II, III, and IV differ only in annual sinking fund payments. Annual deposits to the sinking fund under II, III, and IV must each accumulate to $10,000 at the end of Year 10. As a result, the higher the interest rate in the sinking fund, the smaller the annual deposit. Because the interest rates earned in the sinking fund are 6%, 8%, and 12% respectively in II, III, and IV, the annual deposit to the sinking fund is largest in II, smaller in III, and smallest in IV.

So the correct answer is (E).

Alternatively, you can calculate the annual payment for each of the four plans.
### Plan I

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

$X_{a_{10\%}} = 10,000 \implies X = 1490.29$ (Total annual payment)

### Plan II

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$800$</td>
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<td></td>
</tr>
</tbody>
</table>

$A_{a_{6\%}} = 758.68 \implies X + A = 1,558.68$ (Total annual payment)

### Plan III

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$800$</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$B_{a_{8\%}} = 690.29 \implies X + B = 1,490.29$ (Total annual payment)

### Plan IV

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$800$</td>
<td></td>
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</tr>
</tbody>
</table>

$C_{a_{2\%}} = 569.84 \implies X + C = 1,369.84$ (Total annual payment)
Problem 2 (#17 SOA May 2004 EA-1)

<table>
<thead>
<tr>
<th>Loan term</th>
<th>$100,000 borrowed on 1/1/2004, issued at a 5% annual effective interest rate.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan repayment</td>
<td>No repayments of principal or interest are made on the loan until a sinking fund has accumulated to pay the balance in full. This occurs 12/31/2019.</td>
</tr>
<tr>
<td>Sinking fund</td>
<td>$10,000 annual deposits from 12/31/2004 through 12/31/2011 and $5,000 annual deposits beginning on 12/31/2012. The sinking fund accumulation is $i$% until 12/31/2011 and $k$% thereafter.</td>
</tr>
</tbody>
</table>

During calendar year 2011, interest accrued on the loan and interest earned on the sinking fund are the same.

In what range is $k$%?

[A] Less than 6.10%
[B] 6.10% but less than 6.60%
[C] 6.60% but less than 7.10%
[D] 7.10% but less than 7.60%
[E] 6.60% or more

Solution

This is not a typical sinking fund. In a typical sinking fund, the borrower pays the annual interest due. In addition, he sets up a sinking fund to accumulate the principal at the end of the loan term.

In this problem, however, the borrower sets up a sinking fund to pay, at the end of the loan term, both the principal and the interest due.

You shouldn’t be scared. Just apply the general principal of the time value of money and you should do fine.

To neatly track down the timing of each cash flow, we’ll convert 12/31 of a year to 1/1 of the next year. For example, we convert 12/31/2019 to 1/1/2020. This helps prevent the off-by-one error.

To simplify our calculation, we’ll use $1,000 as the unit money.
We can set up two equations:

- The sinking fund and the loan should accumulate to the same amount at \( t=16 \).

\[
10s_{8\%}^{i\%} + \left(1 + k\%\right)^8 + 5s_{5\%}^{k\%} = 100(1.05)^{16}
\]

- The loan and the sinking fund should generate the same interest amount from \( t=7 \) to \( t=8 \)

\[
100(1.05^7)5\% = 10s_{7\%}^{i\%}
\]

The second equation is easier.

\[
s_{7\%}^{i\%} = \frac{(1+i)^7 -1}{i} = (1+i\%)^7 -1 = 10(1.05^7)5\% , \quad i = 7.907\%
\]

Solving the equation:

\[
10s_{7.907\%}^{(1+k\%)} + 5s_{5\%}^{k\%} = 100(1.05)^{16} , \quad 106.021(1+k\%)^8 + 5s_{5\%}^{k\%} = 218.287
\]

We translate \( 106.021(1+k\%)^8 + 5s_{5\%}^{k\%} = 218.287 \) into the follow cash flow diagram:
In TVM, set PV=106.021, PMT=5, N=8, FV= - 218.287. Use annuity immediate mode. You should get: I/Y=5.98%. So the correct answer is [A].

**Problem 3 (#22 SOA May 2000 EA-1)**

Date of a loan: 1/1/2000
Amount of loan: $X$
Date of the 1st repayment: 12/31/2000
Frequency of repayments: Annually
# of repayments: 10
Amount of each repayment: $1,000

Method of repayment:
One half of the loan is repaid by the amortization method using an interest rate of 7% per annum compounded annually.

The other half is repaid by the sinking fund method where the lender receives 7% per annum, compounded annually, on this portion of the loan, and the sinking fund accumulates at 6% per annum, compounded annually.

Calculate $X$.

**Solution**

One half of the loan is repaid through amortization method. Let $A$ represent the level annual repayment in the amortization method.

\[
\frac{X}{2} = A \cdot a_{10|7\%}, \quad \Rightarrow \quad A = \frac{X}{2 \cdot a_{10|7\%}}
\]
The other half is repaid by the sinking fund method. The annual repayment in this method consists of two parts:

- Annual interest payment of \( \frac{X}{2} \times 7\% \)

- Annual deposit \( B \) into the sinking fund to accumulate to \( \frac{X}{2} \) at \( t=10 \). The interest rate is 6%.

<table>
<thead>
<tr>
<th>Time ( t ) (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>( B )</td>
<td>( B )</td>
<td>( B )</td>
<td>( B )</td>
<td>( B )</td>
</tr>
</tbody>
</table>

\[
\frac{X}{2} = B \frac{s_{10|6\%}}{1}, \quad \Rightarrow \quad B = \frac{X}{2s_{10|6\%}}
\]

So the total annual repayment is:

\[
\frac{X}{2a_{10|7\%}} + \frac{X}{2} \times (7\%) + \frac{X}{2s_{10|6\%}} = 1,000
\]

\[
\Rightarrow \quad \frac{X}{2} \left[ \frac{1}{a_{10|7\%}} + (7\%) + \frac{1}{s_{10|6\%}} \right] = 1,000
\]

\[
a_{10|7\%} = 7.02358154, \quad s_{10|6\%} = 13.18079494
\]

\[
\Rightarrow \quad X = 6,938.53077
\]
Chapter 9  Callable and non-callable bonds

Bond is a standardized loan. The borrower borrows money from a lender. The borrower pays back the borrowed money by paying regular installments called coupons and a final lump sum (called redemption value) in the end.

Key concepts:

Par or face value
- repaid at the end of the bond’s life

Coupon
- the interest rate on the nominal amount of the bond
- fixed throughout the life of the bond

Time to maturity
- the length of time until the bond is redeemed

The market price of a coupon bond is calculated by discounting all the future cash flows at the yield to maturity
- The yield to maturity is the investor’s opportunity cost of capital (i.e. return earned by investing in other assets).
- Because investors demand to earn the prevailing market interest rate, the yield to maturity is the prevailing market interest rate. Here we assume a constant interest rate that doesn’t change with time (i.e. a flat yield curve).
- YTM is the IRR an investor would realize by purchasing the bond, holding it to maturity, and reinvesting coupons at YTM.
- If the bond is purchased at a premium, YTM is less than the coupon rate. Coupons will be reinvested a lower rate; the investor will realize a capital loss when the bond matures.
- If the bond is purchased at a discount, YTM is greater than the coupon rate. Coupons will be reinvested a higher rate; the investor will realize a capital gain when the bond matures.
Bond prices change in the opposite direction from the change of the prevailing market interest rate

- If the market interest rate goes up, the price of a bond goes down; we have to discount the future cash flows at a higher discount rate.

- If the market interest rate goes down, the price of a bond goes up; we have to discount the future cash flows at a lower discount rate.

Selling a bond at par, premium, discount

- When the coupon rate is equal to the prevailing market interest rates, the price of the bond is equal to its par value. The bond sells at par value.

- When coupon rate is below the prevailing market interest rates at a certain point of time, the price of the bond will fall below its par value. The bond sells at a discount.

- When the coupon rate is above the prevailing market interest rates at a certain point of time, the price of the bond will be greater than its par value. The bond sells at a premium.

Bond cash flow diagram – make sure you know how to draw one

<table>
<thead>
<tr>
<th>Unit time = per coupon period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t</td>
</tr>
<tr>
<td>Coupon</td>
</tr>
<tr>
<td>Redemption</td>
</tr>
</tbody>
</table>

- Set per coupon period as the unit time. If a bond pays coupons every 6 months, then the unit time is 6 months.

- Cash flows consist of regular coupons plus a final redemption amount.

- If a bond pays 8% coupons convertible semiannually per $100 face amount, this means that you get $4 once every 6 months, not $8 once every year. Make sure you remember this.
Big ideas:

1. **Bond is a loan just like a student loan or home mortgage.** If you buy a bond issued by AT&T, you are lending your money to AT&T. The principal amount you lend to AT&T is just the present value of the bond cash flows (regular coupons plus face amount at the end of the loan term) discounted at YTM. In turn, AT&T repays the loan through installments (regular coupons plus face amount at the end of the loan term). Everything you learn about loan (such as amortization) applies to a bond. This is the most important concept you need to know about a bond. Knowing this enables you to cut to the chase of many bond problems.

2. **Bond is a standardized loan.**

<table>
<thead>
<tr>
<th>Difference</th>
<th>Bond</th>
<th>Typical loan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Borrower</td>
<td>Government and corporations (called bond issuers). They borrow money to fund special projects or business growth.</td>
<td>Can be anyone. Individuals, corporations, government.</td>
</tr>
<tr>
<td>Lender</td>
<td>Anyone who has idle money and wants to earn interest. When you buy a bond, you become a lender.</td>
<td>Typically banks.</td>
</tr>
<tr>
<td>Payback method</td>
<td>Bond issuers pay back loans through regular payments called coupons and a final payment. Coupons are typically paid once every 6 months. A final payment is made at the end of borrowing period.</td>
<td>Hard for a homeowner to sell his mortgage to other investors.</td>
</tr>
<tr>
<td></td>
<td>In old days before computers, investors actually cut off coupons from a coupon book and mailed it to the issuing firm to claim interest payment. This is the origin of the name coupon.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Some bonds don’t pay coupons. When a bond matures, the borrower pays the principal plus the interest accrued thus far. These bonds are called zero-coupons.</td>
<td></td>
</tr>
<tr>
<td>Can the lender sell the loan to another investor who has idle cash and wants to earn interest?</td>
<td>Bonds are standardized loan and can be easily bought and sold in the open market.</td>
<td></td>
</tr>
</tbody>
</table>

3. **Why standardizing loan (i.e. inventing bond)?**

- One-off borrowing is time-consuming. Corporations and government always need extra money to expand business or finance special projects (such as building highways). If every time they need money they have to walk to a bank and fill out a loan application, the borrowing process is time-consuming.
• One-off borrowing is expensive. Banks often charge extra fees to cover costs associated with loan approvals (such as checking borrower’s credits and negotiating loan terms). As a result, banks often charge a higher interest rate to cover these fees. So government and corporations rather not borrow money from banks. By issuing bonds and borrowing money from the general public, government and corporations lower their borrowing cost.

• Standardized loans are easier for the general public to understand than one-off loans.

• Standardized loans can be sold and traded in the open market (such as New York Stock Exchange and NASDAQ). If a lender wants to back out, he can quickly sell the bond to someone else (may incur a loss though). This helps the sales of the bonds.

• Standardized loans allow multiple investors to split a large amount of loan. If a company wants to borrow $100,000,000 cash, it doesn’t have to spend lot of time looking for one wealthy person or a big bank to lend this amount. Instead, the company can simply issue 100,000 bonds with each bond selling $1,000.

4. **Call a bond** = buy back the bond prior to the bond’s maturity = refinance the loan at a lower coupon rate

• If you ever refinanced your student loan, or if your family refinanced the house, you’ll understand a callable bond.

• Refinancing a student loan. Say you borrowed the U.S. government a $50,000 student loan at 8% interest per year. After you graduated, you started to pay your student loan through monthly payments. You were paying 8% interest per year. Gradually, you reduced your loan balance to $45,000. Then the market interest rate dropped to 4%. You didn’t want to continue paying the U.S. government 8% interest rate -- the market rate was only 4%. What could you do? You walked to a bank, borrowed $45,000 at 4%, and immediately mailed this $45,000 check to the U.S. government. Now you no longer owed the U.S. government anything. You just owed the bank $45,000 but at 4% interest rate.
Call a bond = Refinance the bond. Once the bond is issued, the cash flows of the bond are set. Bond issuers pay the set coupon rates. However, if the interest rate drops in the future, it doesn’t make sense for the bond issuer to continue paying a high coupon rate. As a result, the bond issuer will refinance the bond by backing back the higher coupon bond and by issuing a new bond with a lower coupon rate.

Say AT&T originally took out a loan from you (i.e. you bought a bond issued from AT&T) and paid you annual coupons of 10%. A few years later, the interest rate dropped to 4%. Would AT&T still happily pay you 10% coupons when the market interest rate is only 4%? No. If the bond is callable, AT&T will refinance the loan. AT&T simply borrows money from someone else by issuing a bond that pays a lower coupon rate (such as 4% or 4.5%). Then AT&T immediately pays back what he still owes you at that time using the proceeds it got from issuing the new lower coupon bond. The net result: AT&T gets rid of the 10% coupon bond and replaces it with 4% coupon bond. AT&T saves lot of money by calling a bond (i.e. refinancing a bond).

Before refinancing the bond
(i.e. calling the bond)  AT&T pays 10% coupons.
After refinancing the bond  AT&T pays 4% coupons.

In U.S., most corporate bonds are callable bonds. Call features are attractive to bond issuers because bond issuers can refinance their debt in the event that the interest rate drops.

If the interest rate drops, the issuer of a callable bond can issue a brand new bond in the market at a lower borrowing rate; simultaneously, the bond issuer buys back the old bond using the proceeds generated by the brand new bond.

However, if the interest rate stays level or goes up, the bond issuer will not exercise the call option. It doesn’t make sense to refinance a debt at a higher interest rate. So call feature is an option, not a
duty. The issuer can call a bond if he wants to according to the contract, but it doesn’t have to call.

- However, not every bond can be called. To make a bond callable, the bond issuer must state, in the contract (called bond indenture), that the bond to be issued is callable. Calling a bond is bad for the bond holder. After the bond is called, the bond holder must give up his high-yielding bond and look for another bond (a lower coupon bond) to invest in.

- Typically, when the interest rate drops a lot, callable bonds get called. This is like student loan refinancing. When the interest rate drops a lot, the number of student loans refinanced goes up.

5. **Determine the highest price or the minimum yield of a callable bond.** SOA likes to test this type of problems.

- Pricing approach – assume the bond issuer will choose the redemption date most detrimental to the bond holder (the worse case scenario).

- An investor will pay only the bottom price under the worse case scenario. This way, the investor won’t get burned.

- **How to determine the floor price and the redemption date for a callable bond**

  (1) If the redemption amounts are constant for a range of callable dates

  - **If the modified coupon rate of the bond > the yield rate per coupon period, call the bond ASAP.** This makes intuitive sense. If AT&T pays you 10% coupons semiannually, yet the market rate is only 6% nominal compounding semiannually, AT&T will gladly refinance the original 10% coupon bond by issuing a 6% coupon bond.

  - **If the modified coupon rate of the bond < the yield rate per coupon period, don’t call the bond and let it mature.** This makes intuitive sense. If AT&T pays you 6% coupons semiannually, yet the market rate is 10% nominal compounding semiannually, it will be foolish for AT&T to refinance the original 6% coupon bond by issuing a 10% coupon bond. AT&T will simply ignore its call option.
If the modified coupon rate of the bond = the yield rate per coupon period, calling the bond or not calling the bond makes no difference.

If the redemption amounts are not constant for a range of callable dates,

- Step #1 For each range of callable dates where the redemption amount is constant, apply Rule (1) and determine the redemption date and bond price.

- Step #2 Among the prices calculated in Step #1, choose the minimum price. This is the price to be paid by the investor. The redemption date associated with this price is the worst-case-scenario redemption date.

Sample Problems

Problem 1

| Bond face amount | $1,000 |
| Maturity         | 5 years |
| Coupons          | Zero   |
| Selling price    | $725   |

Calculate the annual rate of return earned by the buyer of the bond.

Solution

This question is about zero coupon bonds.

Key points about zero coupon bonds (also called pure discount bonds):
- Pay stated face or par value at maturity
- Sold at a discount
- Zero coupons are paid in the life of the bond
- Like a saving account
A zero coupon bond is like a saving account. To find the annual return $i$, we solve the following equation:

$$725(1+i)^5 = 1,000 \Rightarrow i = 6.643\%$$

**Problem 2**

<table>
<thead>
<tr>
<th>Bond issue date</th>
<th>1/1/2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity date</td>
<td>1/1/2010</td>
</tr>
<tr>
<td>Face amount</td>
<td>$1,000</td>
</tr>
<tr>
<td>Coupons</td>
<td>8% payable 7/1 and 1/1</td>
</tr>
<tr>
<td>Redemption amount</td>
<td>Par</td>
</tr>
<tr>
<td>Yield to maturity at purchase</td>
<td>10.25% annual effective</td>
</tr>
</tbody>
</table>

Explain whether the bond is a premium bond or discount bond; Calculate the premium or discount. Generate an amortization schedule.

**Solution**

First, we draw a cash flow diagram. Because coupons are paid once every 6 months, we’ll use 6 months as unit time to simply our calculations.

Please note that the term of the bond is 5 years or 10 units of time.

$$PV = 40a_{10\%} + 1,000v^{10} \quad @ \quad i = (1 + 10.25\%)^\frac{1}{2} - 1 = 5\%$$

$$\Rightarrow \quad PV = 922.78$$
Please note that the price of the bond is always equal to PV of the bond; PV of the bond is always equal to the cash flows discounted at YTM (yield to maturity).

\[ PV = 922.78 < \text{Face amount} = 1,000 \]

\[ \Rightarrow \text{The bond is sold at a discount} \]

\[ \text{Discount} = |PV - \text{Face}| = |922.78 - 1,000| = \$77.22 \]

We can also intuitively see why this bond sold at a discount. The coupons of the bond pay the buyer 4% per 6 months, while the market interest rate is only 5% per 6 months. If the bond is still sold at par, the bond issuer (the borrower) will underpay the bond buyer, creating unfairness in the transaction. As a result, the bond issuer charges a price below the par. The amount by which the selling price is below the par amount is the discount.

To generate an amortization schedule of the bond, we’ll treat the bond as a loan. In this loan, the borrower (the bond issuer) borrows the present value (or price) of the bond; he repays the loan by paying ten semiannual payments of $40 each plus a final payment of $1,000 at the end of Year 5.

By treating a bond as a loan, we can amortize a bond the same way we amortize a loan through the following steps:

- Find the outstanding balance \( P_t \) of the bond immediately after repayment \( X(t) \) is made. We can calculate \( P_t \) using the retrospective or prospective method. Please note that the repayment is $40 from \( t = 1 \) to \( t = 9 \) and $1,040 at \( t = 10 \) (the unit time is 6 months).

- Multiple \( P_t \) with the effective interest rate per unit time \( i \) (5% in this problem). This gives us \( \text{Interest}(t) = i \cdot P_t \), the interest portion of \( X(t) \).

- Calculate \( \text{Principal}(t) = X(t) - \text{Interest}(t) \), the principal portion of \( X(t) \).
Following the above procedure, we’ll get the follow amortization schedule:

<table>
<thead>
<tr>
<th>Date</th>
<th>Loan repayment</th>
<th>Interest</th>
<th>Principal</th>
<th>Outstanding balance of the bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2005</td>
<td>$922.78</td>
<td></td>
<td></td>
<td>$922.78</td>
</tr>
<tr>
<td>7/1/2005</td>
<td>$40</td>
<td>$46.14 (1)</td>
<td>-6.14 (2)</td>
<td>928.92 (3)</td>
</tr>
<tr>
<td>1/1/2006</td>
<td>40</td>
<td>46.45</td>
<td>-6.45</td>
<td>935.37</td>
</tr>
<tr>
<td>7/1/2006</td>
<td>40</td>
<td>46.77</td>
<td>-6.77</td>
<td>942.14</td>
</tr>
<tr>
<td>1/1/2007</td>
<td>40</td>
<td>47.11</td>
<td>-7.11</td>
<td>949.25</td>
</tr>
<tr>
<td>7/1/2007</td>
<td>40</td>
<td>47.46</td>
<td>-7.46</td>
<td>956.71</td>
</tr>
<tr>
<td>1/1/2008</td>
<td>40</td>
<td>47.84</td>
<td>-7.84</td>
<td>964.55</td>
</tr>
<tr>
<td>7/1/2008</td>
<td>40</td>
<td>48.23</td>
<td>-8.23</td>
<td>972.78</td>
</tr>
<tr>
<td>1/1/2009</td>
<td>40</td>
<td>48.64</td>
<td>-8.64</td>
<td>981.42</td>
</tr>
<tr>
<td>7/1/2009</td>
<td>40</td>
<td>49.07</td>
<td>-9.07</td>
<td>990.49</td>
</tr>
<tr>
<td>1/1/2010</td>
<td>1,040</td>
<td>49.52</td>
<td>990.48</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>1,400</td>
<td>477.23</td>
<td>922.77</td>
<td></td>
</tr>
</tbody>
</table>

(1) 46.14 = 922.78(5%)
(2) -6.14 = 40 – 46.14
(3) 928.92 = 922.78 - (-6.14)

From the above table, we see that a discount bond has a negative amortization in all the payments except the final one. The earlier periodic repayments do not even cover the interest due, creating negative principal repayments (i.e. increasing the outstanding balance of the loan).

Make sure you can manually create the above table.

Please also note that we can use the amortization method suggested by the textbook:

<table>
<thead>
<tr>
<th>K-th payment</th>
<th>Outstanding balance</th>
<th>Payment</th>
<th>Interest portion</th>
<th>Principal portion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$F\left[1+(r-j)a_{\bar{m}}\right]$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$F\left[1+(r-j)a_{\bar{n}}\right]$</td>
<td>$Fr$</td>
<td>$F\left[ j+(r-j)(1-v_j^m) \right]$</td>
<td>$F(r-j)v_j^m$</td>
</tr>
<tr>
<td>2</td>
<td>$F\left[1+(r-j)a_{\bar{n-2}}\right]$</td>
<td>$Fr$</td>
<td>$F\left[ j+(r-j)(1-v_j^{n-1}) \right]$</td>
<td>$F(r-j)v_j^{n-1}$</td>
</tr>
<tr>
<td>k</td>
<td>$F\left[1+(r-j)a_{\bar{n-k}}\right]$</td>
<td>$Fr$</td>
<td>$F\left[ j+(r-j)(1-v_j^{n-k+1}) \right]$</td>
<td>$F(r-j)v_j^{n-k+1}$</td>
</tr>
<tr>
<td>n-1</td>
<td>$F\left[1+(r-j)a_{\bar{n}}\right]$</td>
<td>$Fr$</td>
<td>$F\left[ j+(r-j)(1-v_j^n) \right]$</td>
<td>$F(r-j)v_j^n$</td>
</tr>
</tbody>
</table>
Under this method, we have \( F = 1,000, \quad r = 4\%, \quad j = 5\%, \quad n = 10 \). Let’s generate an amortization schedule for the first two payments:

<table>
<thead>
<tr>
<th>t</th>
<th>Outstanding balance</th>
<th>Payment</th>
<th>Interest portion</th>
<th>Principal portion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1,000 ( \left[ 1 + (4% - 5%) a_{0\mid 5%} \right] ) ( = 922.78 )</td>
<td>1,000(4%) ( = 40 )</td>
<td>1,000 ( 5% + (4% - 5%) \left( 1 - 1.05^{-10} \right) ) ( = 46.14 )</td>
<td>1,000(4% - 5%) ( 1.05^{-10} ) ( = -6.14 )</td>
</tr>
<tr>
<td>1</td>
<td>1,000 ( \left[ 1 + (4% - 5%) a_{0\mid 5%} \right] ) ( = 928.92 )</td>
<td>1,000(4%) ( = 40 )</td>
<td>1,000 ( 5% + (4% - 5%) \left( 1 - 1.05^{-9} \right) ) ( = 46.45 )</td>
<td>1,000(4% - 5%) ( 1.05^{-9} ) ( = -6.45 )</td>
</tr>
</tbody>
</table>

We can also generate the amortization schedule using BA II Plus/BA II Plus Professional Amortization Worksheet. Refer to Chapter 4 on how to generate the amortization schedule.

**Problem 3**

<table>
<thead>
<tr>
<th>Bond issue date</th>
<th>1/1/2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity date</td>
<td>1/1/2010</td>
</tr>
<tr>
<td>Face amount</td>
<td>$1,000</td>
</tr>
<tr>
<td>Coupons</td>
<td>8% payable 7/1 and 1/1</td>
</tr>
<tr>
<td>Redemption amount</td>
<td>Par</td>
</tr>
<tr>
<td>Yield to maturity at purchase</td>
<td>6.09% annual effective</td>
</tr>
</tbody>
</table>

Explain whether the bond is a premium bond or discount bond; Calculate the premium or discount. Generate an amortization schedule.

**Solution**

\[
PV = 40a_{0\mid 6.09\%} + 1,000v^{10} \quad @ \quad i = \left(1 + 6.09\% \right)^{1/2} - 1 = 3\%
\]
\[
\Rightarrow \quad PV = 1,085.30
\]
PV = 1,085.30 > Face amount = 1,000
⇒ The bond is sold at a premium

Premium = PV – Face amount = 1,085.30 – 1,000 = 85.30

We can also intuitively see why this bond sold at a premium. The coupons of the bond pay the buyer 4% per 6 months, while the market interest rate is only 3% per 6 months. If the bond is still sold at par, the bond issuer (the borrower) will overpay the bond buyer, creating unfairness in the transaction. As a result, the bond issuer charges a price above and beyond the par. The amount by which the selling price exceeds the par amount is the premium.

Amortization schedule:

<table>
<thead>
<tr>
<th>Date</th>
<th>Loan repayment</th>
<th>Interest</th>
<th>Principal</th>
<th>Outstanding balance of the bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2005</td>
<td>$40</td>
<td>$32.56</td>
<td>$7.44</td>
<td>$1,085.30</td>
</tr>
<tr>
<td>7/1/2005</td>
<td>$40</td>
<td>$32.56</td>
<td>$7.44</td>
<td>1,077.86</td>
</tr>
<tr>
<td>1/1/2006</td>
<td>40</td>
<td>32.34</td>
<td>7.66</td>
<td>1,070.20</td>
</tr>
<tr>
<td>7/1/2006</td>
<td>40</td>
<td>32.11</td>
<td>7.89</td>
<td>1,062.31</td>
</tr>
<tr>
<td>1/1/2007</td>
<td>40</td>
<td>31.87</td>
<td>8.13</td>
<td>1,054.18</td>
</tr>
<tr>
<td>7/1/2007</td>
<td>40</td>
<td>31.63</td>
<td>8.37</td>
<td>1,045.81</td>
</tr>
<tr>
<td>1/1/2008</td>
<td>40</td>
<td>31.37</td>
<td>8.63</td>
<td>1,037.18</td>
</tr>
<tr>
<td>7/1/2008</td>
<td>40</td>
<td>31.12</td>
<td>8.88</td>
<td>1,028.30</td>
</tr>
<tr>
<td>1/1/2009</td>
<td>40</td>
<td>30.85</td>
<td>9.15</td>
<td>1,019.15</td>
</tr>
<tr>
<td>7/1/2009</td>
<td>40</td>
<td>30.57</td>
<td>9.43</td>
<td>1,009.72</td>
</tr>
<tr>
<td>1/1/2010</td>
<td>1,040</td>
<td>30.27</td>
<td>1,009.71</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>$1,400</td>
<td>$314.69</td>
<td>$1,085.29</td>
<td></td>
</tr>
</tbody>
</table>

(4) 32.56 = 1,085.30 (3%)
(5) 7.44 = 40 – 32.56
(6) 1,077.86 = 1,085.30 – 7.44
Let’s generate an amortization schedule for the first two payments using the textbook method. We have \( F = 1,000, \ r = 4\%, \ j = 3\%, \ n = 10 \).

<table>
<thead>
<tr>
<th>t</th>
<th>Outstanding balance</th>
<th>Payment</th>
<th>Interest portion</th>
<th>Principal portion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 1,000\left[1 + (4% - 3%)d_{10,3%}\right] )</td>
<td>( 1,085.30 )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( 1,000\left[1 + (4% - 3%)\right]d_{10,3%} )</td>
<td>( 1,077.86 )</td>
<td>( 1,000(4%) = 40 ) ( 1,000\left[3% + (4% - 3%)(1-1.03^{-10})\right] = 32.56 ) ( 1,000(4% - 3%)1.03^{-10} = 7.44 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( 1,000\left[1 + (4% - 3%)d_{10,3%}\right] )</td>
<td>( 1,070.20 )</td>
<td>( 1,000(4%) = 40 ) ( 1,000\left[3% + (4% - 3%)(1-1.03^{-9})\right] = 32.34 ) ( 1,000(4% - 3%)1.03^{-9} = 7.66 )</td>
<td></td>
</tr>
</tbody>
</table>

Make sure you can also generate the amortization schedule using BA II Plus/BA II Plus Professional Amortization Worksheet.

**Problem 4**

<table>
<thead>
<tr>
<th>Face amount of a callable bond</th>
<th>$1,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>8% annual</td>
</tr>
<tr>
<td>Purchase date</td>
<td>7/1/2005</td>
</tr>
<tr>
<td>Call date</td>
<td>Any time between 7/1/2020 and 7/1/2026</td>
</tr>
<tr>
<td>Redemption amount</td>
<td>Par</td>
</tr>
</tbody>
</table>

Calculate the maximum price the buyer of the bond will pay to guarantee a yield of at least 7%.

**Solution**

The bond issuer pays 8% annual coupons, but the buyer of the bond is content to lock in only 7%. Why can't the buyer get 8% return? Because the market interest rate is below 8%. Why is the bond buyer happy to get a minimum return of 7%? Because the market interest rate is volatile, going up and down; by locking in the 7% floor rate, the buyer can sleep well at night assured that he’s getting at least 7%.

Because the buyer is happy to lock in 7%, the bond issuer will gladly refinance the bond at 7% at the earliest callable date. This way, the bond issuer will pay only a 7% annual interest rate on its refinanced debt, instead of continuing paying 8% coupons. As a result, the bond issuer will recall the bond at the earliest callable date, which is 7/1/2020.
So the maximum price the buyer will pay is $1,091.08. If the bond sells above this price, the buyer’s return for investing the bond will be lower than 7%.
Problem 5

<table>
<thead>
<tr>
<th></th>
<th>Bond A</th>
<th>Bond B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Face amount</td>
<td>$100</td>
<td>$100</td>
</tr>
<tr>
<td>Coupons</td>
<td>7% payable semiannually</td>
<td>4% payable semiannually</td>
</tr>
<tr>
<td>Redemption</td>
<td>100% par</td>
<td>150% par</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>10 years</td>
<td>10 years</td>
</tr>
</tbody>
</table>

Bond A and Bond B have the same yield to maturity and sells at the same price. Calculate the price of each bond.

Solution
Unit time = 6 months

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cash flow</td>
<td>$3.5</td>
<td>$3.5</td>
<td>$3.5</td>
<td>...</td>
<td>$3.5</td>
<td>$3.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cash flow</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>...</td>
<td>$2</td>
<td>$2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$150</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>A - B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cash flow</td>
<td>$1.5</td>
<td>$1.5</td>
<td>$1.5</td>
<td>...</td>
<td>$1.5</td>
<td>$1.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$50</td>
</tr>
</tbody>
</table>

\[ PV = 1.5a_{20} - 50v^{20} = 0 \]

\[ \Rightarrow i = 5.07615296\% \] Using BA II Plus/Professional TVM

Then the price of Bond A is: \[ 3.5a_{20} + 100v^{20} = 80.48386442 \]

Then the price of Bond B is: \[ 2a_{20} + 150v^{20} = 80.48386442 \]
**Problem 6 (May 2004 EA-1)**

On 1/1/2005, Smith purchases a 20-year bond with a par value of $1,000. The bond pays semi-annual coupons at an annual rate of 6%. The bond is purchased to yield 5% per annual effective. When each coupon is received, it is immediately reinvested at a rate of interest of 6% per annum convertible quarterly.

In what range is Smith’s effective annual rate of return over the term of the bond?

[A] Less than 5.20%
[B] 5.20% but less than 5.30%
[C] 5.30% but less than 5.40%
[D] 5.40% but less than 5.50%
[F] 5.50% or more

**Solution**

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td>$0</td>
<td>$30</td>
<td>$30</td>
<td>$30</td>
<td>$30</td>
</tr>
</tbody>
</table>

Smith’s total wealth at t=40

\[
30\frac{a_{40\mid 6\%}}{6\%} + 1,000v^{40} \quad @ \quad i = \sqrt{1.05} - 1
\]

\[
\frac{30v_{40\mid 6\%}}{6\%} + \frac{1,000}{\text{Receiving face amount}} \quad @ \quad j = \left(1 + \frac{6\%}{4}\right)^2 - 1
\]

We can solve for \( r \), Smith’s annual rate of return:
Using BA II Plus/BA II Plus Professional TVM, we have:

\[
\begin{align*}
30a_{40|} + 1,000v^{40} &= 1,133.854761 \quad @ i = \sqrt{1.05} - 1 \\
30s_{40|} &= 2,273.610707 \quad @ j = \left(1 + \frac{6\%}{4}\right)^2 - 1 \\
(1 + r)^{40} &= \frac{30s_{40|} + 1,000}{30a_{40|} + 1,000v^{40}} = \frac{2,273.610707 + 1,000}{1,133.854761} = 2.88715171
\end{align*}
\]

The correct answer is D.

**Callable bond**

**Problem 7**

Facts about a callable bond:

<table>
<thead>
<tr>
<th>Face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>6% semiannually</td>
</tr>
<tr>
<td>Redemption value</td>
<td>$100</td>
</tr>
<tr>
<td>Call dates</td>
<td>10th through 15th years at par</td>
</tr>
</tbody>
</table>

Calculate

[1] Find the price of the bond to yield 8% convertible semiannually
[2] Find the price of the bond to yield 4% convertible semiannually

**Solution**

[1] Find the price of the bond to yield 8% convertible semiannually

Assume the bond is called immediately after the \(n\)-th coupon is paid. Because the call dates must be in the 10th through 15th year, we have:

\(n = 20, 21, 22, ..., 30\)

The price of the bond is:

\[
P = 3a_{\frac{n}{2}} + 100v^n \quad @ 4\% \text{ per coupon period}
\]

In the above equation, 3 is the semiannual coupon payment.
We need to minimize $P$, the purchase price of this callable bond. This is the maximum price the investor is willing to pay in order to lock in the minimum return of $i^{(2)} = 8\%$.

To minimize $P$, we need to change the formula:

$$P = 3a_m + 100v^n$$

$$a_m = \frac{1-v^n}{i} \implies v^n = 1-i\,a_m$$

$$\implies P = 3a_m + 100\left(1-i\,a_m\right) = 3a_m + 100\left(1-4\%\,a_m\right) = 100 - a_m$$

$a_m$ is an increasing function with $n$. To minimize $P$, we need to maximize $n$. We choose $n = 30$.

$$\implies P = 3a_{30\,4\%} + 100 \times 1.04^{-30} = 82.708$$

[2] Find the price of the bond to yield 4% convertible semiannually

$$P = 3a_m + 100v^n \quad @ 2\% \text{ per coupon period}$$

$$\implies P = 3a_m + 100\left(1-i\,a_m\right) = 3a_m + 100\left(1-2\%\,a_m\right) = 100 + a_m$$

To minimize $P$, we need to minimize $n$. So we set $n = 20$.

$$\implies P = 3a_{20\,2\%} + 100\left(1.02^{-20}\right) = 116.35$$
Problem 8

Facts about a callable bond:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Face</td>
<td>$1,000</td>
</tr>
<tr>
<td>Coupon</td>
<td>4% semiannually</td>
</tr>
<tr>
<td>Maturity</td>
<td>20 years if not called</td>
</tr>
<tr>
<td>Redemption value</td>
<td>$1,000</td>
</tr>
<tr>
<td>Call dates</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2nd through 5th years at $1,050</td>
</tr>
<tr>
<td></td>
<td>6th through 10th years at $1,025</td>
</tr>
<tr>
<td></td>
<td>11th through 19th years at $1,010</td>
</tr>
<tr>
<td></td>
<td>20th year at $1,000</td>
</tr>
</tbody>
</table>

Calculate the maximum price an investor is willing to pay in order to lock in a yield of 6% convertible semiannually.

Solution

The solution process is similar to the process used for the last problem.

The price of the bond is:

\[
P = 20a_{\overline{n}} + 1050v^n = 1050 + (20 - 1050 \times 3\%) a_{\overline{3\%}} = 1050 - 11.5a_{\overline{3\%}}
\]

Where \( n = 4,5,6,7,8,9,10 \)

\( P \) reaches minimum when \( n = 10 \).

\[
\Rightarrow \quad \text{min } P = 20a_{\overline{10\%}} + 1050(1.03^{-10}) = 951.903
\]

\[
P = 20a_{\overline{n}} + 1025v^n = 1025 + (20 - 1025 \times 3\%) a_{\overline{3\%}} = 1025 - 10.05a_{\overline{3\%}}
\]

Where \( n = 12,13,14,...,20 \)

\( P \) reaches minimum when \( n = 20 \).

\[
\Rightarrow \quad \text{min } P = 20a_{\overline{20\%}} + 1025(1.03^{-20}) = 865.067
\]

\[
P = 20a_{\overline{n}} + 1010v^n = 1010 + (20 - 1010 \times 3\%) a_{\overline{3\%}} = 1010 - 10.3a_{\overline{3\%}}
\]

Where \( n = 22,23,24,...,38 \)

\( P \) reaches minimum when \( n = 38 \).

\[
\Rightarrow \quad \text{min } P = 20a_{\overline{38\%}} + 1010(1.03^{-38}) = 778.328
\]

\[
P = 20a_{\overline{n}} + 1000v^n = 1000 + (20 - 1000 \times 3\%) a_{\overline{3\%}} = 1010 - 10a_{\overline{3\%}}
\]
Where \( n = 39, 40 \)

\[ P \text{ reaches minimum when } n = 40. \]

\[ \Rightarrow \min P = 20a_{40|3\%} + 1000(1.03^{-40}) = 768.85 \]

So 768.85 is the maximum price the investor is willing to pay to lock in a yield of 6\% convertible semiannually.

**Problem 9 (May 2004 EA-1 #32)**

<table>
<thead>
<tr>
<th></th>
<th>Bond A</th>
<th>Bond B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Face amount</strong></td>
<td>$100</td>
<td>$100</td>
</tr>
<tr>
<td><strong>Coupon rate</strong></td>
<td>6%, payable semi-annually</td>
<td>5%, payable semi-annually</td>
</tr>
<tr>
<td><strong>Redemption</strong></td>
<td>Par</td>
<td>$125</td>
</tr>
<tr>
<td><strong>Length of bond</strong></td>
<td>20 years</td>
<td>20 years</td>
</tr>
</tbody>
</table>

Both bonds have the same purchase price and the same yield rate.

Calculate the annual effective yield on these two bonds.

**Solution**

**Bond A**

<table>
<thead>
<tr>
<th>Time t (6 months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>…</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flows</td>
<td>$3</td>
<td>$3</td>
<td>$3</td>
<td>$3</td>
<td>$103</td>
<td></td>
</tr>
</tbody>
</table>

**Bond B**

<table>
<thead>
<tr>
<th>Time t (6 months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>…</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flows</td>
<td>$2.5</td>
<td>$2.5</td>
<td>$2.5</td>
<td>$2.5</td>
<td>$127.5</td>
<td></td>
</tr>
</tbody>
</table>

**Cash flows of** \( A - B \)

<table>
<thead>
<tr>
<th>Time t (6 months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>…</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flows</td>
<td>$0.5</td>
<td>$0.5</td>
<td>$0.5</td>
<td>$0.5</td>
<td>$0.5</td>
<td>$24.5</td>
</tr>
</tbody>
</table>

Because Bond A and B have the same price, the present value of \( A - B \) should be zero.
There are at least two ways to calculate the effective yield.

**Method 1**  use BA II Plus Cash Flow Worksheet

Enter the following into Cash Flow Worksheet:

- \( CF0=0 \)
- \( C01=0.5 \)
- \( F01=39 \)
- \( C02= - 24.5 \)

Press “IRR” “CPT.”

You should get: \( IRR=1.10894155 \)

So the 6-month effective yield is 1.10894155%. The annual effective yield is:

\[
\left(1+1.10894155\%ight)^2 - 1 = 2.23018061\%
\]

**Method 2**  Use BA II Plus TVM Worksheet

We can rewrite the cash flows of \( A - B \) as follows:

<table>
<thead>
<tr>
<th>Time ( t ) (6 months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( \ldots )</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flows</td>
<td>$0.5</td>
<td>$0.5</td>
<td>$0.5</td>
<td>$0.5</td>
<td>( \ldots )</td>
<td>$0.5 - $25</td>
</tr>
</tbody>
</table>

Enter: \( \text{PMT}=0.5, \ \text{FV}= - 25, \ \text{N}=40, \ \text{PV}=0 \).

Press “CPT” “I/Y.”

You should get: \( I/Y= 1.10894155 \).

So the 6-month effective yield is 1.10894155%. The annual effective yield is:

\[
\left(1+1.10894155\%ight)^2 - 1 = 2.23018061\% 
\]
Problem 10 (May 2004 EA-1 #28)

For a given bond:

<table>
<thead>
<tr>
<th>Par value</th>
<th>$1,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Redemption value</td>
<td>$1,100</td>
</tr>
<tr>
<td>Term of bond</td>
<td>10 years</td>
</tr>
<tr>
<td>Coupons</td>
<td>r% per year, payable semiannually</td>
</tr>
<tr>
<td>Issue price</td>
<td>P if yield to maturity is 4%, compounded annually</td>
</tr>
<tr>
<td>Issue price</td>
<td>P – 95.5 if yield to maturity is 5%, compounded annually</td>
</tr>
</tbody>
</table>

Calculate r%.

Solution

<table>
<thead>
<tr>
<th>Time t (6 months)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>…</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flows</td>
<td>5r</td>
<td>5r</td>
<td>5r</td>
<td>5r</td>
<td>1100 + 5r</td>
<td></td>
</tr>
</tbody>
</table>

Bond price is P if yield to maturity is 4%, compounded annually:

\[ P = 5r \, a_{20|i} + (1100)1.04^{-10}, \text{ where } i = 1.04^{\frac{1}{2}} - 1 = 1.98039027\% \]

Bond price is P – 95.5 if yield to maturity is 5%, compounded annually:

\[ P – 95.5 = 5r \, a_{20|j} + (1100)1.05^{-10}, \text{ where } j = 1.05^{\frac{1}{2}} - 1 = 2.46950766\% \]

\[ 5r \, a_{20|i} + (1100)1.04^{-10} = 5r \, a_{20|j} + (1100)1.05^{-10} + 95.5 \]

\[ 5r \left( a_{20|i} - a_{20|j} \right) = 95.5 - 1100 \left( 1.04^{-10} - 1.05^{-10} \right) \]

\[ a_{20|i} = 16.38241895, \quad a_{20|j} = 15.63418569 \]

\[ 95.5 - 1100 \left( 1.04^{-10} - 1.05^{-10} \right) \]

\[ r = \frac{95.5 - 1100 \left( 1.04^{-10} - 1.05^{-10} \right)}{5 \left( a_{20|i} - a_{20|j} \right)} = 7.4\% \]
Problem 11 (May 2000 EA-1 #10)

Issue date of a bond: 1/1/1994  
Term of bond: 15 years  
Par value of bond: $10,000  
Coupons: 8% per year, paid on June 30 and December 31  
Amortized value on July 1, 2001: $13,741.11  
Amortized value on January 1, 2002: $13,629.67  
Calculate the redemption amount to be paid upon maturity.

Solution

<table>
<thead>
<tr>
<th>Cash flows</th>
<th>$400</th>
<th>$400</th>
<th>$400</th>
<th>$400</th>
<th>$400</th>
<th>$400</th>
<th>X + 400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t (6months)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>15</td>
<td>16</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

\[ 13,741.11 = 400 \ a_{\overline{15}\mid} + X \ v^{15} \]
\[ 13,629.67 = 400 \ a_{\overline{14}\mid} + X \ v^{14} \]

Let \( i \) represent the 6-month effective interest rate; \( X \) represent the redemption amount at maturity.

Treat this bond as a loan. Then the coupons are just annual repayments of the loan made by the borrower. The amortized value is just the loan outstanding balance.

At \( t=15 \), the loan balance is $13,741.11. This amount accumulates to $3,741.11(1+i) at \( t=16 \), at which time a loan repayment of $400 is made. Then the loan balance becomes $13,629.67.

\[ \Rightarrow 13,741.11(1+i) - 400 = 13,629.67, \quad i = 2.1\% \]

\[ 13,741.11 = 400 \ a_{\overline{15}\mid} + X \ v^{15}, \quad \Rightarrow X = 11,800 \] (using TVM)

Alternatively, \[ 13,629.67 = 400 \ a_{\overline{14}\mid} + X \ v^{14}, \quad \Rightarrow X = 11,800 \] (using TVM)
Problem 12 (May 2000 EA-1 #11)

Issue date of a bond: 1/1/2001
Coupon dates: 12/31/2002 and every two years thereafter, with the final payment on 12/31/2010.

Coupon amount: $60 each.
Investor’s yield: 8% year annum
Price of the bond at issue: $691.49
Amortized value on 1/1/2005: $A
Amortized value on 1/1/2007: $B

Calculate $|A - B|$.

Solution

<table>
<thead>
<tr>
<th>Cash flow</th>
<th>$60</th>
<th>60</th>
<th>60</th>
<th>60</th>
<th>60</th>
<th>$60 + F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t (2 years)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Here the unit time is 2 years. The effective interest rate per period is:

$$i = 1.08^2 - 1 = 16.64\%$$

We’ll treat this bond as a loan. Then the amortized value of the bond is the outstanding balance of the loan. We’ll use the respective method. Under this method, the outstanding balance at $t = 2$ immediately after coupon payment is the accumulated value of the original loan balance less the accumulated value of the coupon payments:

$$A = 691.49(1+i)^2 - [60(1+i) + 60] = 691.49(1.1664)^2 - [60(1.1664) + 60]$$
$$= 810.7805110$$

Similarly,

$$B = A(1+i) - 60 = 810.7805110(1.1664) - 60 = 885.6943881$$

$$\Rightarrow |A - B| = 74.91$$
Problem 13 (May 2002 EA-1 #9)

Face value of a bond: $1,000
Redemption value: $1,050
Time to maturity: 10 years
Coupon rate: 9% per year, convertible semi-annually
Yield rate: 10.25% per year
The bond is not callable.

Calculate the increase in the book value of the bond during the 3rd year.

Solution

<table>
<thead>
<tr>
<th>Cash flows</th>
<th>$45</th>
<th>$45</th>
<th>$45</th>
<th>$45</th>
<th>$45</th>
<th>$45</th>
<th>$45</th>
<th>$45+1,050</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t (6 months)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>...</td>
</tr>
</tbody>
</table>

The 6-month effective interest rate is \( i = (1 + 10.25\%)^{0.5} - 1 = 5\% \).

The book value of the bond immediately after the 4th coupon payment is:

\[
P_4 = 45 \cdot a_{\frac{i}{2}} + 1,050 \cdot v^{16} = 968.7267
\]

The book value of the bond immediately after the 6th coupon payment is:

\[
P_6 = 45 \cdot a_{\frac{i}{2}} + 1,050 \cdot v^{14} = 975.7602
\]

The increase in the book value of the bond during the 3rd year is:

\[
P_6 - P_2 = 975.7602 - 968.7267 = 7.043
\]
Chapter 10  Valuation of stocks

Key points:

- If you own a stock, you are entitled to receive future dividends. In addition, you can sell the stock in the future.

- You can think of a stock as a series of cash flows:

  
<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>T</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$D_1$</td>
<td>$D_2$</td>
<td>...</td>
<td>$D_T + P_T$</td>
<td>$P_0$</td>
<td></td>
</tr>
</tbody>
</table>

- The price of the stock is equal to the PV of future cash flows.

  $$P_0 = \frac{D_1 + P_1}{1+i}$$  (assume you buy the stock at t=0 and sell it at t=1)

  $$P_1 = \frac{D_2 + P_2}{1+i}$$  (assume the next owner buys the stock at t=1 and sells it at t=1)

  $$P_0 = \frac{D_1 + P_1}{1+i} + \frac{D_2 + P_2}{(1+i)^2}$$

  $$P_2 = \frac{D_3 + P_3}{1+i}$$  (assume the next owner buy the stock at t=2 and sell it at t=3)

  We can continue this line of thinking.

  $$P_0 = \frac{D_1}{1+i} + \frac{D_2}{(1+i)^2} + \ldots + \frac{D_T}{(1+i)^T} + \ldots = \sum_{t=1}^{\infty} \frac{D_t}{(1+i)^t}$$

  $$\Rightarrow$$  Stock price = PV of future dividends.
Constant growth model

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>T</th>
<th>…</th>
</tr>
</thead>
</table>

Cash flow

\[ D_1 \quad D_2 = D_1(1 + g) \quad D_T = D_1(1 + g)^{T-1} \]

\[ P_0 \]

Dividend paid at the end of Year 1 = \( D_1 \)
Dividend paid at the end of Year 2 = \( D_2 = D_1(1 + g) \)
Dividend paid at the end of Year 3 = \( D_3 = D_1(1 + g)^2 \)

......
Dividend paid at the end of Year T= \( D_T = D_1(1 + g)^{T-1} \)

If the interest rate \( r \) is greater than the dividend growth rate \( g \)

\[
\Rightarrow P_0 = \frac{D_1}{1+i} + \frac{D_1(1+g)}{(1+i)^2} + ... + \frac{D_1(1+g)^{T-1}}{(1+i)^T} + ... = \frac{D_1}{r-g}
\]

Problems

Problem 1

<table>
<thead>
<tr>
<th>Stock price</th>
<th>$50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividend at the end of Year 1</td>
<td>$2</td>
</tr>
<tr>
<td>Dividends growth rate per year forever</td>
<td>8%</td>
</tr>
</tbody>
</table>

Calculate the return expected by investors.

Solution

\[
P_0 = \frac{D_1}{r-g} \quad \Rightarrow r = g + \frac{D_1}{P_0} = 8% + \frac{2}{50} = 12%
\]

So investors are expecting 12% return annually.
**Problem 2**

<table>
<thead>
<tr>
<th>Stock purchase date</th>
<th>9 months before the next dividend is due</th>
</tr>
</thead>
<tbody>
<tr>
<td>Next dividend</td>
<td>$2 per share</td>
</tr>
<tr>
<td>Dividend payment</td>
<td>Annual</td>
</tr>
<tr>
<td>Dividend growth rate</td>
<td>5% per year in perpetuity</td>
</tr>
</tbody>
</table>

Calculate the purchase price of the stock if investors want an 8% return annual effective.

**Solution**

Please note that in the following diagram, we need to find the purchase price of the stock at \( t = 0.25 \) (purchase date), not \( t = 0 \). Please note that 9 months before \( t = 1 \) is \( t = 0.25 \) (the distance between \( t = 0.25 \) and \( t = 1 \) is 0.75.)

Method #1

<table>
<thead>
<tr>
<th>Time ( t ) (year)</th>
<th>0</th>
<th>0.25</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash</td>
<td>$2</td>
<td>( 2 \times 1.05 )</td>
<td>( 2 \times 1.05^2 )</td>
<td>...</td>
<td>( 2 \times 1.05^{t-1} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{2}{1.05} a_{\overline{\infty}|j} = \frac{2}{1.05j} = \frac{2}{1.05(2.85714\%)} = 66.66667 \quad \text{(geometric annuity shortcut)}
\]

where \( j = \frac{8\% - 5\%}{1+5\%} = 2.85714\% \)

The purchase price of the bond at \( t = 0.25 \) is: \( 66.66667 \times (1.08^{0.25}) = 67.96 \)

Method #2

<table>
<thead>
<tr>
<th>Time ( t ) (year)</th>
<th>0</th>
<th>0.25</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash</td>
<td>$2</td>
<td>( 2 \times 1.05 )</td>
<td>( 2 \times 1.05^2 )</td>
<td>...</td>
<td>( 2 \times 1.05^{t-1} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
2 \overline{a}_{\overline{\infty}|j} = \frac{2}{d_j} = \frac{2}{1 - \frac{1}{1 + 2.85714\%}} = 72
\]

Then the price of the bond at \( t = 0.25 \) is: \( 72(1.08^{-0.75}) = 67.96 \)
Chapter 11  Price of a bond sold between two coupon payments

Exam FM Sample Questions have one question (#50) on this topic.

**Procedure to determine the market price sold between coupon dates:**

- Determine the purchase price of the bond assuming the 2nd owner has 100% ownership of the next coupon
- Determine the accrued interest
- Determine the market price of the bond using the following equation

\[
\text{Bond (quoted) market price} = \text{Bond purchase price (assuming 100% ownership of the next coupon)} - \text{Accrued interest as of the last coupon date}
\]

**Explanation**

To calculate the bond's market price, we first ignore the fact that the original owner deserves a portion of the next coupon. We pretend that the 2nd owner possesses 100% of the next coupon (plus all the other cash flows). We then calculate the purchase price of the bond assuming the 2nd owner has 100% of the next coupon.

The purchase price of a bond (assuming 100% ownership of the next coupon)

= PV of the bond’s future cash flows (including the next coupon) discounted at the market interest rate.

Remember, whenever we discount a cash flow, we implicitly assume the 100% ownership of this cash flow. If the ownership is not 100%, we can NOT discount this cash flow by its full amount.

Next, we consider the fact that the original owner deserves a fraction of the next coupon.

If a bond is sold between two coupon dates, the buyer of the bond must compensate the seller for the fraction of the next coupon payment the seller deserves but will not receive. This amount is called accrued interest.
There are different ways of calculating the accrued interest. Let’s not worry about it now.

Finally, we calculate the real value of the bond by subtracting the accrued interest from the purchase price of the bond. The logic here is simple. The purchase price of the bond is overstated; it’s calculated on the assumption that the 2\textsuperscript{nd} owner gets the next coupon 100\% and the original owner gets none. Since the 2\textsuperscript{nd} buyer must pay the original owner the accrued interest, we need to deduct the accrued interest from the purchase price of the bond.

Now you know that the market price of a bond sold between two coupon dates is the PV of bond’s future cash flows minus the accrued interest. However, the actual calculation is messy because the next coupon occurs at a fractional time. We’ll use the following steps to calculate the market price of the bond:

\begin{itemize}
\item **Step #1**  \hspace{1cm} \textbf{Find the fractional time}
\end{itemize}

$k = \frac{\text{days between settlement date and last coupon date}}{\text{Total days per coupon period}} = \frac{a}{b}$

\begin{center}
\begin{tabular}{c}
| Last coupon date | Settlement date (i.e. Bond purchase date) | Next coupon date |
\end{tabular}
\end{center}
We need to use one of the two day count methods to calculate $k$.

### Day count method

<table>
<thead>
<tr>
<th>Day count method</th>
<th>Explanation</th>
<th>Example</th>
<th>When used</th>
</tr>
</thead>
<tbody>
<tr>
<td>360/360</td>
<td>Assume every month has 30 days and every year has 360 days.</td>
<td>The # of days between 2/1/2005 and 3/1/2005 is 30 days.</td>
<td>For municipal and corporate bonds</td>
</tr>
<tr>
<td>Actual/Actual</td>
<td>Use the actual # of days in a month.</td>
<td>The # of days between 2/1/2005 and 3/1/2005 is 28 days.</td>
<td>For treasury bonds</td>
</tr>
</tbody>
</table>

In Exam FM, if a problem doesn’t specify which day count method to use, use Actual/Actual method.

### Step #2 Calculate the PV of the bond’s future cash flows.

Time zero is the last coupon payment date.

<table>
<thead>
<tr>
<th>Time t</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>k</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>…</td>
</tr>
<tr>
<td>t</td>
</tr>
<tr>
<td>…</td>
</tr>
<tr>
<td>n</td>
</tr>
</tbody>
</table>

Cash flow

\[
PV_x = \sum_{t=1}^{n} CF(t) \cdot v^{t-k} = \sum_{t=1}^{n} \frac{CF(t)}{(1+i)^{t-k}} = \frac{CF(1)}{(1+i)^{1-k}} + \frac{CF(2)}{(1+i)^{2-k}} + \ldots + \frac{CF(n)}{(1+i)^{n-k}}
\]
If we don’t like fractional discounting periods such as $1 - k$, $2 - k$, …, we can calculate the PV of the future cash flows at $t=1$ and then discount the this PV to $t=k$ (see the diagram below):

Time zero is the last coupon payment date.

$\begin{align*}
\text{Cash flow} & \quad CF(1) \quad CF(2) \quad CF(3) \quad \ldots \quad CF(t) \quad \ldots \quad CF(n) \\
\text{PV}_i & = CF(1) + \frac{CF(2)}{1+i} + \frac{CF(3)}{(1+i)^2} + \ldots + \frac{CF(n)}{(1+i)^{n-1}} \\
\text{PV}_i & = \text{Bond's book value @ the next coupon date}
\end{align*}$

$\text{PV}_k = \frac{\text{PV}_i}{(1+i)^{k}} = (\text{Bond's book value @ the next coupon date})^{1-k}$

Alternatively, we can calculate the PV at $t=0$ by discounting all future cash flows to $t=0$. Next, we accumulate this PV at $t=0$ to $t=k$ (see the diagram below):

$\begin{align*}
\text{Cash flow} & \quad CF(1) \quad CF(2) \quad CF(3) \quad \ldots \quad CF(t) \quad \ldots \quad CF(n) \\
\text{PV}_k & = \text{PV}_0(1+i)^k \\
& = (\text{Bond's book value @ the last coupon date})(1+i)^k \\
\text{PV}_0 & = \frac{CF(1)}{1+i} + \frac{CF(2)}{(1+i)^2} + \ldots + \frac{CF(n)}{(1+i)^n} = \text{Bond's book value @ the last coupon date}
\end{align*}$
Step #3 Calculate the accrued interest.

\[
k = \frac{\text{days between settlement date and last coupon date}}{\text{Total days per coupon period}} = \frac{a}{b}
\]

\[
\text{Accrued Interest} = k \times \text{coupon} = k( Fr ) = \frac{a}{b}( Fr )
\]

Step #4 Calculate the (quoted) market price.

Market Price
\[
= \text{Purchase price of the bond at settlement} - \text{accrued interest.}
\]

The method outlined in the above four steps is the method most often used in the real world. Broverman described this method in his textbook, except he didn’t mention the day count method (perhaps to keep the concept simple).

Kellison, however, described a myriad of ways to calculate the bond market price in Table 7.4 (page 223).

<table>
<thead>
<tr>
<th>Method</th>
<th>Flat price (i.e. the bond purchase price)</th>
<th>Accrued interest</th>
<th>Market price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical method</td>
<td>( B_i (1+i)^k )</td>
<td>( ( Fr ) s_{\frac{a}{b}} )</td>
<td>( B_i (1+i)^k - ( Fr ) s_{\frac{a}{b}} )</td>
</tr>
<tr>
<td>Practical method</td>
<td>( B_i (1+k i) )</td>
<td>( k(Fr) )</td>
<td>( B_i (1+k i) - k(Fr) )</td>
</tr>
<tr>
<td>Semi-theoretical method</td>
<td>( B_i (1+i)^k )</td>
<td>( k(Fr) )</td>
<td>( B_i (1+i)^k - k(Fr) )</td>
</tr>
</tbody>
</table>

Under the theoretical method, \( s_{\frac{a}{b}} \) (where \( k \) is a fraction) doesn’t have an intuitive explanation. However, SOA loves the complex concept of \( s_{\frac{a}{b}} \). As
a result, you need to memorize the theoretical method of calculating the accrued interest rate.

Please note that TVM Worksheet in BA II Plus/BA II Plus Professional allows N (the # of compounding periods) to be a fraction. So you can use TVM to calculate $s^k_{\overline{1}|}$ even when $k$ is a fraction.

Let's look at Kellison's method:

<table>
<thead>
<tr>
<th>Method</th>
<th>Flat price (i.e. the bond purchase price)</th>
<th>Accrued interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical method</td>
<td>Compounding interest</td>
<td>Compounding interest</td>
</tr>
<tr>
<td>Practical method</td>
<td>Simple interest</td>
<td>Simple interest</td>
</tr>
<tr>
<td>Semi-theoretical method</td>
<td>Compounding interest</td>
<td>Simple interest</td>
</tr>
</tbody>
</table>

So under Kellison's method, both the flat price and the accrued interest can be calculated using either a simple interest rate or a compounding interest rate. There is a total of three methods to calculate the market price – theoretical, practical, and semi-theoretical.

Because Kellison’s three methods are in the syllabus, you might want to memorize his three methods.

**Example 1**

<table>
<thead>
<tr>
<th>Bond face amount</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon rate</td>
<td>4% semiannual</td>
</tr>
<tr>
<td>Coupon payments</td>
<td>July 1 and December 31</td>
</tr>
<tr>
<td>Issue date</td>
<td>1/1/2000</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>12/31/2004</td>
</tr>
<tr>
<td>Date when bond is resold</td>
<td>5/1/2002</td>
</tr>
<tr>
<td>Yield to maturity to the 2nd buyer</td>
<td>6% semiannual</td>
</tr>
</tbody>
</table>

What’s the market price of the bond at the settlement date assuming the 2nd buyer has 100% ownership of the next coupon and all the other cash flows? Use a compounding interest rate
### Solution

Unit time = 0.5 year

The interest rate per coupon period \( i = \frac{6\%}{2} = 3\% \)

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>……</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>12/31</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
<tr>
<td>5/1</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
<tr>
<td>7/1</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
<tr>
<td>12/31</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
<tr>
<td>2001</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
<tr>
<td>2002</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
<tr>
<td>2002</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
<tr>
<td>12/31</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
<tr>
<td>2004</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>……</td>
<td>$100</td>
</tr>
</tbody>
</table>

We’ll discount future cash flows occurring at \( t=1, 2, \ldots, 6 \) to \( t=x \) at a discount rate of 3\% per coupon period. The PV of these cash flows is the purchase price of the bond at \( t=x \) if we ignore that fact that the previous owner deserves any portion of the next coupon.

\[
PV_x = \frac{2}{(1.03)^{1-x}} + \frac{2}{(1.03)^{2-x}} + \ldots + \frac{102}{(1.03)^{6-x}} = 1.03^x \left[ \frac{2}{(1.03)^{1}} + \frac{2}{(1.03)^{2}} + \ldots + \frac{102}{(1.03)^{6}} \right]
\]

\[\Rightarrow PV_x = 1.03^x \left( 2a_{\overline{6}|3\%} + 100v^6 \right)\]

Next, we need to find \( x \).

\[
x = \frac{\text{days between settlement date and last coupon date}}{\text{Total days per coupon period}}
\]

Method #1 – 360/360 day count method (for municipal and corporate bonds)

Under this method:

\# of days between 12/31/2001 and 5/1/2002 is 121 days.
# of days between 12/31/2001 and 7/1/2002 is 181 days. How do we actually count the days? Though we can do the math ourselves assuming a month has 30 days, we will let BA II Plus/BA II Plus Professional Date Worksheet count the days for us.

Key strokes in BA II Plus/BA II Plus Professional Date Worksheet for calculating the # of days between 12/31/2001 and 5/1/2002:

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystrokes</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Select Date Worksheet</td>
<td>2^{nd} [Date]</td>
<td>DT1= (old content)</td>
</tr>
<tr>
<td>Clear worksheet</td>
<td>2^{nd} [CLR Work]</td>
<td>DT1 = 12-31-1990</td>
</tr>
<tr>
<td>Enter 1^{st} date</td>
<td>12.3101</td>
<td>DT1 = 12-31-2001</td>
</tr>
<tr>
<td>Enter 2^{nd} date</td>
<td>5.0102</td>
<td>DT2 = 5-01-2002</td>
</tr>
<tr>
<td>Choose 360/360 day count method</td>
<td>↓ ↓ 2^{nd} Set</td>
<td>360</td>
</tr>
<tr>
<td>Compute days between dates</td>
<td>↑ CPT</td>
<td>DBD = 121</td>
</tr>
</tbody>
</table>

So there are 121 days between 12/31/2001 and 5/1/2002 under 360/360 day count method.

Similarly, we find that there are 181 days between 12/31/2001 and 7/1/2002 under 360/360 day count method.

\[
\Rightarrow x = \frac{\text{days between settlement date and last coupon date}}{\text{Total days per coupon period}} = \frac{121}{181}
\]

\[
\Rightarrow PV_x = 1.03^x \left( 2a_{6|3\%} + 100v^6 \right) = 1.03^{\frac{121}{181}} \left( 2a_{6|3\%} + 100v^6 \right)
\]

\[
2a_{6|3\%} + 100v^6 = 94.58280856 \quad \text{(using TVM)}
\]

\[
\Rightarrow PV_x = 1.03^x \left( 2a_{6|3\%} + 100v^6 \right) = 1.03^{\frac{121}{181}} (94.58280856) \approx 96.47038172
\]

Other methods to calculate \( PV_x \):

**Alternative Method A**

\[
\Rightarrow PV_x = 1.03^x PV_0
\]

\[
PV_0 = 2a_{6|3\%} + 100v^6
\]

\[
\Rightarrow PV_x = 1.03^x \left( 2a_{6|3\%} + 100v^6 \right) \approx 96.47038172
\]
Alternative Method B

\[ PV_x = v^{1-x} PV_i \]

\[ PV_i = 2 \dd_{\overline{6}|35} + 100v^5 \]

\[ PV_x = v^{1-x} \left( 2 \dd_{\overline{6}|35} + 100v^5 \right) = v^{-x} \left[ v \left( 2 \dd_{\overline{6}|35} + 100v^5 \right) \right] \]

\[ = v^{-x} \left( 2a_{\overline{7}|3%} + 100v^5 \right) = 1.03^x \left( 2a_{\overline{7}|3%} + 100v^5 \right) \approx 96.47038172 \]

Method #2 – Actual/Actual day count method (for Treasury bonds)

Since SOA doesn’t allow us to bring a calendar in the exam room, we can NOT look up a calendar and count the actual days. Once again, we’ll let BA II Plus/BA II Plus Professional do the work for us.

Key strokes in BA II Plus/BA II Plus Professional Date Worksheet for calculating the # of days between 12/31/2001 and 5/1/2002:

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Keystrokes</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Select Date Worksheet</td>
<td>2nd [Date]</td>
<td>DT1= (old content)</td>
</tr>
<tr>
<td>Clear worksheet</td>
<td>2nd [CLR Work]</td>
<td>DT1 = 12-31-1990</td>
</tr>
<tr>
<td>Enter 1st date</td>
<td>12.3101</td>
<td>DT1 = 12-31-2001</td>
</tr>
<tr>
<td>Enter 2nd date</td>
<td>5.0102</td>
<td>DT2 = 5-01-2002</td>
</tr>
<tr>
<td>Choose 360/360 day count method</td>
<td>↓ ↓ 2nd Set</td>
<td>ACT</td>
</tr>
<tr>
<td>Compute days between dates</td>
<td>↑ CPT</td>
<td>DBD = 121</td>
</tr>
</tbody>
</table>

So there are 121 days between 12/31/2001 and 5/1/2002 under Actual/Actual day count method.

Similarly, we find that there are 182 days between 12/31/2001 and 7/1/2002 under Actual/Actual day count method.

\[ x = \frac{\text{days between settlement date and last coupon date}}{\text{Total days per coupon period}} = \frac{121}{182} \]

In this problem, there is little different between the two day count methods. However, in some problems, the difference between the two methods can be bigger.

Finally, we are ready to calculate the purchase price of the 2nd buyer:
An exam problem may not tell you which date count method to use. In that case, use Actual/Actual method.

Example 2

<table>
<thead>
<tr>
<th>Bond face amount</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon rate</td>
<td>4%</td>
</tr>
<tr>
<td>Coupon payments</td>
<td>July 1 and December 31</td>
</tr>
<tr>
<td>Issue date</td>
<td>1/1/2000</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>12/31/2004</td>
</tr>
<tr>
<td>Date when bond is resold</td>
<td>5/1/2002</td>
</tr>
<tr>
<td>Yield to maturity to the 2nd buyer</td>
<td>6% nominal compounding semiannually</td>
</tr>
<tr>
<td>Day count</td>
<td>Actual/Actual</td>
</tr>
</tbody>
</table>

What’s the market price of the bond at the settlement date assuming the 2nd buyer has 100% ownership of the next coupon and all the other cash flows? Use a simple interest rate.

Solution

Unit time = 0.5 year

The interest rate per coupon period \( \frac{6\%}{2} = 3\% \)

Time 0 1 2 ...... 6

|-----------|---------|---------|-----------|-----------|

\[ PV_0 + PV_X + PV_1 \]

\[ PV_X = 1.03^{18.5} \left( 2a_{6.3\%} + 100v^6 \right) \approx 96.45990820 \]
We already calculated that under Actual/Actual

\[ x = \frac{\text{days between settlement date and last coupon date}}{\text{Total days per coupon period}} = \frac{121}{182} \]

To find the bond purchase price, we first we discount the future cash flows to t=0 (the prior coupon date):

\[ B_0 = PV_0 = 2a_{\frac{3\%}{2}} + 100v^6 \]

Next, we accumulate \( B_0 \) to time \( x \) using a simple interest rate:

\[ \Rightarrow B_x = PV_x = (1 + 0.03x)PV_0 = (1 + 0.03x)\left(2a_{\frac{3\%}{2}} + 100v^6\right) \]
\[ \Rightarrow B_x = PV_x = \left(1 + \frac{3\% \times 121}{182}\right)\left(2a_{\frac{3\%}{2}} + 100v^6\right) \]
\[ \Rightarrow B_x = PV_x = \left(1 + 3\% \times \frac{121}{182}\right)(94.58280856) = 96.46926787 \]

**Example 3**

<table>
<thead>
<tr>
<th>Bond face amount</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon rate</td>
<td>4% semiannual</td>
</tr>
<tr>
<td>Coupon payments</td>
<td>July 1 and December 31</td>
</tr>
<tr>
<td>Issue date</td>
<td>1/1/2000</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>12/31/2004</td>
</tr>
<tr>
<td>Date when bond is resold</td>
<td>5/1/2002</td>
</tr>
<tr>
<td>Yield to maturity to the 2\text{nd} buyer</td>
<td>6% nominal compounding semiannually</td>
</tr>
</tbody>
</table>

Day count: Actual/Actual

Calculate the market price of the bond immediately after the bond is sold on 5/1/2002, under the following methods:

- Theoretical method
- Practical
- Semi-theoretical method
Solution

\[ \text{Unit time} = 0.5 \text{ year} \]

The interest rate per coupon period \( i = \frac{6\%}{2} = 3\% \)

\[
\begin{array}{cccccc}
12/31 & 5/1 & 7/1 & 12/31 & 12/31 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>......</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2$</td>
<td>$2$</td>
<td>$2$</td>
<td>$2$</td>
<td>$100$</td>
<td></td>
</tr>
</tbody>
</table>

We already calculated that under Actual/Actual day count,

\[
x = \frac{\text{days between settlement date and last coupon date}}{\text{Total days per coupon period}} = \frac{121}{182}
\]

\[
B_x = PV_x = 1.03^x PV_0 = 1.03^{\frac{121}{182}} \left( 2a_{\frac{1}{6}3\%} + 100v^6 \right) \approx 96.45990820
\]

\[
AI = (\text{coupon})s_{\frac{1}{3}} = 2s_{\frac{1}{3}} = 2 \left( \frac{1+3\%}{3\%} \right)^{\frac{121}{182}} - 1 = 1.32307317
\]

Alternative method to calculate \( 2s_{\frac{1}{3}} \):

Enter the following into BA II Plus/BA II Plus Professional TVM:

\[
PMT=2, \ N=\frac{121}{182}, \ I/Y=3. \quad \Rightarrow \quad FV= -1.32307317
\]

Bond market price= purchase price – accrued interest

\[
= 96.45990820 - 1.32307317 = 95.14
\]
Practical method

\[ B_x = P V_x = (1 + 0.03x) P V_0 = (1 + 0.03x) \left( 2a_{63 \%} + 100v^6 \right) \]

\[ = \left( 1 + 3\% \times \frac{121}{182} \right) \left( 2a_{63 \%} + 100v^6 \right) \]

\[ = \left( 1 + 3\% \times \frac{121}{182} \right) (94.58280856) = 96.46926787 \]

\[ AI = (coupon) x = \left( 2 \right) \frac{121}{182} = 1.32967033 \]

Bond market price = purchase price – accrued interest

\[ = 96.46926787 - 1.32967033 = 95.14 \]

Semi-theoretical method

\[ B_x = P V_x = 1.03^x P V_0 = 1.03^{182} \left( 2a_{63 \%} + 100v^6 \right) \approx 96.45990820 \]

\[ AI = (coupon) x = \left( 2 \right) \frac{121}{182} = 1.32967033 \]

Bond market price = purchase price – accrued interest

\[ = 96.45990820 - 1.32967033 = 95.13 \]

The results under the three methods are very close.

Reference to SOA problems

Exam FM Sample Questions #50

Explanation of SOA’s solution to FM Sample Questions #50. This problem asks for the purchase price, not the (quoted) market price. So there’s no need to subtract the accrued interest. The problem also tells you to use a simple interest between bond coupon dates.

First, SOA calculates the bond price \( P = P V_0 = 906.32 \) on the previous coupon date of 4/15/2005 using \( i^{(2)} = 7\% \). Next, using the simple interest rate, we calculate the purchase price of the bond on 6/28 as

\[ B_x = P V_x = (1 + 0.035x) P V_0 = \left( 1 + 0.035 \times \frac{74}{183} \right) 906.32 = 919.15. \]
Chapter 12  Time weighted return and dollar weighted return

Time weighted return

- To calculate the time weighted return of a fund over a time horizon \([0,t]\), we first break down \([0,t]\) into \(n\) sub-periods. These \(n\) sub-periods can be of equal lengths or unequal lengths.

- For example, to calculate the time weighted return of a fund’s performance in a year, we can break down a year into 4 quarters. We can also break down a year into 3 sub-periods: sub-period 1 consists of January, sub-period 2 consists of 4 months from February to May, and sub-period 3 consists of 7 months from June to December. (An SOA problem will tell you the whole period is broken down into how many sub-periods.)

- Next, we calculate the return for each sub-period as follows:

\[
R_k = \text{Return of } k\text{-th sub-period} = \frac{\text{Ending asset value of } k\text{-th sub-period} - 1}{\text{Beginning asset value of } k\text{-th sub-period}}
\]

- We then calculate the time weighted return \(R\) for the whole period \([0,t]\) by solving the following equation:

\[
(1 + R)^t = (1 + R_1) (1 + R_2) \cdots (1 + R_n)
\]

- The time weighted return is the geometric average of all the sub-periods’ returns.

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>(0)</th>
<th>(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\leftarrow \quad 1 + R_1 \rightarrow )</td>
<td>(\leftarrow \quad 1 + R_2 \rightarrow )</td>
<td>(\leftarrow \quad 1 + R_n \rightarrow )</td>
</tr>
<tr>
<td>(\leftarrow \quad \text{Period 1} \rightarrow )</td>
<td>(\leftarrow \quad \text{Period 2} \rightarrow )</td>
<td>(\leftarrow \quad \text{Period } n \rightarrow )</td>
</tr>
<tr>
<td>(\leftarrow \quad (1 + R)^t = (1 + R_1)(1 + R_2) \cdots (1 + R_n) \rightarrow )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Dollar weighted return

- To calculate the dollar weighted return of a fund over a time horizon \([0,t]\), we first find the complete history of all the cash inflows and cash outflows during \([0,t]\).

1. We need to know how much money the fund has at time zero. We call this \(CF(0)\). This is the beginning account value.

2. We need to know how much money the fund has at time \(t\). We call this \(CF(t)\). This is the ending account value.

3. We need to know what happened in between. If any money is added to or withdrawn from the fund during \([0,t]\), we need to find out (1) when this happened, (2) how much money is added to or withdrawn from the fund.

4. For example, we record a cash flow of \(CF(t_i)\) at time \(t_i\) where \(0 < t_i < t\). If \(CF(t_i)\) is an inflow (i.e. more money is added to the fund), we will make \(CF(t_i)\) positive; If \(CF(t_i)\) is an outflow (i.e. money is withdrawn from the fund), we will make \(CF(t_i)\) negative.

- Next, we translate the fund history during \([0,t]\) into the following cash flow diagram:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>(t_1)</th>
<th>(t_2)</th>
<th>(t_3)</th>
<th>(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>(CF(0))</td>
<td>(CF(t_1))</td>
<td>(CF(t_2))</td>
<td>(CF(t_3))</td>
<td>(CF(t))</td>
</tr>
</tbody>
</table>

- Finally, we find the dollar weighted return \(R\) by solving the following equation (we assume a simple interest rate \(R\))

\[
CF(0)(1+R)+CF(1)[1+(t-t_1)R]+CF(2)[1+(t-t_2)R]+...=CF(t)
\]

Guo FM, fall 2009
Problem 1

<table>
<thead>
<tr>
<th>Date</th>
<th>Fun’s balance before deposits and withdrawals</th>
<th>Deposit</th>
<th>Withdrawal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2004</td>
<td>$100,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/31/2004</td>
<td>$120,000</td>
<td>$30,000</td>
<td></td>
</tr>
<tr>
<td>6/30/2004</td>
<td>$140,000</td>
<td></td>
<td>$50,000</td>
</tr>
<tr>
<td>10/31/2004</td>
<td>$198,000</td>
<td>$90,000</td>
<td>$70,000</td>
</tr>
<tr>
<td>12/31/2004</td>
<td>$220,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Find the fund’s time weighted return and dollar weighted return during 2004.

Solution

Find the time weighted return

\[
(1 + R) = \frac{120,000}{100,000} \times \frac{140,000}{120,000 + 30,000} \times \frac{198,000}{140,000 - 50,000} \times \frac{220,000}{198,000 + 90,000 - 70,000}
\]

\[
R = 148.66\%
\]

We calculate the return from 1/1 to 3/31, the beginning fund value of is $100,000 and the ending value is $120,000. When we calculate the return from 3/31 to 6/30, however, the beginning fund value is $120,000 + $30,000 = $150,000

We add $30,000 because $30,000 fresh money flows into the fund at the end of 3/31, perhaps because the fund has earned 20% return from 1/1 to 3/31.

Find the dollar weighted return

To calculate the dollar weighted return, we need to know the beginning fund value, the ending fund value, and all the cash flows in-between. However, we don’t need to know the interim fund balances. To simply our calculation, let’s first remove the interim fund balances.
In addition, to help us track down the time neatly, we convert a month-end dates to a month-begin date. Specifically, we change 3/31/2004 to 4/1/2004; 6/30 to 7/1; 10/31 to 11/1. We'll keep 12/31.

This way, we know that the distance between 3/31 and 1/1 is the distance between 4/1 and 1/1 (3 months apart). If we don’t convert 3/31 to 4/1, we might incorrectly conclude 3/31 and 1/1 are 2 months apart.

Is it OK to convert 3/31 to 4/1? Yes. When we count the fund value on 3/31, we actually count the fund value at the end of 3/31, which is the same as the fund value counted in the beginning of 4/1.
In the above expression, \(220,000 - 100,000 - 30,000 + 50,000 - 20,000 = 120,000\) is the total interest earned in 2004. \(100,000 + 30,000\left(\frac{9}{12}\right) - 50,000\left(\frac{6}{12}\right) + 20,000\left(\frac{2}{12}\right)\) is the total principal that generated the \(\$120,000\) interest. The ratio represents the interest earned during 2004.

In Exam FM, SOA wants us to calculate the time weighted return using a simple interest rate; we shall do so in the exam.

In the real world, however, the standard practice is to calculate the time weighted return using a compound interest rate.

Let’s calculate the time weighted return using a compound interest rate.

We set up the following equation:

\[
100,000(1+i) + 30,000(1+i)^{\frac{9}{12}} - 50,000(1+i)^{\frac{6}{12}} + (90,000 - 70,000)(1+i)^{\frac{2}{12}} = 220,000
\]

\[
\Rightarrow 100,000(1+i) + 30,000(1+i)^{\frac{9}{12}} - 50,000(1+i)^{\frac{6}{12}} + 20,000(1+i)^{\frac{2}{12}} = 220,000
\]

We can’t solve this equation manually; we need to use BA II Plus/BA II Plus Professional Cash Flow Worksheet.

Because BA II Plus/BA II Plus Professional Cash Flow Worksheet can not use fractional time (all cash flow times need to be entered as a non-negative integer), we will use a month as the unit time and calculate the monthly effective interest rate \(r\).

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$100,000</td>
<td>$30,000</td>
<td>- $50,000</td>
<td></td>
<td>$20,000</td>
<td>- $220,000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\Rightarrow 100,000(1+r)^{12} + 30,000(1+r)^{9} - 50,000(1+i)^{6} + 20,000(1+i)^{2} = 220,000
\]

\[
\Rightarrow 100,000 + 30,000(1+r)^{-3} - 50,000(1+i)^{-6} + 20,000(1+i)^{-10} - 220,000(1+i)^{-12} = 0
\]

To solve the above equation, we enter the following into Cash Flow Worksheet:
To speed up our calculation, we use $1,000 as the unit money. If a cash flow is $30,000, we enter 30. Try to learn this technique in the exam.

Cash flow diagram (set $1,000 as one unit of money):

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$100</td>
<td>30</td>
<td>- 50</td>
<td>20</td>
<td>- 220</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Cash Flow Worksheet

- **CF0 = 100**: 1st cash flow at t=0 is $100.
- **C01=0, F01=2**: Indicate cash flows at t=1, 2 are zero. If a cash flow is zero, still need to press “Enter.”
- **C02=30**: Indicate cash flow at t=3 is 30. No need to set F02=1; Cash Flow Worksheet automatically use 1 as the default cash flow frequency.
- **C03=0, F03=2**: Indicate cash flows at t=4, 5 are zero
- **C04= - 50**: Indicate cash flow at t=6
- **C05=0, F05=3**: Indicate cash flows at t=7, 8, 9 are zero
- **C06=20**: Indicate cash flow at t=10
- **C07 =0**: Indicate cash flows at t=11 is zero
- **C08= - 220**: Indicate cash flow at t=12

Let Cash Flow Worksheet solve for IRR. Press “IRR” “CPT.”

**IRR=6.6794%**

Finally, we convert this monthly effective rate into an annual effective rate:

\[
(1 + 0.066794)^{12} - 1 \approx 117.25%
\]

**Problem 2**

A fund has the following transactions:

<table>
<thead>
<tr>
<th>Date</th>
<th>Account Value (in millions)</th>
<th>Contribution (in millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2000</td>
<td>1.35</td>
<td></td>
</tr>
<tr>
<td>12/31/2001</td>
<td>X</td>
<td>3.95</td>
</tr>
<tr>
<td>1/1/2002</td>
<td>5.8</td>
<td></td>
</tr>
<tr>
<td>1/1/2003</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
For the period from 1/1/2000 to 1/1/2003, the dollar weighted return using the compound interest rate and the time weighted return are the same.

Calculate $X$

**Solution**

**Dollar weighted return**

When calculating the dollar weighted return, we ignore the interim account values. So $X$ is not needed.

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.35</td>
<td>3.95</td>
<td>5.8</td>
<td></td>
</tr>
</tbody>
</table>

$1.35(1+i)^3 + 3.95(1+i) = 5.8$

This is a difficult equation to solve. Fortunately, we can use the IRR functionality in BA II Plus or BA II Plus Professional Cash Flow Worksheet.

Multiplying both sides $1.35(1+i)^3 + 3.95(1+i) = 5.8$ with $v^3 = (1+i)^{-3}$, we have:

$1.35 + 3.95v - 5.8v^3 = 0$

This equation translates to the following cash flow diagram:

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.35</td>
<td>3.95</td>
<td>-5.8</td>
<td></td>
</tr>
</tbody>
</table>

Please note the cash flow at $t=3$ is a negative 5.8.
Enter the following into Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>CF0</th>
<th>C01</th>
<th>C02</th>
<th>C03</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.35</td>
<td>0</td>
<td>3.95</td>
<td>-5.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>F01</th>
<th>F02</th>
<th>F03</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Press “IRR” “CPT.” You should get: IRR=6.06031163

So the dollar weighted return using a compounding interest rate is:

\[ i = 6.06\% \]

**Time weighted return**

Come back to the table.

<table>
<thead>
<tr>
<th>Date</th>
<th>Account Value (in millions)</th>
<th>Contribution (in millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2000</td>
<td>1.35</td>
<td></td>
</tr>
<tr>
<td>12/31/2001</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>1/1/2002</td>
<td>X</td>
<td>3.95</td>
</tr>
<tr>
<td>1/1/2003</td>
<td>5.8</td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{X}{1.35} \times \frac{5.8}{X + 3.95} = (1 + j)^3
\]

We are told that \( j = i = 6.06\% \)

\[
\Rightarrow \frac{X}{1.35} \left( \frac{5.8}{X + 3.95} \right) = 1.06^3, \quad X = 1.515
\]

**Problem 3 (SOA May 2005 EA-1 #1)**

A fund has the following transactions

<table>
<thead>
<tr>
<th>Date</th>
<th>Account Value</th>
<th>Contributions</th>
<th>Benefits payments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2005</td>
<td>$1,000</td>
<td>( C_1 )</td>
<td>0</td>
</tr>
<tr>
<td>3/31/2005</td>
<td>0</td>
<td>$100</td>
<td></td>
</tr>
</tbody>
</table>
The time weighted rate of return in 2005 is 6.25%. The dollar weighted rate of return in 2005 is 6.00%.

In what range is $C_1$?

- (A) Less than $405
- (B) $405 but less than $430
- (C) $430 but less than $455
- (D) $455 but less than $480
- (E) $480 or more

**Solution**

This problem is simple conceptually, but it involves messy calculations.

**Time weighted return**

We break down Year 2005 into the two segments: [1/1, 3/31], [4/1, 12/31].

<table>
<thead>
<tr>
<th>Segment</th>
<th>1/1 to 3/31</th>
<th>4/1 to 12/31</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin acct value</td>
<td>1,000 + $C_1$</td>
<td>1,300</td>
</tr>
<tr>
<td>Ending acct value</td>
<td>1,300 + 100 = 1,400 (^{(1)})</td>
<td>1,700 + 150 − $C_2$ = 1,850 − $C_2$ (^{(2)})</td>
</tr>
</tbody>
</table>

1. On 4/1, the account value is $1,300. Because $100 is taken out on 3/31, the account value on 3/31 is 1,400.

2. On 1/1/2006, the account value is $1,700. On 12/31/2005, $C_2$ amount of new money comes in and $150 flows out. So the account value on 12/31/2005 is 1,700 + 150 − $C_2$ = 1,850 − $C_2$.

We are given that the time weighted return during 2005 is 6.25%.

\[
\frac{1,400}{1,000 + C_1} \times \frac{1,850 - C_2}{1,300} = 1.0625 \quad \text{(Equation 1)}
\]

**Dollar weighted return**

When calculating the dollar weighted return, we need to throw away the intermediate account values. So we don’t need to use the account value
on 4/1/2005. We only care about the account value on 1/1/2005; the account value on 12/31/2005; and the money that flows in or comes out during Year 2005.

The Account value on 1/1/2005 is $1,000 + C_1$. This earns a full year interest. On 3/31/2005, $100$ flows out; this earns 9 months negative interest. The account value on 12/31/2005 is $1,700 + 150 - C_2 = 1,850 - C_2$.

We are told that the timed weighted return during 2005 is 6%:

\[(1,000 + C_1) \times 1.06 - 100 \left( 1 + \frac{9}{12} \times 6\% \right) = 1,850 - C_2 \quad \text{(Equation 2)}\]

Solving these two equations, we have:

\[C_1 = 423.84, \quad C_2 = 445.23\]

So the correct answer is B.

**Problem 4 (SOA May 2004 EA #10)**

<table>
<thead>
<tr>
<th>Date</th>
<th>Market value of fund</th>
<th>Contributions</th>
<th>Withdrawals</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2004</td>
<td>$100,000</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>4/1/2004</td>
<td>$85,000</td>
<td>$30,000</td>
<td>None</td>
</tr>
<tr>
<td>8/1/2004</td>
<td>$100,000</td>
<td>None</td>
<td>$20,000</td>
</tr>
<tr>
<td>12/31/2004</td>
<td>$80,000</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

Market value of the fund is prior to contributions and withdrawals.

*A* = Time weighted return  
*B* = Dollar weighted return

In what range is the absolute value of *A* + *B*?

(A) Less than 45%  
(B) 45% but less than 48%  
(C) 48% but less than 51%  
(D) 51% but less than 54%  
(E) 54% or more

**Solution**
To speed up our calculation, we use $1000 as one unit of money. Then the original table becomes:

<table>
<thead>
<tr>
<th>Date</th>
<th>Market value of fund</th>
<th>Contributions</th>
<th>Withdrawals</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2004</td>
<td>$100</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>4/1/2004</td>
<td>$85</td>
<td>$30</td>
<td>None</td>
</tr>
<tr>
<td>8/1/2004</td>
<td>$100</td>
<td>None</td>
<td>$20</td>
</tr>
<tr>
<td>12/31/2004</td>
<td>$80</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

Time weighted return

\[
\frac{85}{100} \times \frac{100}{85 + 30} \times \frac{80}{100 - 20} = 1 + i
\]

⇒ \( i = -26.09\% \)

Dollar weighted return

We first delete the interim fund values:

<table>
<thead>
<tr>
<th>Date</th>
<th>Market value of fund</th>
<th>Contributions</th>
<th>Withdrawals</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2004</td>
<td>$100</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>4/1/2004</td>
<td>$85</td>
<td>$30</td>
<td>None</td>
</tr>
<tr>
<td>8/1/2004</td>
<td>$100</td>
<td>None</td>
<td>$20</td>
</tr>
<tr>
<td>12/31/2004</td>
<td>$80</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

\[
100(1 + j) + 30\left(1 + \frac{9}{12} j\right) - 20\left(1 + \frac{5}{12} j\right) = 80, \ \Rightarrow \ j = -26.28\%
\]

⇒ \(|A + B| = |i + j| = |-26.09\% - 26.28\%| = 52.37\%

So the answer is D.

Problem 5 (SOA May 2003 EA-1 #5)
All assets of a pension plan are invested by manager Smith and manager Jones. There are no other plan assets. The following chart shows the market value of the plan’s assets with each manager:

<table>
<thead>
<tr>
<th>Date</th>
<th>Smith</th>
<th>Jones</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance 12/31/2001</td>
<td>$2,500,000</td>
<td>$2,500,000</td>
</tr>
<tr>
<td>Contribution 1/1/2002</td>
<td>$0</td>
<td>$1,500,000</td>
</tr>
<tr>
<td>Balance 6/30/2002</td>
<td>$2,800,000</td>
<td>$4,500,000</td>
</tr>
<tr>
<td>Transfer 7/1/2002</td>
<td>$1,000,000</td>
<td>($1,000,000)</td>
</tr>
<tr>
<td>Balance 12/31/2002</td>
<td>$4,180,000</td>
<td>$3,500,000</td>
</tr>
</tbody>
</table>

\[ X = \text{one-half of the sum of both manager's time-weighted percentage returns for 2002.} \]

\[ Y = \text{dollar weighted percentage return for 2002 for the entire pension plan.} \]

\[ Z = Y - X \]

In what range is \( Z \)?

(A) Less than 0.09%
(B) 0.09% but less than 0.18%
(C) 0.18% but less than 0.27%
(D) 0.27% but less than 0.36%
(E) 0.36% or more

**Solution**

We’ll use $1,000,000 as one unit of money. Doing so will greatly simplifies our calculation and reduces chances of errors. You should learn this technique.

Then the table is simplified as follows:

<table>
<thead>
<tr>
<th>Date</th>
<th>Smith</th>
<th>Jones</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance 12/31/2001</td>
<td>$2.5</td>
<td>$2.5</td>
</tr>
<tr>
<td>Contribution 1/1/2002</td>
<td>$0</td>
<td>$1.5</td>
</tr>
<tr>
<td>Balance 6/30/2002</td>
<td>$2.8</td>
<td>$4.5</td>
</tr>
<tr>
<td>Transfer 7/1/2002</td>
<td>$1</td>
<td>($1)</td>
</tr>
<tr>
<td>Balance 12/31/2002</td>
<td>$4.18</td>
<td>$3.5</td>
</tr>
</tbody>
</table>

**Time-weighted return**

Smith:
2.8 \overline{2.5} \times \overline{4.18 \overline{2.8+1}} = 1+i, \Rightarrow i = 23.2%

Jones:

\frac{4.5}{2.5+1.5} \times \frac{3.5}{4.5-1} = 1+j, \Rightarrow i = 12.5%

\Rightarrow X = \frac{1}{2}(i + j) = \frac{1}{2}(23.2\% + 12.5\%) = 17.85\%

Dollar weighted return for the entire pension plan

We combine the transactions by Smith and by Jones into one:

<table>
<thead>
<tr>
<th></th>
<th>Date</th>
<th>Smith</th>
<th>Jones</th>
<th>Entire Pension Plan (Smith + Jones)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance</td>
<td>12/31/2001</td>
<td>$2.5</td>
<td>$2.5</td>
<td>$5</td>
</tr>
<tr>
<td>Contribution</td>
<td>1/1/2002</td>
<td>$0</td>
<td>$1.5</td>
<td>$1.5</td>
</tr>
<tr>
<td>Balance</td>
<td>6/30/2002</td>
<td>$2.8</td>
<td>$4.5</td>
<td>$7.3</td>
</tr>
<tr>
<td>Transfer</td>
<td>7/1/2002</td>
<td>$1</td>
<td>($1)</td>
<td>$0</td>
</tr>
<tr>
<td>Balance</td>
<td>12/31/2002</td>
<td>$4.18</td>
<td>$3.5</td>
<td>$7.68</td>
</tr>
</tbody>
</table>

The interim balance of $7.3 on 6/30/2002 is not needed for the calculation.

5(1+k) + 1.5(1+k) = 7.68, \Rightarrow k = 18.15\%, \Rightarrow Y = k = 18.15\%

\Rightarrow Z = Y - X = 18.15\% - 17.85\% = 0.3\%

So the answer is D.

Problem 6 (SOA May 2000 EA-1 #17)

Market value of a pension fund:

<table>
<thead>
<tr>
<th>Date</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2000</td>
<td>$50,000</td>
</tr>
<tr>
<td>3/31/2000</td>
<td>$60,000</td>
</tr>
<tr>
<td>6/30/2000</td>
<td>$45,000</td>
</tr>
<tr>
<td>9/30/2000</td>
<td>$40,000</td>
</tr>
</tbody>
</table>
12/31/2000 | $65,000

Contribution and benefit payments:

<table>
<thead>
<tr>
<th>Date</th>
<th>Contributions</th>
<th>Benefit Payments</th>
</tr>
</thead>
<tbody>
<tr>
<td>4/1/2000</td>
<td>$0</td>
<td>$P</td>
</tr>
<tr>
<td>7/1/2000</td>
<td>$17,000</td>
<td>P</td>
</tr>
<tr>
<td>10/1/2000</td>
<td>$55,000</td>
<td>P</td>
</tr>
</tbody>
</table>

Dollar-weighted rate of return for 2000 using simple interest: 7%.

In what range is the time weighted rate of return for 2000?

**Solution**

We'll use $1,000 as one unit of money. First, let's combine the two tables into one.

<table>
<thead>
<tr>
<th>Date</th>
<th>Value</th>
<th>Contributions</th>
<th>Benefit Payments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2000</td>
<td>$50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/31/2000</td>
<td>$60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4/1/2000</td>
<td>$0</td>
<td>$P</td>
<td></td>
</tr>
<tr>
<td>6/30/2000</td>
<td>$45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/1/2000</td>
<td>$17</td>
<td>P</td>
<td></td>
</tr>
<tr>
<td>9/30/2000</td>
<td>$40</td>
<td></td>
<td>P</td>
</tr>
<tr>
<td>10/1/2000</td>
<td>$55</td>
<td>P</td>
<td></td>
</tr>
<tr>
<td>12/31/2000</td>
<td>$65</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Dollar weighted return

\[
50(1+i) - P\left(1+\frac{9}{12}i\right) + (17-P)\left(1+\frac{6}{12}i\right) + (55-P)\left(1+\frac{3}{12}i\right) = 65
\]

We are given: \(i = 7\%\).

\[
50(1.07) - P\left(1+\frac{9}{12}\times0.07\right) + (17-P)\left(1+\frac{6}{12}\times0.07\right) + (55-P)\left(1+\frac{3}{12}\times0.07\right) = 65
\]

Solving the above equation, we get: \(P = 19.986 \approx 20\)

Time weighted return
\[
\frac{60}{50} \times \frac{45}{60-P} \times \frac{40}{45+17-P} \times \frac{65}{40+55-P} = 1 + j
\]

\[
\Rightarrow \frac{60}{50} \times \frac{45}{60-20} \times \frac{40}{45+17-20} \times \frac{65}{40+55-20} \approx 1.11
\]

\[
\Rightarrow j = 11\%
\]

So the answer is D.

**Problem 6 (FM May 2005 # 16)**

At the beginning of the year, an investment fund was established with an initial deposit of 1000. A new deposit of 1000 was made at the end of 4 months. Withdrawals of 200 and 500 were made at the end of 6 months and 8 months, respectively. The amount in the fund at the end of the year is 1560.

Calculate the dollar-weighted (money-weighted) yield rate earned by the fund during the year.

(A) 18.57%
(B) 20.00%
(C) 22.61%
(D) 26.00%
(E) 28.89%

**Solution**

\[
1000(1 + i) + 1000\left(1 + \frac{8}{12} i\right) - 200\left(1 + \frac{6}{12} i\right) - 500\left(1 + \frac{4}{12} i\right) = 1560
\]

\[
\Rightarrow i = 18.571\%
\]

So the answer is A.
Chapter 13  Investment year & portfolio method

SOA problems on this topic tend to be simple and straightforward. Here are supplemental explanations.

1. When insurance companies set their premium rates, they do so after taking into the account that premiums collected are invested somewhere earning interests. If premium dollars can be invested prudently and earn a higher interest rate, premium rates will be lower and agents can sell the insurance products more easily. Conversely, if premiums earn a lower interest rate, then premium rates will be higher and the insurance products will be less competitive in the market place.

2. Insurance companies (especially life insurance companies) often declare two interest rates, the guaranteed interest rate and the current interest rate. The guaranteed interest rate is set permanently and can not be changed. The current interest rate is the actual interest rate credited to the policyholder and is adjusted on the on-going basis.

3. For example, a whole life insurance policy may have a guaranteed interest rate of 3% and the current interest rate of 5.5%.

   • The guaranteed 3% means that the policyholder will earn at least 3% per year no matter what.

   • The guaranteed interest rate is written in the insurance contract; once set, it can not be changed.

   • This guaranteed rate is just the lowest interest possible. It is written in the contract to protect the insurance company against really bad investment experiences. Unless the investment experience is really bad, the insurance company doesn’t really use this rate.

   • The current interest rate of 5.5% is the rate actually credited to the policyholder.

   • Insurance companies change their current rates periodically (ex every 6 months or every year) to be line with the prevailing market interest rate.
4. Potential buyers of insurance products often want to know the interest rate earned by their premiums. They often shop around and compare interest rates earned by premium dollars.

5. When setting the on-going current interest rates periodically, the insurance company often uses one of the two methods: the portfolio method and the investment year method.

The portfolio method

- Assets are combined for different products (ex. whole life and universal life), or different time periods when premiums come to the insurance company (ex. all the premiums collected last year, whether in the 1st quarter or other quarters).

- A single interest rate is used for different products, different periods of time.

- Simple for the insurance company’s IT department to manage. For example, the IT department does not need to use complex software to keep track of when premiums come in and premiums are for what product.

The investment year method (or the new money method)

- Assets are segmented for different products, or different time periods during which funds are received. Different interest rates are used.

- Disadvantage - IT department needs to use complex software to keep track of when new funds come in and new funds are for what products.

- Advantage – The interest rate credited to premiums or other funds depends when premiums or funds come in. If the interest rate is higher when new premiums come to the insurance company, new premiums will automatically get the higher interest rate.

SOA problems on the portfolio method and the investment year method are simple, requiring candidates to look up the correct interest rate from a given interest rate table. Please refer to Sample FM #8 to learn how to solve this type of problems.
Chapter 14  Short Sales

Short sales of stocks are covered in two areas: (1) theory of interest textbooks (such as Sam Broverman’s textbook Mathematics of Investment and credit section 8.2.2), and (2) Derivatives Markets (section 1.4).

According to the SOA’s syllabus, you just need to short sales as explained in Derivatives Markets.

You don’t need to worry about short sales as explained in any of the four recommended textbooks for the theory of interest. As a matter of fact, SOA removed Sample FM #38, #39, and #40. All these problems are short sales of stocks as explained in the theory of interest textbooks.

This chapter is not on the syllabus. However, I include this chapter to help you better understand Derivatives Markets section 1.4. If you understand Derivatives Markets section 1.4, that’s great and you can just skip this chapter.

Company A’s stocks are currently selling $100 per share. You have a good reason to believe that this stock price is not going to last long and will go down soon. Perhaps you heard the rumor that Company A uses aggressive accounting to exaggerate its sales. Or perhaps you learned from the Wall Street Journal that Company A’s top management is incompetent. Or perhaps you found that one of Company A’s chief competitors has just designed a revolutionary product which will make Company A’s main product line obsolete.

For whatever reason, you believe that Company A’s stock price will go down in the very near future, say in 15 days. How can you make money on Company A’s stocks based on your analysis?

You can make money this way. You can use an internet brokerage firm to sell short 1,000 shares of Company A stocks. Or if you prefer traditional brokerage firm, you can ask your broker to sell 1,000 shares of Company A stocks short for you. Selling short means that you sell something you don’t have. Currently you don’t own any stocks from Company A, but you want to sell 1,000 shares of Company A stocks. So you are selling short. Essentially, you are borrowing 1,000 shares from a brokerage firm and sell them in the open market.

Just as you have predicted, 15 days later, the price of Company A’s stocks drops to $80 per share. You buy 1,000 shares of Company A
stocks in the open market at $80 per share and return the 1,000 shares to the brokerage firm.

Let’s see how much money you have made. You sell 1,000 shares at $100 per share. Your cost is $80 per share. So you have earned $20 per share. Your total profit is ($20)(1,000)=$20,000. Nice job.

This is the stripped-down version of short sales; the actual transactions are a little more complex. Let’s analyze short sales.

First, in short sales, you sell shares high now and buy shares back later. You hope to buy shares back in the near future at a lower price. However, if the share price goes up, you’ll have to buy shares back at a higher price and incur a loss. For example, 15 days later, Company A’s stocks sell $110 per share. All the rumors you heard about Company turn out to be false. Company A is doing quite well. You wait for a while, but Company A stock prices stay stable at $110 per share. Finally, you have to buy back 1,000 shares at $110 per share in the open market and return them to the brokerage firm which did the short sale for you. You incurred a total loss of 1,000($110-$100)=$10,000.

Second, laws require that short sales can take place only if the last recorded change in the price of a stock is positive (i.e. the stock price went up last time). You can’t short sell a stock after the stock price went down. You can short sell a stock after the stock price went up. This rule attempts to discourage wild speculations on stocks.

Third, in short sales, your liability to the brokerage firm is the number of shares borrowed, not the value of the borrowed shares at the time when short sales take place. In our example, you borrow 1,000 shares of Company A stocks from the brokerage firm at $100 per share. Your liability to the brokerage firm is 1,000 shares of stocks, not $100,000. In the future, you just need to return 1,000 shares to the brokerage firm. If you can return 1,000 shares of stocks to the brokerage firm at a lower price, you’ll make a profit. If you return 1,000 shares at a higher price, you have a loss.

Forth, you can borrow shares for long time before paying them back. Have you wondered where the brokerage firm gets the 1,000 shares of Company A stocks to lend them to you? The brokerage firm simply borrows 1,000 shares of Company A stocks from another investor who holds 1,000 shares of Company A stocks in his account at the brokerage firm. This investor will not even know that the brokerage firm has borrowed 1,000 shares from his account. If this investor wants to sell 1,000 shares of Company A stocks after you have short sold his 1,000 shares of stocks, the brokerage firm simply borrows 1,000 shares of
Company A stocks from another investor. As a result, you can borrow someone else’s shares for a long time.

However, in some situations you short sold somebody else’s 1,000 shares of stocks. Then the original owner demanded 1,000 shares of stocks (because he wanted to sell them), but the brokerage firm couldn’t find another investor who had 1,000 shares of stocks. If this happens, the brokerage firm will ask you to immediately purchase 1,000 shares from the open market and return them to the brokerage firm.

Fifth, if the borrowed stocks pay dividends after the short sale and before you return the borrowed stocks to the brokerage firm, you are required to pay the dividend to the original owner of the stocks. When you wanted to short sell 1,000 shares of Company A stocks, your brokerage firm decided borrow 1,000 shares from an investor (we call him Smith) and lent them to you. You short sold these 1,000 shares of stocks at $100 per share to someone (we call him John). One month later, Company A decided to pay a dividend of 5 cents per share. Company A found out that John was the owner of the 1,000 shares. At this time you had not paid back the 1,000 borrowed shares to the brokerage firm.

Company A would pay the dividend worth a total of $0.05(1,000)=$50 to the new owner John, not to the original owner Smith. However, Smith still owned 1,000 shares of Company A stocks and should get a total of $50 dividend. Consequently, you needed to pay the brokerage firm $50 dividend and the brokerage firm would give the $50 to the original owner Smith.

Does this mean that you paid $50 dividend to Smith out of your own pocket? No. The dividend of 5 cents per share was reflected in the stock price. In other words, when you short sold 1,000 shares of stocks at $100 per share, the $100 price already reflected the possibility that Company would distribute certain amount of dividend to its shareholders. Because Company A stocks could possibly generate a dividend of 50 cents per share, you could short sell Company A stocks at $100 per share. If Company was not expected to distribute any dividend in the near future, the sales price of Company A stocks would have been less than $100 per share.

Sixth, you must meet the initial margin requirement before you can short sell. For example, the brokerage firm requires a 50% initial margin. This means that before a short sale can take place, your collateral account in the brokerage firm must hold at least 50% of the then current market value of 1,000 shares of Company A stocks.
Seventh, you must meet the ongoing maintenance margin requirement. The maintenance margin requirement provides another layer of protection to the brokerage firm. For example, the brokerage firm requires a 40% maintenance margin. This means that from the moment when a short sale takes place to the moment immediately before you finally return 1,000 shares to the brokerage firm, your collateral account in the brokerage firm must hold at least 40% of the then current market value of 1,000 shares of Company A stocks.

Mathematically, at any time $t$ (where $t$ is between when a short sale takes place to the moment immediately before you finally return the borrowed shares to the brokerage firm)

$$\frac{\text{Your equity}(t)}{\text{market value}(t)\text{of the stocks borrowed}} \geq \text{Maintenance margin \%}$$

$$\Rightarrow \quad \text{Your equity}(t) \geq [\text{Maintenance margin \%}]\left[\text{market value}(t)\text{of the stocks borrowed}\right]$$

Equity$(t) = \text{Asset}(t) - \text{Liability}(t)$

Please note that the 40% margin requirement can be met by cash, stocks and bonds. If your account at the brokerage firm has cash, stocks of another company, or bonds worth less than $40,000, you will get a margin call, in which cash you must deposit additional cash to your collateral account.

For additional information, refer to [http://www.investopedia.com/university/shortselling/](http://www.investopedia.com/university/shortselling/)

For a detailed description of how a short sale works, refer to [http://webcomposer.pace.edu/pviswanath/notes/investments/shortsal.html#logistics](http://webcomposer.pace.edu/pviswanath/notes/investments/shortsal.html#logistics)

**Sample problems**

**Problem 1** (SOA Course 6 #3 May 2001)

(a) With respect to short sales of a security:
   (1) Describe the process for executing a short sale
   (2) Outline an investor’s motivation for executing such a transaction

(b) You are given the following:

<table>
<thead>
<tr>
<th>Date</th>
<th>Corporate Z Share Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 1, 2001</td>
<td>60</td>
</tr>
<tr>
<td>January 15, 2001</td>
<td>63</td>
</tr>
</tbody>
</table>
Corporation Z paid a dividend of 1 on January 15, 2001
The maximum price of Corporation Z shares during the month of January 2001 was 63

On January 1, 2001, Investor A sold short 100 shares of Corporation Z
On January 31, 2001, Investor A covered the short position

The initial margin requirement was 50%
The maintenance margin requirement was 40%

There were no other transaction costs
No interest was earned on the balance with the broker

(3) Outline the transaction on January 1, 2001
(4) Outline the transaction on January 31, 2001
(5) Determine whether a margin call was necessary

Show all the work.

Solution

Download the official solution from the SOA website.

Please note some of the required concepts in FM are from Course 6.

Problem 2 (SOA Course 6 #3 May 2003)

You are given the following information:

- margin requirement on short sales: 50%
- maintenance margin: 30%
- an investor’s account with a broker currently holds:
  - value of T-bills: 10,000
  - number of shares of XYZ stocks: 500
- stock prices:

<table>
<thead>
<tr>
<th>Date</th>
<th>ABC Stock Price</th>
<th>XYZ Stock Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>June 2, 2003</td>
<td>103</td>
<td>75</td>
</tr>
<tr>
<td>June 3, 2003</td>
<td>102</td>
<td>76</td>
</tr>
<tr>
<td>June 4, 2003</td>
<td>99</td>
<td>77</td>
</tr>
</tbody>
</table>
The investor tells the broker to short 1,000 shares of the ABC stock on June 3, 2003. The broker executes the order on the first day allowed. Shares are traded once per day.

(a) Calculate the additional cash (if any) necessary to satisfy the margin requirement.

(b) Calculate the amount of the margin calls (if any) between June 3, 2003 and June 10.

Show all the work.

Solution

Download the official solution from the SOA website.

Problem 3

You short sell 100 shares of Company XYZ, which currently sell for $50 per share. The initial margin requirement is 40%.

1. What’s your rate of return if one year later Company XYZ’s stocks sell $55, $50, or $45 per share? Assume you earn no interest on your margin account. Also assume that Company XYZ doesn’t pay dividend.

2. If the maintenance margin is 30%, how high can XYZ’s stock price go up before you get a margin call?

3. Redo the above two questions assuming that XYZ pays dividend of $1 per share at the end of year 1.

Solution

1. To satisfy the margin requirement, you must deposit the following amount of cash into the margin account at \( t = 0 \):

\[
$50(100)(40%) = $2,000
\]

Your rate of return is:
\[ r = \frac{\text{profit}}{\text{Initial deposit}} = \frac{100(50 - P)}{2,000}, \text{ where } P \text{ is the share price at } t=1 \]

Your returns are:

- If \( P = 55 \), \( r = \frac{100(50 - 55)}{2,000} = -25\% \)
- If \( P = 50 \), \( r = \frac{100(50 - 50)}{2,000} = 0\% \)
- If \( P = 45 \), \( r = \frac{100(50 - 45)}{2,000} = 25\% \)

2. Your asset consists of:
   - proceeds from short sales: \( 100(50) = $5,000 \)
   - initial deposit into the margin account: $2,000

Your total asset is $7,000

Your liability is to return 100 shares of stocks: $100P

Your equity is: \( 7,000 - 100P \)

You get a margin call if the ratio of your equity to the then market value of the stocks is less than the maintenance margin:

\[ \frac{7000 - 100P}{100P} < 30\% \quad \Rightarrow \quad P > \frac{70}{1.3} = $53.85 \]

3. Now you have to pay dividend to your broker. The dividend amount is \( 100($1\text{ per share}) = $100 \). Now your return is

\[ r = \frac{\text{profit}}{\text{Initial deposit}} = \frac{100(50 - P) - 100}{2,000} \]

Your returns are:

- If \( P = 55 \), \( r = \frac{100(50 - 55) - 100}{2,000} = -30\% \)
- If \( P = 50 \), \( r = \frac{100(50 - 50) - 100}{2,000} = -5\% \)
If \( P = 45 \), \( r = \frac{100(50 - 45) - 100}{2000} = 20\% \)

Your asset consists of:
- proceeds from short sales: \( 100(50) = $5,000 \)
- initial deposit into the margin account: \$2,000 \)
Total asset: \$7,000 \)

Your liability:
- To return 100 shares of stocks: \$100P \)
- To pay the dividend: \$100 \)
Total liability: \$100P + 100 \)

Your equity is: \( 7,000 - 100P - 100 \)

You get a margin call if the ratio of your equity to the market value of the stocks sold is less than the maintenance margin:

\[ \frac{7000 - 100P - 100}{100P} < 30\% , \quad \Rightarrow \quad P = \frac{69}{1.3} = $53.08 \]

**Problem 4  (#17, Nov 2005 FM)**

Theo sells a stock short with a current price of 25,000 and buys it back for \( X \) at the end of 1 year. Governmental regulations require the short seller to deposit margin of 40% at the time of the short sale. The prevailing interest rate is an 8% annual rate, and Theo earns a 25% yield on the transaction.

Calculate \( X \)

**Solution**

At \( t = 0 \), Theo deposited \( 25,000(40\%) = 10,000 \) into the margin account. At \( t = 1 \) when the short sale is closed out, Theo’s wealth is

\[ FV \text{ of the initial deposit + gain from the short sale - dividend paid} = 10,000(1.08) + (25,000 - X) - 0 \]

Theo earned 25% yield on the transaction:
10,000(1.08) + 25,000 − X = 10,000(1.25), \Rightarrow X = 23,300

**Problem 5  (#36, May 2003 Course 2) (also Sample FM #38)**

(SOA removed Sample #38 through Sample #44 from the Sample FM Questions. As a result, you can skip the following questions. I included these questions for completeness.)

Eric and Jason each sell a different stock short at the beginning of the year for a price of 800. The margin requirement for each investor is 50% and each will earn an annual effective interest rate of 8% on his margin account.

Each stock pays a dividend of 16 at the end of the year. Immediately thereafter, Eric buys back his stock at a price of $800 − 2X$, and Jason buys back his stock at a price of $800 + X$.

Eric’s annual effective yield, $i$, on the short sale is twice Jason’s annual effective yield.

Calculate $i$.

**Solution**

At $t = 0$, Eric deposits 800*50%=400.

At $t = 1$, Eric’s wealth is:

\[
FV \text{ of the initial deposit } + \text{ gain from the short sale } - \text{ dividend paid} \\
= 400(1.08) + \left[800 - (800 - 2X)\right] - 16 \\
= 400(1.08) + 2X - 16
\]

Eric’s annual effective yield from the short sale is $i$. Then

\[
400(1+i) = 400(1.08) + 2X - 16, \quad i = \frac{2X + 16}{400}
\]

At $t = 0$, Jason deposits 800*50%=400.

At $t = 1$, Jason’s wealth is:

\[
FV \text{ of the initial deposit } + \text{ gain from the short sale } - \text{ dividend paid} \\
= 400(1.08) + \left[800 - (800 + X)\right] - 16 \\
= 400(1.08) - X - 16
\]

Jason’s annual effective yield from the short sale is $j$. Then

\[
400(1+j) = 400(1.08) - X - 16, \quad j = \frac{16 - X}{400}
\]

We are told that $i = 2j$: 
\[
\frac{2X + 16}{400} = 2\left(\frac{16 - X}{400}\right), \quad X = 4
\]

\[
i = \frac{2X + 16}{400} = \frac{2(4)+16}{400} = 6\%
\]

**Sample FM Problem #39**

Jose and Chris each sell a different stock short for the same price. For each investor, the margin requirement is 50% and interest on the margin debt is paid at an annual effective rate of 6%. Each investor buys back his stock one year later at a price of 760. Jose’s stock paid a dividend of 32 at the end of the year while Chris’s stock paid no dividends. During the 1-year period, Chris’s return on the short sale is \(i\), which is twice the return earned by Jose. Calculate \(i\).

**Solution**

Let \(X\) represent the price of the stock when the short sale takes place (i.e. at \(t=0\)).

At \(t=0\), Jose deposits \(0.5X\).

At \(t=1\), Jose’s wealth is:

FV of the initial deposit + gain from the short sale – dividend paid

\[
= 0.5X (1.06) + (X - 760) - 32 = 1.53X - 792
\]

Jose’s annual effective yield from the short sale is \(i\). Then

\[
0.5X (1 + j) = 1.53X - 792, \quad j = \frac{1.53X - 792}{0.5X} - 1
\]

At \(t=0\), Chris deposits \(0.5X\).

At \(t=1\), Chris’s wealth is:

FV of the initial deposit + gain from the short sale – dividend paid

\[
= 0.5X (1.06) + (X - 760) = 1.53X - 760
\]

Chris’s annual effective yield from the short sale is \(i\). Then

\[
0.5X (1 + j) = 1.53X - 760, \quad i = \frac{1.53X - 760}{0.5X} - 1
\]

We are told that \(i = 2j\):

\[
2\left(\frac{1.53X - 792}{0.5X} - 1\right) = \frac{1.53X - 760}{0.5X} - 1, \quad X = 800
\]
Sample FM Problem #40
Bill and Jane each sell a different stock short for a price of 1000. For both investors, the margin requirement is 50%, and interest on the margin is credited at an annual effective rate of 6%. Bill buys back his stock one year later at a price of $P$. At the end of the year, the stock paid a dividend of $X$. Jane also buys back her stock after one year, at a price of $(P - 25)$. At the end of the year, her stock paid a dividend of $2X$. Both investors earned an annual effective yield of 21% on their short sales. Calculate $P$.

Solution

At $t = 0$, Bill deposits $1000 \times 50\% = 500$.
At $t = 1$, Bill’s wealth is:
\[
\text{FV of the initial deposit + gain from the short sale – dividend paid} = 500(1.06) + (1000 - P) - X = 1530 - P - X
\]

Bill’s annual effective yield from the short sale is 21%. Then
\[
500(1 + 21\%) = 1530 - P - X
\]

At $t = 0$, Jane deposits $1000 \times 50\% = 500$.
At $t = 1$, Jane’s wealth is:
\[
\text{FV of the initial deposit + gain from the short sale – dividend paid} = 500(1.06) + [1000 - (P - 25)] - 2X = 1555 - P - 2X
\]

Jane’s annual effective yield from the short sale is 21%. Then
\[
500(1 + 21\%) = 1555 - P - 2X
\]

\[
1530 - P - X = 1555 - P - 2X, \quad X = 25
\]

\[
500(1 + 21\%) = 1530 - P - 25, \quad P = 900
\]
Chapter 15  Term structure of interest rate, spot rate, forward rate, and arbitrage

Key points:

1. Law of one price (no arbitrage principle)
   - Two bonds (or other securities) with identical cash flows should sell for an identical price.
   - If they don’t sell for the same price (i.e. if one bond sells at a higher price than another bond with identical cash flows), anyone can make money by buying the lower priced bond, turning around, and selling it at a higher price.
   - The strategy of exploiting loopholes to make money is called arbitrage.
   - Characteristics of arbitrage:
     (1) Profit is made with 100% certainty
     (2) Profit is made with zero cost
     (3) Profit is made with zero risks taken
   - No arbitrage principle assumes there are no transaction costs such as tax and commissions.

2. Term structure of interest rates
   - Term structure of interest refers to the phenomenon that a bond’s yield-to-maturity changes as the bond’s maturity changes.
   - A hypothetical example

<table>
<thead>
<tr>
<th>Maturity of a bond</th>
<th>Yield to maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>7%</td>
</tr>
<tr>
<td>2 year</td>
<td>8%</td>
</tr>
<tr>
<td>3 years</td>
<td>8.75%</td>
</tr>
<tr>
<td>4 years</td>
<td>9.25%</td>
</tr>
<tr>
<td>5 years</td>
<td>9.5%</td>
</tr>
</tbody>
</table>
• Term structure of interest is also called the yield curve, which is a 2-D graph showing how the yield to maturity \( (Y) \) axis is a function of the bond’s maturity \( (X) \) axis.

• Different theories explain why a longer-term bond has a higher or lower yield-to-maturity than does a short-term bond. (No need to learn those theories)

• Because a traded bond’s yield of maturity is often used as the market’s fair discount rate (the rate used to discount future cash flows), the discount rate is a function of the length of the investment.

3. Spot rate.

• Spot rate is the return you can lock in at time zero. It’s the return you can get by lending (or depositing) your money at \( t = 0 \) and having your money repaid in a future time \( t_1 \).

\[
\text{Sign contract; Lend money} \quad \quad \text{Loan repaid}
\]

\[
\begin{array}{c|c|c}
\text{Time} & 0 & \text{spot rate} \to t_1 \\
\end{array}
\]

• For example, at \( t = 0 \) you bought a two year zero-coupon bond with $100 face amount for a price of $85. In this transaction, you lent $85 at \( t = 0 \). Your money is to be repaid at \( t = 2 \). The return you have locked in at \( t = 0 \) can be solved as follows:

\[
85(1+r)^2 = 100 \quad \Rightarrow \quad r = 8.465\%
\]

So at \( t = 0 \) you locked in an annual return of 8.465% for two years. \( r = 8.465\% \) is the 2-year spot rate.

• The word “spot” comes from the phrase “the spot market.” The spot market (or called cash market) is where the seller immediately delivers the product to the buyer on the spot. Example. You pay $10 to a farmer and he immediately gives you 15 tomatoes on the spot. Remember, spot market = immediate delivery. (In the real world, however, many spot transactions are completed within a few hours or a couple of days.)
The phrase “spot rate” is used because at time zero when you buy the investment, the invest vehicle (a bond, a CD, or any other investment opportunity) is immediately delivered to you at time zero.

A spot rate answers this question, “If a one buys a bond now at time zero from a market, what return can he get?”

Spot rate is often denoted as $s_t$. This is the return you can get if you invest your money at time zero for $t$ years.

4. Forward rate

At time zero, you sign an investment contract, which requires you to lend your money at a future time $t_1$. Your money is to be repaid in another future time $t_2$ where $t_2 > t_1$. The return you earn by lending your money from $t_1$ to $t_2$ is a forward rate.

![Time Chart]

Sign contract | Lend money | Loan repaid
--- | --- | ---
| | | |
Time 0 | $t_1$ | ← forward rate → $t_2$

The word “forward” comes from the forward market. In a forward market, an agreement is made at time zero but the delivery date is at a future time $t_1$. Example. You pay $10 to a farmer today for him to deliver 15 tomatoes to you in 3 months. You do so perhaps you really use tomatoes and you worry that tomato price may go up in the future. Most likely, people who buy forward products are not directly consumers, but are profit makers. They hope to resell a forward contract at a higher price.

Forward market = future delivery.

A forward rate answers the question, “If a $n$-year term bond (or another investment vehicle) is delivered at $t_1$ (where $t_1 > 0$) to an investor, what return will the investor get for lending money from $t_1$ to $t_1 + n$? ” ($n$ is not necessarily an integer)

Forward rate is often denoted as $f_{jk}$. It’s the interest rate charged for lending money from $j$ to $j+k$. 

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Guo FM, fall 2009

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5. Relationship between spot rates and forward rates (assuming no arbitrage)

| ← | ← | ← | ← | → |
| ← | ← | ← | → |
| ← | ← | → |
| ← | ← | ← | ← | → |

\[
\begin{align*}
1 + s_1 &= 1 + f_{0,1} \\
(1 + s_1)^2 &= (1 + s_1)(1 + f_{1,1}) = (1 + f_{0,1})(1 + f_{1,1}) \\
(1 + s_2)^3 &= (1 + s_2)^2(1 + f_{2,1}) = (1 + f_{0,1})(1 + f_{1,1})(1 + f_{2,1}) \\
(1 + s_3)^4 &= (1 + s_3)^3(1 + f_{3,1}) = (1 + f_{0,1})(1 + f_{1,1})(1 + f_{2,1})(1 + f_{3,1}) \\
\cdots \\
(1 + s_n)^n &= (1 + s_{n-1})^{n-1}(1 + f_{n-1,1}) = (1 + f_{0,1})(1 + f_{1,1})(1 + f_{2,1}) \cdots (1 + f_{n-1,1})
\end{align*}
\]

- Example. \( s_1 = 13.4\% \), \( s_2 = 15.51\% \). What's \( f_{1,1} \)?

You have two options:

1. Invest $1 at time zero. Pull your money out at the end of Year 2. You should reap an annual return of 15.51%.

2. Invest $1 at time zero for one year and pull your money ($1.134) out at the end of Year 1, reaping 13.4% return. Then reinvest your $1.134 during Year 2, reaping a return of \( f_{1,1} \) during Year 2.

The above two options should produce identical wealth. If not, loopholes exist for people to exploit.
Sample problems

Problem 1

You are given the following three situations:

| #1 | You earned $1,000,000 in a gambling game at Las Vegas last week. |
| #2 | The exchange rate at New York is $2 = £1 |
|    | The exchange rate at London is £2 = $6. |
| #3 | Programmers in a software company were working around the clock to finish building a new software package. To boost the morale of his over-worked programmers, the CEO of the company set an “$50 per bug” incentive plan. Under this plan, a programmer was to be awarded $50 for each bug he found in his codes. |

Explain which situations represent arbitraging opportunities.

Solution

#1 is NOT an arbitrage. In an arbitrage, one makes money with zero cost; sure profits are made without any risks taken. In gambling, however, a gambler takes lot of risks, yet the profit is not certain.

#2 opens to arbitrage. For example, you can change $2 into £1 at New York. Immediately, you change £1 back to $3 at London. You can do these two transactions over the internet and earn $1 profit with zero cost and zero risk.

#3 opens to arbitrage. This is a not fake story. A software company actually did something like this. The CEO had a good intention to
encourage his programmers to quickly discover and fix bugs. However, after the incentive policy was implemented, some programmers purposely added bugs into their codes so they would qualify for more rewards. Soon the CEO found out about the loophole and cancelled the incentive plan.

**Problem 2**

Consider two bonds $A$ and $B$ in a financial market. $A$ is a one-year bond with 10% annual coupon and $100$ par value; $A$ sells for $97$. $B$ is a two-year bond with 8% annual coupon and $100$ par value bond; $B$ sells for $88$.

Calculate $s_1$ and $s_2$, the 1-year spot and 2-year spot rate.

**Solution**

$s_1$ is the spot rate for one year, from $t=0$ to $t=1$. This is the return for investing money for one year. The cash flows at the end of Year 1 should use $s_1$ as the discount rate.

$s_2$ is the spot rate for two years, from $t=0$ to $t=2$. This is the return for investing money for two years. The cash flows at the end of Year 2 should use $s_2$ as the discount rate.

We need to solve the following two equations:

\[
\begin{align*}
97 &= \frac{110}{1+s_1} \\
88 &= \frac{8}{1+s_1} + \frac{108}{(1+s_2)^2}
\end{align*}
\]

$\Rightarrow s_1 = 13.4\%, \ s_2 = 15.51\%$
Problem 3

3 months from now (i.e. at $t = 0.25$), you need to borrow $5,000 for six months to fund your six-month long vacation. Worrying that the interest rate may go up a lot in three months, you want to lock in a borrowing rate right now at $t = 0$.

You are given the following facts:

- The 3-month spot rate is 4.5% annual effective.
- The 9-month spot rate is 5.7% annual effective.

Explain how you can lock in, at $t = 0$, a $5,000 loan for six months with a guaranteed interest rate. Calculate the guaranteed borrowing rate.

Solution

Lock-in strategy:

At $t = 0$, borrow $5,000 \times (1 + 4.5\%)^{0.25} = 4,945.28$ for 9-months @ 5.7% annual effective. And immediately deposit the borrowed $4,945.28 for 3 months @ 4.5% annual effective.

At $t = \frac{3}{12} = 0.25$ (3 months later), your $4,945.28 will grow into:

$$4,945.28 \times (1 + 4.5\%)^{0.25} = 5,000$$

You will use this $5,000 to fund your 6-month long vacation. Effectively, you have borrowed $5,000 at $t = 0.25$.

At $t = \frac{9}{12} = 0.75$ (9 months later), your vacation is over. Your loan of $4,945.28 is due. You pay off this loan with a following payment at $t = 0.75$:

$$4,945.28 \times (1 + 5.7\%)^{0.75} = 5,196.92$$

The interest rate you locked from $t = 0.25$ to $t = 0.75$ can be solved as follows:

$$4,945.28 \times (1 + 4.5\%)^{0.25} \times (1 + r)^{0.5} = 4,945.28 \times (1 + 5.7\%)^{0.75}$$

$$\Rightarrow \quad (1 + r)^{0.5} = (1 + 5.7\%)^{0.75} \times (1 + 4.5\%)^{-0.25} = 1.031044 \Rightarrow \quad r = 6.3052\%$$
The rate you locked in, $r = 6.3052\%$, is a 6-month forward rate @ $t = 0.25$.

**Problem 4**

3 months from now (i.e. at $t = 0.25$), an insurance company will send you $5,000 cash. You want to invest $5,000 immediately after you receive it and invest it for 6 months. Worrying that the interest rate may go down a lot in three months, you want to lock in an interest rate right now at $t = 0$.

You are given the following facts:
- The 3-month spot rate is 4.5% annual effective.
- The 9-month spot rate is 5.7% annual effective.

Explain how you can lock in, at $t = 0$, an investment opportunity where you can lend your future $5,000 for six months with a guaranteed interest rate. Calculate the guaranteed earning rate.

**Solution**

This problem is similar to Problem 3. The only difference is that this time we want to lend money and earn a guaranteed interest rate.

Lock-in strategy:

At $t = 0$, borrow $5,000(1 + 4.5\%)^{0.25} = 4,945.28$ for 3 months @ 4.5% annual effective. And immediately deposit the borrowed $4,945.28 for 9-months @ 5.7% annual effective.

At $t = \frac{3}{12} = 0.25$ (3 months later), your borrowed amount of $4,945.28 will grow into:

$$4,945.28(1 + 4.5\%)^{0.25} = 5,000$$
At $t = 0.25$, you can pay this loan amount using the payment of $5,000 sent to you by the insurance company. As a result, your loan is paid off. At $t = 0.75$ (9 months later), your original deposit of $4,945.28 will grow into:

$$4,945.28(1 + 5.7\%)^{0.75} = 5,196.92$$

The interest rate you locked from $t = 0.25$ to $t = 0.75$ can be solved as follows:

$$4,945.28(1 + 4.5\%)^{0.25}(1 + r)^{0.5} = 4,945.28(1 + 5.7\%)^{0.75}$$

$$\Rightarrow (1 + r)^{0.5} = (1 + 5.7\%)^{0.75}(1 + 4.5\%)^{-0.25} = 1.031044$$

$$\Rightarrow r = 6.3052\%$$

<table>
<thead>
<tr>
<th>Time t (year)</th>
<th>0</th>
<th>0.25</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>← $(1 + 4.5%)^{0.25}$ →</td>
<td>← $(1 + r)^{0.5}$ →</td>
<td>← $(1 + 5.7%)^{0.75}$ →</td>
</tr>
</tbody>
</table>

The rate you locked in, $r = 6.3052\%$, is a 6-month forward rate @ $t = 0.25$.

**Problem 5**

You are given the following force of interest:

$$\delta(t) = 5\% + 0.12\%t$$

Calculate

- $s_5$, the 5 year spot rate.
- $s_6$, the 6 year spot rate
- $f_{5,1}$, the one year forward rate from $t = 5$ to $t = 6$

**Solution**
\( s_5 \) is the annual return you can lock in for investing money at time zero for 5 years.

\[
(1 + s_5)^5 = \exp \left[ \int_0^5 \delta(t) \, dt \right]
\]

\[
\int_0^t \delta(x) \, dx = \int_0^t (5\% + 0.12\% x) \, dx = 5\% t + 0.06\% t^2
\]

\[
\Rightarrow (1 + s_5)^5 = \exp \left[ \int_0^5 \delta(t) \, dt \right] = \exp \left[ 5\% t + 0.06\% t^2 \right]_{t=5} = e^{26.5\%} = 1.30343098
\]

\[
\Rightarrow s_5 = 5.44296\%
\]

Similarly,

\[
(1 + s_6)^6 = \exp \left[ \int_0^6 \delta(t) \, dt \right] = \exp \left[ 5\% t + 0.06\% t^2 \right]_{t=6} = e^{32.16\%} = 1.37933293
\]

\[
\Rightarrow s_6 = 5.50625\%
\]

To find \( f_{5,1} \), we use the no arbitrage principle. Compare the following two options:

- At time zero, lock in a 6 year investment and get \( s_6 = 5.50625\% \) annual return.

- At time zero, lock in a 5 year investment and get \( s_5 = 5.44296\% \) annual return. Then reinvest the total money starting from the end of Year 5 and ending at the end of Year 1 (i.e. during Year 6), reaping an annual return of \( f_{5,1} \)

These two options should generate the same amount of wealth.
Problem 6

You are given the following facts about three securities A, B, and C:

<table>
<thead>
<tr>
<th>Security</th>
<th>Selling price at t=0</th>
<th>Cash flow at t=1</th>
<th>Cash flow at t=2</th>
<th>Cash flow beyond t=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$7,570.03</td>
<td>$5,000</td>
<td>$4,000</td>
<td>$0</td>
</tr>
<tr>
<td>B</td>
<td>$16,274.27</td>
<td>$10,000</td>
<td>$9,500</td>
<td>$0</td>
</tr>
<tr>
<td>C</td>
<td>$0</td>
<td>$10,000</td>
<td>$0</td>
<td>$0</td>
</tr>
</tbody>
</table>

Calculate
(1) The price of Security C at time zero assuming no arbitrage.

(2) If Security C sells for $7,200 at time zero, design an arbitrage strategy and calculate how much profit you can make.

(3) If Security C sells for $8,000 at time zero, design an arbitrage strategy and calculate how much profit you can make.

Solution

Calculate the price of Security C at time zero assuming no arbitrage.

Let \( s_1 \) represent the one year spot rate.
Let \( s_2 \) represent the two year spot rate.
\[ 7,570.03 = \frac{5,000}{1+s_1} + \frac{4,000}{(1+s_2)^2} \quad 16,274.27 = \frac{10,000}{1+s_1} + \frac{9,500}{(1+s_2)^2} \]

Set \( \frac{1}{1+s_1} = x \) and \( \frac{1}{(1+s_2)^2} = y \).

\[ 7,570.03 = 5,000x + 4,000y \quad 16,274.27 = 10,000x + 9,500y \]

\[ \Rightarrow \ x = 1.1^{-1}, \quad y = 1.15^{-2} \]

\[ \Rightarrow \ \text{The price of Security C at time zero if no arbitrage:} \]

\[ \frac{10,000}{(1+s_2)^2} = 10,000 (1.15^{-2}) = 7,561.44 \]

**If Security C sells for $7,200 at time zero, design an arbitrage strategy and calculate how much profit you can make.**

If Security C sells for $7,200 at time zero, then this price is below its market fair price of $7,561.44, creating an arbitrage opportunity.

To exploit this opportunity, we first synthetically create a 2-year zero-coupon bond with $10,000 par. This bond is \((B-2A)\).

<table>
<thead>
<tr>
<th>Security</th>
<th>Selling price at t=0</th>
<th>Cash flow at t=1</th>
<th>Cash flow at t=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$7,570.03</td>
<td>$5,000</td>
<td>$4,000</td>
</tr>
<tr>
<td>B</td>
<td>$16,274.27</td>
<td>$10,000</td>
<td>$9,500</td>
</tr>
<tr>
<td>B-2A</td>
<td>$16,274.27-2(7,570.03) = $1,134.21</td>
<td>$10,000-2(5,000) = $0</td>
<td>$9,500-2(4,000) = $1,500</td>
</tr>
<tr>
<td>((\frac{2}{3})(B-2A))</td>
<td>((\frac{2}{3}))1,134.21 = $7,561.40</td>
<td>(\frac{2}{3})$0</td>
<td>(\frac{2}{3})$1,500 = $10,000</td>
</tr>
</tbody>
</table>

Based on the above table, a 2-year zero-coupon bond with $10,000 par can be created by buying (or selling short) \(\frac{2}{3}\) units of Security B’s and simultaneously short selling (or buying) \(\frac{2}{3}\times2 = 13\frac{1}{3}\) units of Security A’s.

This synthetically created 2-year zero coupon $10,000 par bond is worth $7,561.40 at time zero.
Our arbitrage strategy if Security $C$ sells for $7,200 at time zero:

- The arbitrage strategy is always “buy low, sell high.”
- At time zero, we spend $7,200 and buy Security $C$. This is buy low.
- Simultaneously at time zero, we short sell the synthetically created security $\left( \frac{2}{3} \right)(B - 2A)$, which has the same cash flow as Security $C$ but is worthy $7,561.4$, $361.4$ more than Security $C$. This is sell high.
- Short selling $\left( \frac{2}{3} \right)(B - 2A)$ means short selling $\left( \frac{2}{3} \right)B$ and immediately buying $\left( \frac{2}{3} \right)A$ from the market.
- At $t = 2$, we get $10,000 from Security $C$. We'll use this $10,000 to pay security $\left( \frac{2}{3} \right)(B - 2A)$’s cash flow at $t = 2$, closing our short position.
- The net result: at time zero, we earn $7,561.40 - $7,200 = $361.40 sure profit per transaction above. Of course, if we can make 1,000 such transactions in a day, we’ll make $361,400 in a day.

**If Security $C$ sells for $8,000 at time zero, design an arbitrage strategy and calculate how much profit you can make**

- At time zero, we sell short Security $C$. We earn $8,000 cash.
- Simultaneously at time zero, we buy the synthetically created security $\left( \frac{2}{3} \right)(B - 2A)$, which means buying $\left( \frac{2}{3} \right)B$ and short selling $\left( \frac{2}{3} \right)(2A)$.
- At $t = 2$, we get $10,000 from the synthetically created security $\left( \frac{2}{3} \right)(B - 2A)$. We’ll use this $10,000 to pay Security $C$’s cash flow at $t = 2$ and close out our short position.
The net result: at time zero, we earn $8,000 - $7,561.40 = $438.60 sure profit per transaction above. Of course, if we make 1,000 such transactions in a day, we'll make $438,600 in a day.

Problem 7

You are given the following spot rates:

\[ s_t = 5\% + \frac{t\%}{5}, \text{ where } t = 1, 2, 3 \]

Calculate \( f_{1,1} \) and \( f_{2,1} \), the one year forward rates at \( t = 1 \) and at \( t = 2 \).

Solution

\[
(1 + s_1)(1 + f_{1,1}) = (1 + s_2)^2
\]

\[
(1 + s_2)^2 (1 + f_{2,1}) = (1 + s_3)^3
\]

Let's look at the meaning of these two equations.

The 1st equation says that if you invest your money year by year, you should have the same wealth at \( t = 2 \) as you initially lock in a 2-year investment opportunity. Otherwise, an arbitrage opportunity exits.

```
Time t     0     1     2

← 1+s₁  → ← 1+f₁₁  →

← (1+s₂)^2  →
```

Similarly, the 2nd equation compares two investment options.

- **Option 1.** At time zero, investment $1 and lock in a two year investment opportunity and accumulate to \( (1+s_2)^2 \) at \( t = 2 \). Next, immediately reinvest your wealth of \( (1+s_2)^2 \) in a one year investment opportunity and earn a forward rate \( f_{2,1} \) during Year 2. Option 1 accumulates a total of \( (1+s_2)^2 (1+f_{2,1}) \) at \( t = 3 \).
• **Option 2.** At time zero, investment $1 in a 3 year investment opportunity, locking in an annual return of $s_3$. This accumulates a total wealth of $(1 + s_3)^3$ at $t = 3$.

• No arbitrage principle requires that the two options generate an identical amount of wealth at $t = 3$.

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>← $(1 + s_2)^2$ → $1 + f_{2,1}$ →</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>← $(1 + s_3)^3$ →</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Once you understand the meaning of these two equations, the remaining work is purely algebra.

$s_1 = 5\% + \frac{t \%}{5}$

$\Rightarrow s_1 = 5\% + \frac{1%}{5} = 5.2\%$

$\Rightarrow s_2 = 5\% + \frac{2%}{5} = 5.4\%$

$\Rightarrow s_2 = 5\% + \frac{3%}{5} = 5.6\%$

$f_{1,1} = \frac{(1 + s_2)^2}{1 + s_1} - 1 = \frac{(1 + 5.4\%)^2}{1 + 5.2\%} - 1 \approx 5.60\%$

$f_{2,1} = \frac{(1 + s_3)^3}{(1 + s_2)^2} - 1 = \frac{(1 + 5.6\%)^3}{(1 + 5.4\%)^2} - 1 \approx 6.40\%$

**Problem 8**

You are given the following information with respect to a bond:

- par amount: $1,000
- term to maturity: 3 years
- annual coupon rate 8% payable annually
- 1-year continuous spot rate is 5%
- 1-year forward rate @ $t=1$ is 6%
- 1-year forward rate @ $t=2$ is 7%
Calculate YTM (yield to maturity) of the bond.

**Solution**

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$80</td>
<td>$80</td>
<td>$1,080</td>
<td></td>
</tr>
</tbody>
</table>

First, we need to calculate the PV of the bond. PV of the bond is sum of each cash flow discounted at an appropriate spot rate.

\[
PV = \sum_{t=1}^{3} \frac{CF(t)}{(1 + s_t)^t} = \frac{80}{1 + s_1} + \frac{80}{(1 + s_2)^2} + \frac{1,080}{(1 + s_3)^3}
\]

So we need to calculate the 1-year, 2-year, and 3-year spot rates.

We are given: 

\[1 + s_1 = e^{0.05}\]

Please note that we are given a continuous spot rate \(\delta = 5\%\) in the 1st year.

We are also given: 

\[f_{1,1} = 6\%,\quad f_{2,1} = 7\%\]

Using the relationship between spot rates and forward rates, we have:

\[
(1 + s_2)^2 = (1 + s_1)(1 + f_{1,1}) = e^{0.05}(1.06)
\]

\[
(1 + s_3)^3 = (1 + s_1)(1 + f_{1,1})(1 + f_{2,1}) = e^{0.05}(1.06)(1.07)
\]

\[
\Rightarrow PV = \frac{80}{e^{0.05}} + \frac{80}{e^{0.05}(1.06)} + \frac{1,080}{e^{0.05}(1.06)(1.07)} = 1,053.66
\]

YTM can be solved in the following equation:

\[
1,053.66 = \frac{80}{1 + y} + \frac{80}{(1 + y)^2} + \frac{1,080}{(1 + y)^3}
\]
To solve this equation, use BA II Plus/BA II Plus Professional.

Enter PMT=80, N=3, FV=1,000, PV=-1,053.66. Let the calculator solve for I/Y.

We should get: I/Y=5.99279%

So the yield to maturity is 5.99279%.

**Problem 9**

Short term, one-year annual effective interest rates are currently 10%; they are expected to be 9% one year from now, 8% two years from now, 7% three years from now, and 6% four years from now.

Calculate

- The spot yield of 1-year, 2-year, 3-year, 4-year, and 5-year zero coupon bonds respectively.

- The annual effective yield of a bond redeemed at $100 par value in 5 years and pays 8% coupon annually.

**Solution**

First, let’s draw a diagram:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$8</td>
<td>$8</td>
<td>$8</td>
<td>$8</td>
<td>$8</td>
<td>$108</td>
</tr>
<tr>
<td>$1+s_1$</td>
<td>$1+f_{1,1}$</td>
<td>$1+f_{2,1}$</td>
<td>$1+f_{3,1}$</td>
<td>$1+f_{4,1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>9%</td>
<td>8%</td>
<td>7%</td>
<td>6%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(1+s_2)^2$</td>
<td>$(1+s_3)^3$</td>
<td>$(1+s_4)^4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Let $s_t$ represent the spot yield over $t$ years.

$s_1 = 10\%

(1 + s_2)^2 = 1.1 \times 1.09 \Rightarrow s_2 = 9.499\%

(1 + s_3)^3 = 1.1 \times 1.09 \times 1.08 \Rightarrow s_3 = 8.997\%

(1 + s_4)^4 = 1.1 \times 1.09 \times 1.08 \times 1.07 \Rightarrow s_4 = 8.494\%

(1 + s_5)^5 = 1.1 \times 1.09 \times 1.08 \times 1.07 \times 1.06 \Rightarrow s_5 = 7.991\%

Next, let's calculate the annual effective yield of a bond redeemed at par value of $100 in 5 years and pays 6% coupon annually.

The PV of the bond is:

\[ \sum_{t=1}^{5} CF(t)(1+s_t)^{-t} = \frac{8}{1 + 10\%} + \frac{8}{(1 + 9.499\%)^2} + \frac{8}{(1 + 8.997\%)^3} + \frac{8}{(1 + 8.494\%)^4} + \frac{108}{(1 + 7.991\%)^5} = 99.43227 \]

Next, we need to solve the equation:

\[ 99.43227 = 8 a_{3\bar{m}} + 100 v^5 \]

In BA II Plus TVM, Enter PV= -99.43227, PMT = 8, FV =100, N=5.

Press "CPT" "I/Y." You should get: I/Y=8.143%

So the annual effective yield is 8.143%.
Problem 10  (November 2005 FM #6)
Consider a yield curved defined by the following equation

\[ i_k = 0.09 + 0.002k - 0.001k^2 \]

here \( i_k \) is the annual effective rate of return of zero coupon bonds with maturity of \( k \) years.

Let \( j \) be the one-year effective rate during year 5 that is implied by this yield curve.

Calculate \( j \).

Solution
First, let's draw a diagram:

\[
\begin{array}{c|c|c|c|c|c|c}
\text{Time } t & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\leftarrow & 1+i_1 & \rightarrow & 1+f_{1,1} & \rightarrow & 1+f_{2,1} & \rightarrow \\
\leftarrow & (1+i_2)^2 & \rightarrow \\
\leftarrow & (1+i_3)^3 & \rightarrow \\
\leftarrow & (1+i_4)^4 & \rightarrow \\
\leftarrow & (1+i_5)^5 & \rightarrow \\
\end{array}
\]

We are asked to find \( j = f_{4,1} \). Please note that Year 5 is from \( t=4 \) to \( t=5 \),

\[
(1+i_4)^4 (1+f_{4,1}) = (1+i_5)^5
\]
\[ j = f_{1,1} = \frac{(1+i_5)^5}{(1+i_4)} - 1 = \frac{\left[1 + 0.09 + 0.002(5) - 0.001(5)^2\right]^5}{\left[1 + 0.09 + 0.002(4) - 0.001(4)^2\right]^4} - 1 = \frac{1.075^5}{1.082^4} - 1 = 4.74\% \]

**Problem 11**

The one-year spot interest rate at \( t = 0 \) is 6% annual effective.

The annual effective yield of a two year bond issued at \( t = 0 \) that pays 4% annual coupons and that is redeemed at par value of $100 is 7% annual effective.

The issue price at \( t = 0 \) of a three-year bond that pays 8% coupons annually is $102 per $100 nominal.

Calculate

- \( f_{1,1} \), the one-year spot rate at \( t = 1 \)
- \( f_{2,1} \), the one-year forward rate at \( t = 2 \).

**Solution**

Let’s first calculate the PV of the bond issued at \( t = 0 \) that pays 4% annual coupons and that is redeemed at par value of $100 is 7% annual effective.

\[ PV = 4 a_{\overline{2}\%} + 100(1.07)^2 = 94.576 \]

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \leftarrow 1+i_1 \rightarrow )</td>
<td>( \leftarrow 1+f_{1,1} \rightarrow )</td>
<td>( \leftarrow 1+f_{2,1} \rightarrow )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \leftarrow 1+6% \rightarrow )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \leftarrow (1+i_2)^2 \rightarrow )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \leftarrow (1+i_3)^3 \rightarrow )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ 94.576 = \frac{4}{1+i_1} + \frac{104}{(1+i_2)^2} = \frac{4}{1+6\%} + \frac{104}{(1+6\%)(1+f_{1,1})} \]
\[ f_{1,1} = 8.051\% \]

The issue price at \( t = 0 \) of a three-year bond that pays 8% coupons annually is $102 per $100 nominal:

\[
102 = \frac{8}{1+i_1} + \frac{8}{(1+i_2)^2} + \frac{8+100}{(1+i_3)^3}
\]

\[
\Rightarrow 102 = \frac{8}{1+6\%} + \frac{8}{(1+6\%)(1+8.051\%)} + \frac{8+100}{(1+6\%)(1+8.051\%)(1+f_{2,1})}
\]

\[
\Rightarrow f_{2,1} = 7.805\%
\]

**Problem 12**

The \( n \) year spot rates at \( t=0 \) are defined as follows:

\[
i_n = 0.05 + \frac{n}{1000}, \text{ where } 1, 2, 3
\]

Calculate the implied one-year forward rate at \( t=1 \) and \( t=2 \).

**Solution**

We are asked to find \( f_{1,1} \) and \( f_{2,1} \).

<table>
<thead>
<tr>
<th>Time ( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \leftarrow 1+i_1 \rightarrow )</td>
<td>( \leftarrow 1+f_{1,1} \rightarrow )</td>
<td>( \leftarrow 1+f_{2,1} \rightarrow )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \leftarrow (1+i_2)^2 \rightarrow )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \leftarrow (1+i_3)^3 \rightarrow )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[(1+i_2) = (1+i_1)(1+f_{1,1}), \quad (1+i_3) = (1+i_2)^2(1+f_{2,1})\]

\[i_1 = 0.05 + \frac{1}{1000} = 0.051, \quad i_2 = 0.05 + \frac{2}{1000} = 0.052, \quad i_3 = 0.05 + \frac{3}{1000} = 0.053\]

\[\Rightarrow f_{1,1} = \frac{(1+i_2)^2}{1+i_1} - 1 = \frac{1.052^2}{1.051} - 1 = 5.3\%\]

\[\Rightarrow f_{2,1} = \frac{(1+i_3)^3}{(1+i_2)^2} - 1 = \frac{1.053^3}{1.052^2} - 1 = 5.5\%\]

**Problem 13**

The following \(n\)-year spot rates are observed at \(t = 0\):

- 1-year spot rate is 2%
- 2-year spot rate is 3%
- 3-year spot rate is 4%
- 4-year spot rate is 5%
- 5-year spot rate is 6%
- 6-year spot rate is 7%
- 7-year spot rate is 8%
- 8-year spot rate is 9%
- 9-year spot rate is 10%

Calculate \(f_{5,4}\), the 4-year forward rate at \(t = 5\).

**Solution**

\[(1+s_5)^5(1+f_{5,4})^4 = (1+s_9)^9\]

\(s_5 = 6\%\), \(s_9 = 10\%\)

\[\Rightarrow (1+f_{5,4})^4 = \frac{(1+s_9)^9}{(1+s_5)^5} = \frac{(1+10\%)^9}{(1+6\%)^5}\]

\[\Rightarrow f_{5,4} = 15.21\%\]
**Problem 14**
The spot yield curve on 1/1/2006 is defined as follows:

\[ s_t = 0.04e^{-0.06t} + 0.03 \]

Calculate \( f_{3,2} \).

**Solution**

\[
(1 + s_3)^3 (1 + f_{3,2})^2 = (1 + s_5)^5
\]

\[
s_3 = 0.04e^{-0.06(3)} + 0.03 = 0.0634108
\]

\[
s_2 = 0.04e^{-0.06(5)} + 0.03 = 0.059633
\]

\[
(1 + f_{3,2})^2 = \frac{(1 + s_5)^5}{(1 + s_3)^5} = \frac{0.0634108^5}{0.059633^5} \Rightarrow f_{3,2} = 5.4\%
\]

**Problem 15**
You are given the following forward rates:

\[ f_{0,1} = 5\% , \quad f_{1,1} = 6\% , \quad f_{2,1} = 7\% \]

Calculate the annual effective yield of a 3-year bond that pays 8% annual coupon with face amount of $100.

**Solution**

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flows</td>
<td>$8</td>
<td>$8</td>
<td>$108</td>
<td></td>
</tr>
</tbody>
</table>

\[
\leftarrow 1 + i_1 \rightarrow \leftarrow 1 + f_{1,1} \rightarrow \leftarrow 1 + f_{2,1} \rightarrow
\]

\[
\leftarrow (1 + i_2)^2 \rightarrow
\]

Guo FM, fall 2009
The price of the bond (i.e. the PV) is:

\[
\frac{8}{1+i_1} + \frac{8}{(1+i_2)^2} + \frac{8+100}{(1+i_3)^3} = \frac{8}{1.05} + \frac{8}{1.05(1.06)} + \frac{8+100}{1.05(1.06)(1.07)} = 105.49378
\]

Next, we solve the following equation:

\[
105.49378 = 8a_{\overline{n|}} + 100v^3
\]

Using BA II Plus TVM, we should get:

\[
i = 5.94675\%
\]

**Problem 16  (Sample FM #33)**
You are given the following information with respect to a bond:

- Par amount: $1,000
- Term to maturity: 3 years
- Annual coupon: 6% payable annually

<table>
<thead>
<tr>
<th>Term</th>
<th>Annual spot interest rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7%</td>
</tr>
<tr>
<td>2</td>
<td>8%</td>
</tr>
<tr>
<td>3</td>
<td>9%</td>
</tr>
</tbody>
</table>

Calculate the value of the bond.

**Solution**

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flows</td>
<td>60</td>
<td>60</td>
<td>1,060</td>
<td></td>
</tr>
</tbody>
</table>

\[
\left(1+i_1\right)^2 \rightarrow (1+i_2)^3 \rightarrow \left(1+f_{1,1}\right) \rightarrow \left(1+f_{2,1}\right)
\]
The bond is worth its present value. The present value is:

\[
\frac{60}{1+i_1} + \frac{60}{(1+i_2)^2} + \frac{60+100}{(1+i_3)^3} = \frac{60}{1+7\%} + \frac{60}{(1+8\%)^2} + \frac{60+100}{(1+9\%)^3} = 926.03
\]

**Problem 17  (Sample FM #34)**

You are given the following information with respect to a bond:
- Par amount: $1,000
- Term to maturity: 3 years
- Annual coupon: 6% payable annually

<table>
<thead>
<tr>
<th>Term</th>
<th>Annual spot interest rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7%</td>
</tr>
<tr>
<td>2</td>
<td>8%</td>
</tr>
<tr>
<td>3</td>
<td>9%</td>
</tr>
</tbody>
</table>

Calculate the annual effective yield rate for the bond if the bond is sold at a price equal to its value.

**Solution**

From Problem 16, we know the present value of the bond is 926.03.

\[
926.03 = 60 \cdot a_{\overline{3}|i} + 1000v^3
\]

Using BA II Plus TVM, we get:

\[
i = 8.918\%
\]
Chapter 16  Macaulay duration, modified duration, convexity

An asset has the following cash flows:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>......</th>
<th>k</th>
<th>......</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>CF(0)</td>
<td>CF(1)</td>
<td>CF(k)</td>
<td>CF(n)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
P = \sum_{t=0}^{n} CF(t) v^t = \sum_{t=0}^{n} \frac{CF(t)}{(1+r)^t}
\]

Asset price = Present value of the future cash flows

We want to find out how \( P \), the asset price, is sensitive to the change of the interest rate \( r \), the effective interest per year.

We define the following term:

Macaulay duration = negative price elasticity relative to \((1+r)\)

\[
D_{MAC} = -\text{Asset Price Elasticity relative to } (1+r) = -\frac{\% \text{ change in price}}{\% \text{ change in } (1+r)}
\]

Often time, Macaulay duration is simply called duration. If you are asked to calculate the duration, just calculate Macaulay duration.

We make the Macaulay duration equal to the negative price elasticity. This way, the Macaulay duration becomes a positive number -- we like positive numbers better. To see why the Macaulay duration is positive, notice that the present value of an asset is inversely related to the interest rate. If \( r \) goes up (i.e. \% change of \( r \) is positive), then we discount cash flows at a higher discount rate, causing the asset price to go down (i.e. \% change of the asset price becomes negative). Similarly, if \( r \) goes down (i.e. \% change of \( r \) is negative), then we discount cash flows at a lower discount rate, causing the asset price to go up (i.e. \% change of the
asset price becomes positive). By setting the Macaulay duration equal to the negative price elasticity, we'll get a positive number.

\[ D_{MAC} = -\frac{dp}{d(1+r)\frac{1}{1+r}} = -\frac{dp}{dr} = -(1+r)\left[ \frac{1}{p} \frac{dp}{dr} \right] \]

\[ P = \sum_{i=0}^{n} CF(t) v^i = \sum_{i=0}^{n} \frac{CF(t)}{(1+r)^i} \]

\[ \Rightarrow \frac{dP}{dr} = \frac{d}{dr} \sum_{i=0}^{n} CF(t)(1+r)^{-i} = -\frac{1}{1+r} \sum_{i=0}^{n} t CF(t)(1+r)^{-i} = -\frac{1}{1+r} \sum_{i=0}^{n} t CF(t) v^i \]

\[ \Rightarrow D_{MAC} = -(1+r)\left[ \frac{1}{p} \frac{dp}{dr} \right] = -(1+r)\left[ \frac{1}{p} - \frac{1}{1+r} \sum_{i=0}^{n} t CF(t) v^i \right] \]

\[ D_{MAC} = \frac{\sum_{i=0}^{n} t CF(t) v^i}{P} = \frac{\sum_{i=0}^{n} t CF(t) v^i}{\sum_{i=0}^{n} CF(t) v^i} \]

Observation:
A zero coupon bond’s duration is simply its maturity. A zero coupon bond has only one cash flow at its maturity \( n \).

\[ \Rightarrow D_{MAC} = \frac{n CF(n) v^n}{CF(n) v^n} = n \]

We define the 2\(^{nd}\) term:

Modified Duration\(D_{MOD} = \frac{1}{p} \frac{1}{p} \frac{dP}{dr} \)

\[ \Rightarrow D_{MOD} = -\frac{1}{p} \frac{dP}{dr} = -\frac{1}{p} \left[ \frac{1}{1+r} \sum_{i=0}^{n} t CF(t) v^i \right] = \frac{1}{1+r} \left( \frac{1}{p} \right) \sum_{i=0}^{n} t CF(t) v^i \]

\[ \Rightarrow D_{MOD} = \frac{1}{1+r} \left( \frac{1}{p} \right) \sum_{i=0}^{n} t CF(t) v^i = \frac{1}{1+r} D_{MAC} \]
Wall Street method of calculating the modified duration:

\[
D_{MOD} = D_{MAC} = \frac{1}{1+y/m} \sum_{t=0}^{n} \frac{t CF(t) v^t}{1+y/m} \sum_{t=0}^{n} CF(t) v^t
\]

\( y \) is the yield to maturity expressed as a nominal interest rate compounding as often as coupons are paid; \( m \) is the # of coupons per year. Consequently, \( \frac{y}{m} \) is the effective interest per coupon period.

To understand the Wall Street formula for the modified duration, please note that when Wall Street talks about a bond yield, it uses a nominal interest rate compounding as frequently as coupons are paid, not the annual effective interest. For example, for a bond that pays coupons semiannually, if the yield to maturity is 10.25% annual effective, then Wall Street will quote the bond yield as \( y = i^{(2)} \).

\[
\left(1 + \frac{y}{2}\right)^2 = 1 + 10.25\% \quad \Rightarrow \quad y = 10\%
\]

So Wall Street quotes the yield to maturity as 10%.

Then why does Wall Street use \( D_{MOD} = D_{MAC} = \frac{1}{1+y/m} \sum_{t=0}^{n} \frac{t CF(t) v^t}{1+y/m} \sum_{t=0}^{n} CF(t) v^t \)?

Please note that here we start off with \( t = 1 \), not \( t = 0 \). This is because a bond’s first coupon does not start off with \( t = 0 \).

Let \( y \) represent the yield to maturity expressed as the nominal interest rate compounding as frequently as coupons are paid. Assume that coupons are paid \( m \)-thly. So essentially, \( y = i^{(m)} \) and the effective interest rate \( r = \left(1+\frac{y}{m}\right)^m - 1 \). Assume the bond’s term to maturity is \( n \). The present value of the bond is:
\[ P = \sum_{t=1}^{n} \frac{CF(t)}{(1+r)^t} = \sum_{t=1}^{n} \frac{CF(t)}{\left(1+\frac{y}{m}\right)^{mt}} \]

\[ \Rightarrow \frac{1}{P} \frac{dP}{dy} = \frac{1}{P} \frac{d}{dy} \sum_{t=1}^{n} \frac{CF(t)}{\left(1+\frac{y}{m}\right)^{mt}} = \frac{1}{P} \left( \frac{1}{m} \right) \sum_{t=1}^{n} \frac{t CF(t)}{\left(1+\frac{y}{m}\right)^{mt+1}} \]

\[ = -\frac{1}{1+\frac{y}{m}} \sum_{t=1}^{n} \frac{CF(t)}{\left(1+\frac{y}{m}\right)^{mt}} = \frac{1}{1+\frac{y}{m}} \left( \frac{1}{P} \sum_{t=1}^{n} \frac{CF(t)}{\left(1+\frac{y}{m}\right)^{mt}} \right) = \frac{1}{1+\frac{y}{m}} D_{MAC} \]

\[ \Rightarrow D_{MOD} = -\frac{1}{P} \frac{dP}{dy} = \frac{1}{1+\frac{y}{m}} D_{MAC} \]

So the Wall Street formula and the textbook formula are different:

**Wall Street:** \[ D_{MOD, Wall Street} = -\frac{1}{P} \frac{dP}{dy} \]
(derivative of price relative to nominal yield)

**Textbook:** \[ D_{MOD, textbook} = -\frac{1}{P} \frac{dP}{dr} \]
(derivative of price relative to effective yield)

Please note that Bond Worksheet in BA II Plus Professional uses the Wall Street method to calculate the bond price and the modified duration. If you use Bond Worksheet to calculate the bond price and its modified duration, make sure that you use the nominal interest rate. You’ll get a wrong result if you enter the annual effective yield to maturity into Bond Worksheet.

In Exam FM, use the textbook definition of the modified duration:

\[ D_{MOD} = \frac{1}{1+r} D_{MAC} \] (where \( r \) is the annual effective yield to maturity)

Don’t use the Wall Street definition.
For a bond paying coupons annually, these two methods produce the same modified duration. This is because we have \( m = 1 \) and \( y = i^{(1)} = r \).

We define the 3rd term:

\[
\text{Convexity} = \frac{1}{p} \frac{d^2 P}{dr^2} = \frac{1}{p} \frac{d}{dr} \left[ -\sum_{i=1}^{n} t CF(t)(1+r)^{-t} \right] = \frac{1}{p} \sum_{i=1}^{n} t(t+1) CF(t)(1+r)^{-(t-2)}
\]

\[
\Rightarrow \quad \text{Convexity} = \frac{1}{p} \frac{1}{(1+r)^2} \sum_{i=1}^{n} t(t+1) CF(t) v^i
\]

\[
= \frac{1}{(1+r)^2} \left[ \frac{1}{p} \sum_{i=1}^{n} t^2 CF(t) v^i + \frac{1}{p} \sum_{i=1}^{n} t CF(t) v^i \right]
\]

\[
= \frac{1}{(1+r)^2} \left[ \frac{1}{p} \sum_{i=1}^{n} t^2 CF(t) v^i + \text{Duration} \right]
\]

\[
= v^2 \frac{1}{p} \sum_{i=1}^{n} t^2 CF(t) v^i + \text{Duration}
\]

How the price of an asset changes if the interest changes by a small amount:

\[
P(r) = \sum_{i=1}^{n} CF(t) v^i = \sum_{i=1}^{n} \frac{CF(t)}{(1+r)^i}
\]

\[
\Rightarrow \quad \Delta P = \frac{dP}{dr} \Delta r + \frac{1}{2} \frac{d^2 P}{dr^2} (\Delta r)^2 + \ldots \quad \text{(Taylor series)}
\]

Divide by \( P \):

\[
\Rightarrow \quad \frac{\Delta P}{P} = \frac{1}{P} \frac{dP}{dr} \Delta r + \frac{1}{2} \frac{d^2 P}{dr^2} (\Delta r)^2 + \ldots
\]

\[
= -D_{\text{MOD}} \Delta r + \frac{1}{2} (\text{Convexity})(\Delta r)^2 + \ldots
\]

\[
= -\frac{1}{1+r} D_{\text{MAC}} \Delta r + \frac{1}{2} (\text{Convexity})(\Delta r)^2 + \ldots
\]

If the interest rate change \( \Delta r \) is small, we can set \( (\Delta r)^2 \approx 0 \)
Sample problems

Problem 1

<table>
<thead>
<tr>
<th>Bond face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>4% semiannual</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>3 years</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>5% annual effective</td>
</tr>
</tbody>
</table>

Use BA II Plus/BA II Plus Professional Cash Flow Worksheet, calculate the duration (i.e. Macaulay duration), modified duration, and convexity of the bond.

Solution

First, we draw a cash flow diagram.

Unit time = 1 year

Time t | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3
Cash flow | $2 | $2 | $2 | $2 | $2 | $2 | $100

For bonds, often it’s easier if we set the unit time = the payment period:

Unit time = 0.5 year

Time t | 0 | 1 | 2 | 3 | 4 | 5 | 6
Cash flow | $2 | $2 | $2 | $2 | $2 | $2 | $100

\[
\Rightarrow D_{MAC} = \frac{\sum_{t=1}^{3} t \cdot CF(t) \cdot v^t}{\sum_{t=1}^{3} CF(t) \cdot v^t}
\]

(where \( t=0.5, 1, 1.5, 2, 2.5, 3 \))
\[
\frac{1}{2} \sum_{t=1}^{6} t CF(t) v^t = \frac{\sum_{t=1}^{6} CF(t) v^t}{\sum_{t=1}^{6} v^t} \quad \text{where } t=1, 2, 3, 4, 5, 6
\]

Unit time is half a year \( v = 1.05 \frac{1}{2} \)

We set up the following table (we are using 6 months a one unit time):

<table>
<thead>
<tr>
<th>( t )</th>
<th>( CF(t) )</th>
<th>( t \ CF(t) )</th>
<th>( t^2 )</th>
<th>( t^2 CF(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
<td>9</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>10</td>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td>6</td>
<td>102</td>
<td>612</td>
<td>36</td>
<td>3,672</td>
</tr>
</tbody>
</table>

First, we'll find the bond’s price. We enter \( CF(t) \) into Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( CF(t) )</th>
<th>Cash flows</th>
<th>Cash flow frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2</td>
<td>CF1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$2</td>
<td>CF2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$2</td>
<td>CF3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$2</td>
<td>CF4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>$2</td>
<td>CF5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>$102</td>
<td>CF6</td>
<td>1</td>
</tr>
</tbody>
</table>

Interest Rate per coupon period \( i = \sqrt{1.05 - 1} = 2.4695\% \)

Calculate NPV \( 97.41125361 \)

\[ NPV = \sum_{t=1}^{6} CF(t) v^t = 97.41125361 \]
Next, we calculate $\sum_{t=1}^{6} t \cdot CF(t) \cdot v^t$. We enter the following into Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$CF(t)$</th>
<th>$t \cdot CF(t)$</th>
<th>Cash flows</th>
<th>Cash flow frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>CF1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>CF2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
<td>CF3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>8</td>
<td>CF4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>10</td>
<td>CF5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>102</td>
<td>612</td>
<td>CF6</td>
<td>1</td>
</tr>
</tbody>
</table>

Interest rate  
\[ i = \sqrt[2]{1.05} - 1 = 2.4695\% \]

Calculate NPV
\[ \text{NPV}=556.1144433 \]

\[ \Rightarrow \sum_{t=1}^{6} t \cdot CF(t) \cdot v^t = 556.1144433 \]

\[ \Rightarrow D_{MAC} = \frac{1}{2} \sum_{t=1}^{6} CF(t) \cdot v^t = \frac{1}{2} \cdot 556.1144433 \cdot \frac{1}{97.41125361} = 2.85446713 \]

Next, we’ll calculate the modified duration.

Wall Street method:

\[ D_{MOD\text{ Wall Street}} = \frac{1}{1 + \text{effective yield per coupon period}} \cdot D_{MAC} \]

\[ = \frac{1}{\sqrt{1.05}} \cdot D_{MAC} = \frac{2.85446713}{\sqrt{1.05}} = 2.78567468 \]

Textbook method:

\[ D_{MOD\text{ Textbook}} = \frac{1}{1 + \text{annual effective yield rate}} \cdot D_{MAC} \]

Guo FM, fall 2009
Next, we’ll calculate the convexity using the following formula:

\[
\text{Convexity} = \frac{1}{p} \left( \frac{1}{(1+i)^2} \sum_{t=1}^{3} t(t+1) CF(t) v^t \right)_{\text{Unit time is one year}} = \frac{1}{p} \left( \frac{1}{(1+5\%)^2} \sum_{t=1}^{3} t^2 CF(t) (1.05)^{-t} + \sum_{t=1}^{3} t CF(t) (1.05)^{-t} \right)_{\text{Unit time is one year}}
\]

\[
= \frac{1}{p} \left( \frac{1}{(1+5\%)^2} \left[ \sum_{t=1}^{3} \left( \frac{t}{2} \right)^2 CF(t) v^t + \sum_{t=1}^{3} \frac{t}{2} CF(t) v^t \right] \right)_{\text{Unit time is half a year}}
\]

\[
= \frac{1}{p} \left( \frac{1}{(1+5\%)^2} \left[ \sum_{t=1}^{3} t^2 CF(t) v^t + \sum_{t=1}^{3} \frac{t}{2} CF(t) v^t \right] \right)_{\text{Unit time is half a year}}
\]

To calculate \( \sum_{t=1}^{6} t^2 CF(t) v^t \), we enter \( t^2 CF(t) \) (bold numbers below) into the Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( CF(t) )</th>
<th>( i^2 )</th>
<th>( i^2 CF(t) )</th>
<th>Cash flows</th>
<th>Cash flow frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>$2$</td>
<td>CF1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>$8$</td>
<td>CF2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>$18$</td>
<td>CF3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>16</td>
<td>$32$</td>
<td>CF4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>25</td>
<td>$50$</td>
<td>CF5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>102</td>
<td>36</td>
<td>$3,672$</td>
<td>CF6</td>
<td>1</td>
</tr>
</tbody>
</table>

Interest Rate per coupon period

\[ i = \sqrt{1.05 - 1} = 2.4695\% \]

Calculate NPV

\[ 3,271.595675 \]

\[ \Rightarrow \sum_{t=1}^{6} t^2 CF(t) v^t = 3,271.595675 \]
We already know that

\[
\sum_{t=1}^{6} t \cdot CF(t) \cdot v^t = 556.1144433
\]

\[
P = \sum_{t=1}^{6} CF(t) \cdot v = 97.41125361
\]

\[
\Rightarrow \text{Convexity} = \frac{1}{P} \frac{1}{(1+5\%)^2} \left[ \frac{1}{4} \sum_{t=1}^{6} t^2 \cdot CF(t) \cdot v^t + \frac{1}{2} \sum_{t=1}^{6} t \cdot CF(t) \cdot v^t \right]
\]

Unit time is half a year

\[
v = (1+i)^{-1}, i = 2.4695\%
\]

\[
= \frac{1}{97.41125361} \frac{1}{(1+5\%)^2} \left[ \frac{1}{4} \left( \frac{3,271.595675}{97.41125361} + \frac{1}{2} (556.1144433) \right) \right] = 10.2048222
\]

Alternatively, convexity is:

\[
v^2 \left[ \frac{1}{P} \sum_{t=1}^{n} t^2 \cdot CF(t) \cdot v^t + \text{Duration} \right] = \frac{1}{(1+5\%)^2} \left[ \frac{1}{P} \sum_{t=1}^{6} \frac{1}{4} t^2 \cdot CF(t) \cdot v^t + \text{duration} \right]
\]

Unit time is half a year

\[
v = (1+i)^{-1}, i = 2.4695\%
\]

\[
= \frac{1}{(1+5\%)^2} \left[ \left( \frac{1}{4} \right) \frac{3,271.595675}{97.41125361} + 2.85446713 \right] = 10.2048222
\]

Though the problem looks complex and intimidating, the solution process is really quick and simple.

**Let’s summarize the calculation steps:**

<table>
<thead>
<tr>
<th>If a bond has</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupons ( C ) payable ( m )-thly per year</td>
</tr>
<tr>
<td>Yield to maturity = ( r ) annual effective</td>
</tr>
<tr>
<td>Term to maturity = ( n )</td>
</tr>
<tr>
<td>Face amount = ( F )</td>
</tr>
</tbody>
</table>

Steps to calculate the bond duration, modified duration, and convexity using BA II Plus/BA II Plus Professional:
**Step 1** – Set the per coupon payment period $\frac{1}{m}$ as the unit time. Convert the annual effective interest rate into the effective rate per coupon period:

$$i = (1 + r)^\frac{1}{m} - 1$$

**Step 2** - Set up the cash flow master table:

<table>
<thead>
<tr>
<th>$t$ (Unit time=$\frac{1}{m}$)</th>
<th>$CF(t)$</th>
<th>$t \cdot CF(t)$</th>
<th>$t^2$</th>
<th>$t^2 \cdot CF(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C</td>
<td>C</td>
<td>1</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
<td>2C</td>
<td>4</td>
<td>4C</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>3C</td>
<td>9</td>
<td>9C</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$mn-1$</td>
<td>C</td>
<td>$(mn-1)C$</td>
<td>$(mn-1)^2$</td>
<td>$(mn-1)^2 \cdot C$</td>
</tr>
<tr>
<td>$mn$</td>
<td>$F+C$</td>
<td>$mn(F+C)$</td>
<td>$(mn)^2$</td>
<td>$(mn)^2 \cdot (F+C)$</td>
</tr>
</tbody>
</table>

**Step 3** - Manually enter the cash flows into Cash Flow Worksheet. Set $I = 100i = 100 \left[ (1 + r)^\frac{1}{m} - 1 \right]$. Calculate the following 3 items:

$$P = \sum_{t=1}^{mn} CF(t) \cdot v^t, \quad \sum_{t=1}^{mn} t \cdot CF(t) \cdot v^t, \quad \sum_{t=1}^{mn} t^2 \cdot CF(t) \cdot v^t$$

where $v = (1 + r)^\frac{1}{m}$

**Step 4** – Calculate the duration and convexity using the following formulas:

$$D_{MAC} = \frac{1}{m} \sum_{t=1}^{mn} t \cdot CF(t) \cdot v^t \cdot \frac{1}{P}$$

$$\text{Convexity} = \frac{1}{P} \frac{1}{(1 + r)^2} \left[ \frac{1}{m^2} \sum_{t=1}^{mn} t^2 \cdot CF(t) \cdot v^t + \frac{1}{m} \sum_{t=1}^{mn} t \cdot CF(t) \cdot v^t \right]$$

$$D_{MOD \text{ Textbook}} = \frac{1}{1 + r} D_{MAC}$$
Alternative method to calculate convexity:

\[
Convexity = \frac{1}{(1 + r)^2} \left[ \frac{1}{p} \sum_{t=1}^{n} \frac{1}{m} \sum_{t=1}^{n} t^2 CF(t) v' + \frac{1}{p} \sum_{t=1}^{n} t CF(t) v' \right]
\]

\[
= \frac{1}{(1 + r)^2} \left[ \frac{1}{p} \sum_{t=1}^{n} \frac{1}{m} \sum_{t=1}^{n} t^2 CF(t) v' + \text{duration} \right]
\]

**Problem 2**

<table>
<thead>
<tr>
<th>Bond face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>4% semiannual</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>3 years</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>5% annual effective.</td>
</tr>
</tbody>
</table>

Use **BA II Plus Professional** Bond Worksheet, calculate the duration (i.e. Macaulay duration) and the modified duration.

**Solution**

Please note that Bond Worksheet in BA II Plus can NOT directly calculate the modified duration. BA II Plus Professional does. For this reason, you might want to buy BA II Plus Professional.

BA II Plus Professional Bond Worksheet uses the Wall Street convention in quoting a bond by using a nominal yield to maturity. As a result, when Bond Worksheet calculates the price and the modified duration, it uses the nominal yield that compounds as frequently as coupons are paid.

We need to find, \( y \), the nominal yield compounding twice a year (coupons are paid twice a year):

\[
\left(1 + \frac{y}{2}\right)^2 = 1 + 5\% \quad \Rightarrow \quad y = 4.939\%
\]
Key strokes in BA II Plus Professional:

<table>
<thead>
<tr>
<th>Key Stroke</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd Bond</td>
<td>This activates Bond Worksheet.</td>
</tr>
<tr>
<td>Enter SDT=1.0100</td>
<td>SDT = settlement date (i.e. purchase date of the bond)</td>
</tr>
<tr>
<td>Enter CPN=4</td>
<td>Set coupon = 4% of par.</td>
</tr>
<tr>
<td>Enter RDT=1.0103</td>
<td>RDT = redemption date (or bond’s maturity)</td>
</tr>
<tr>
<td>Enter RV=100</td>
<td>Redemption value. Because the bond is redeemed at par and the par=100, we set RV=100.</td>
</tr>
<tr>
<td>Day counting method</td>
<td>Use 360 counting method (i.e. assume a year has 360 days). Don’t use the actual day counting method.</td>
</tr>
<tr>
<td>Coupon frequency</td>
<td>2/Y (i.e. twice a year)</td>
</tr>
<tr>
<td>YTD=4.939015332</td>
<td>Enter 4.939015332, not 4.93901532% (i.e. don’t enter the % sign).</td>
</tr>
<tr>
<td>CPT PRI (compute price of the bond)</td>
<td>We get PRI=97.41125361. This is the bond price.</td>
</tr>
<tr>
<td>Al=0</td>
<td>Accrued interest is zero. Don’t worry about this feature. We don’t need it to pass Exam FM.</td>
</tr>
<tr>
<td>DUR</td>
<td>DUR=2.78567468 Remember this is the modified duration under the Wall Street method.</td>
</tr>
</tbody>
</table>

\[ D_{MOD}^{Wall Street} = 2.78567468 \]

Next, we will convert the modified duration (Wall Street method) into Macaulay duration.

\[ D_{MOD}^{Wall Street} = \left[ \frac{1}{1 + \frac{YLD(m)}{m}} \right] D_{MAC} \]
Finally, we’ll find the modified duration during the textbook definition:

\[ D_{MOD}^{Textbook} = \frac{1}{1 + \text{annual effective yield}} \cdot D_{MAC} \]

\[ = \frac{1}{1.05} \cdot D_{MAC} = \frac{2.85446713}{1.05} = 2.71854012 \]

Please note that Bond Worksheet can not calculate the convexity. To calculate convexity, we have to use Cash Flow Worksheet or use a formula-driven approach.

**Problem 3**

<table>
<thead>
<tr>
<th>Bond face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>4% semiannual</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>3 years</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>5% annual effective.</td>
</tr>
<tr>
<td>Redemption</td>
<td>$110</td>
</tr>
</tbody>
</table>

Use **BA II Plus Professional** Bond Worksheet, calculate the duration (i.e. Macaulay duration) and the modified duration.

**Solution**

Compared with Problem 2, Problem 3 has a bond not deemed at par. When calculating the duration of a bond not deemed at par using BA II Plus Professional Bond Worksheet, we need to convert such a bond to a bond redeemed at par. Such as conversion is needed because BA II Plus Professional Bond Worksheet can NOT calculate the duration of a bond not redeemed at par.

If we don’t convert a non-par bond to a par bond, BA II Plus will give a wrong result. Let’ see.
Key strokes in BA II Plus Professional Bond Worksheet:

<table>
<thead>
<tr>
<th>2nd Bond</th>
<th>This activates Bond Worksheet.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Enter SDT=1.0100</td>
<td>We arbitrarily set the purchase date of the bond is to 1/1/2000.</td>
</tr>
<tr>
<td>This sets SDT=1-01-2000</td>
<td></td>
</tr>
<tr>
<td>Enter CPN=4</td>
<td>Set coupon = 4% of par.</td>
</tr>
<tr>
<td>Enter RDT=1.0103</td>
<td>RDT = redemption date</td>
</tr>
<tr>
<td>This sets RDT=1-01-2003</td>
<td>We set RDT=1-01-2003 (the bond has a 3 year maturity).</td>
</tr>
<tr>
<td>Enter RV=110</td>
<td>Redemption value.</td>
</tr>
<tr>
<td>Day counting method</td>
<td>Use 360 counting method</td>
</tr>
<tr>
<td>Coupon frequency</td>
<td>2/Y (i.e. twice a year)</td>
</tr>
<tr>
<td>YLD=4.939015332</td>
<td>Don’t enter 4.93901532%</td>
</tr>
<tr>
<td>CPT PRI (compute price of the bond)</td>
<td>We get PRI=106.0496293. This is the bond price. This price is correct.</td>
</tr>
<tr>
<td>AI=0</td>
<td>Accrued interest is zero.</td>
</tr>
<tr>
<td>DUR</td>
<td>DUR=2.78567468</td>
</tr>
</tbody>
</table>

Notice anything strange here? Even though the bond in Problem 3 is different from the bond in Problem 2, BA II Plus gives us the same modified duration (Wall Street version of modified duration). Something must be wrong.

If we use this modified duration (Wall Street version), then the Macaulay duration under the textbook definition is:

\[
D_{MOD}^{Wall Street} = \frac{1}{1+\frac{YLD^{(m)}}{m}}D_{MAC}
\]

\[
\Rightarrow D_{MAC} = D_{MOD}^{Wall Street}\left[1+\frac{YLD^{(m)}}{m}\right]
\]

\[
= 2.78567468\left(1+\frac{4.93901532\%}{2}\right) = 2.85446713
\]

We know this figure is wrong. Next, let’s calculate the bond’s real duration using Cash Flow Worksheet.
Cash flow diagram:

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$110</td>
</tr>
</tbody>
</table>

The effective interest rate per coupon period is \( i = \sqrt{1.05} - 1 = 0.024695 \) or 2.4695%.

We'll use the generic procedure described in Problem 2 to find the duration. First, we come up with following cash flow table (we are using 6 months a one unit time):

<table>
<thead>
<tr>
<th>( t )</th>
<th>( CF(t) )</th>
<th>( t CF(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>112</td>
<td>672</td>
</tr>
</tbody>
</table>

Using Cash Flow Worksheet, we find:

\[ \sum_{t=1}^{6} CF(t) v^t = 106.04962959, \quad \sum_{t=1}^{6} t CF(t) v^t = 607.94469918 \]

\[ \Rightarrow D_{MAC} = \frac{\sum_{t=1}^{6} t CF(t) v^t}{\sum_{t=1}^{6} CF(t) v^t} = \frac{607.94469918}{106.04962959} = 2.866321653 \]

So the correct duration is 2.866321653. The modified duration is:

\[ D_{MOD,Textbook} = \frac{1}{1 + \text{annual effective yield}} D_{MAC} = \frac{1}{1.05} \times 2.866321653 = 2.72983 \]

We see that BA II Plus Professional calculates a bond’s duration assuming a bond is always deemed at par, no matter what redemption value is. For example, in this problem you can enter RV=0 or any other non-negative number and you'll still get DUR=2.78567468. You can check this for yourself.

We can overcome this issue by converting a non-par bond into a par bond. In this problem, the conversion goes like this:
\[
\text{new coupon rate} = \frac{\text{original coupon rate} \times \text{par}}{\text{redemption value}} = \frac{4\% \times 100}{110} = 3.6366364\%
\]

This will give us the right duration.

Revised key strokes in BA II Plus Professional Bond Worksheet:

<table>
<thead>
<tr>
<th>2\textsuperscript{nd} Bond</th>
<th>This activates Bond Worksheet.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Enter SDT=1.0100</td>
<td>We arbitrarily set the purchase date of the bond is to 1/1/2000.</td>
</tr>
<tr>
<td>This sets SDT=1-01-2000</td>
<td></td>
</tr>
<tr>
<td>Enter CPN=3.63636364</td>
<td>Revised coupon rate</td>
</tr>
<tr>
<td>Enter RDT=1.0103</td>
<td>RDT = redemption date</td>
</tr>
<tr>
<td>This sets RDT=1-01-2003</td>
<td>We set RDT=1-01-2003 (the bond has a 3 year maturity).</td>
</tr>
<tr>
<td>Enter RV=110</td>
<td>Redemption value. You can even enter RV=0. This won’t affect the result.</td>
</tr>
<tr>
<td>Day counting method</td>
<td>Use 360 counting method</td>
</tr>
<tr>
<td>Coupon frequency</td>
<td>2/Y (i.e. twice a year)</td>
</tr>
<tr>
<td>YLD=4.939015332</td>
<td>Don’t enter 4.93901532%</td>
</tr>
<tr>
<td>PRI</td>
<td>If you compute the bond’s price, you’ll get a garbage number because we convert the original bond. So ignore PRI.</td>
</tr>
<tr>
<td>AI=0</td>
<td>Accrued interest is zero.</td>
</tr>
<tr>
<td>DUR</td>
<td>DUR=2.79721351 (This is Wall Street’s version of the modified duration)</td>
</tr>
</tbody>
</table>

Next, we'll convert the Wall Street’s modified duration into Macaulay duration:

\[
D_{MOD}^{Wall\ Street} = \frac{1}{1 + \frac{YLD^{(m)}}{m}} D_{MAC}
\]

\[
\Rightarrow D_{MAC} = D_{MOD}^{Wall\ Street} \left[ 1 + \frac{YLD^{(m)}}{m} \right] = 2.79721351 \left( 1 + \frac{4.93901532\%}{2} \right) = 2.79721351 \sqrt{1.05} = 2.86636165
\]
This number matches the duration calculated using Cash Flow Worksheet.

**Key point to remember:**

If you ever use BA II Plus Professional Bond Worksheet to calculate a bond’s duration, whether the bond is redeemed at par or not, always calculate the new coupon rate:

\[
\text{new coupon rate} = \frac{\text{original coupon rate} \times \text{par}}{\text{redemption value}}
\]

If the bond is redeemed at par, the new coupon rate = original coupon rate.

Enter this new coupon rate into Bond Worksheet. Next, convert the Wall Street modified duration into Macaulay duration.

**Problem 4**

<table>
<thead>
<tr>
<th>Bond face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>4% semiannual</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>3 years</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>5% annual effective.</td>
</tr>
<tr>
<td>Redemption Value</td>
<td>$95</td>
</tr>
</tbody>
</table>

Calculate the bond’s duration using BA II Plus Professional Bond Worksheet.

**Solution**

\[
\text{new coupon rate} = \frac{\text{original coupon rate} \times \text{par}}{\text{redemption value}} = \frac{4\% \times 100}{95} = 4.21052632\%
\]
2nd Bond
Enter SDT=1.0100
This sets SDT=1-01-2000
Enter CPN=4.21052632
Revised coupon rate
Enter RDT=1.0103
This sets RDT=1-01-2003
We set RDT=1-01-2003 (the bond has a 3 year maturity).
Enter RV= any non-negative number (such as zero or 0.1)
Day counting method
Use 360 counting method
Coupon frequency
2/Y (i.e. twice a year)
YLD=4.939015332
Don’t enter 4.93901532%
PRI
If you compute the bond’s price, you’ll get a garbage number because we convert the original bond. So ignore PRI.
AI=0
Accrued interest is zero.
DUR
DUR=2.77908513 (This is Wall Street’s version of the modified duration)

Next, we’ll convert the Wall Street’s modified duration into Macaulay duration:

\[ D_{MOD}^{Wall \ Street} = \frac{1}{1 + \frac{YLD^{(m)}}{m}} D_{MAC} \]

\[ \Rightarrow D_{MAC} = D_{MOD}^{Wall \ Street} \left[ 1 + \frac{YLD^{(m)}}{m} \right] \]

\[ = 2.77908513 \left( 1 + \frac{4.93901532\%}{2} \right) = 2.77908513 \sqrt{1.05} = 2.84771485 \]

### Problem 5

<table>
<thead>
<tr>
<th>Bond face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>4% semiannual</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>3 years</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>5% annual effective.</td>
</tr>
</tbody>
</table>
Use the formula driven approach, calculate the duration, the modified duration, and convexity.

Solution

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$2</td>
<td>$100</td>
</tr>
</tbody>
</table>

The effective interest rate per coupon period is \( i = \sqrt{1.05} - 1 = 2.4695\% \).

\[
D_{MAC} = \frac{1}{2} \sum_{t=1}^{6} t \cdot CF(t) \cdot v^t \div \sum_{t=1}^{6} CF(t) \cdot v^t
\]

\[
\sum_{t=1}^{6} CF(t) \cdot v^t = 2a_{6|} + 100v^6 \quad @ i = 2.4695\%
\]

\[
\sum_{t=1}^{6} t \cdot CF(t) \cdot v^t = 2(Ia)_{6|} + 6(100)v^6 \quad @ i = 2.4695\%
\]

\[
\Rightarrow D_{MAC} = \frac{1}{2} \frac{\sum_{t=1}^{6} t \cdot CF(t) \cdot v^t}{\sum_{t=1}^{6} CF(t) \cdot v^t} = \left( \frac{1}{2} \right) \frac{2(Ia)_{6|} + 6(100)v^6}{2a_{6|} + 100v^6}
\]

\[
= \left( \frac{1}{2} \right) \frac{556.1144433}{97.41125361} = 2.85446713
\]

\[
D_{MOD, Textbook} = \frac{1}{1 + \text{annual effective yield}} D_{MAC}
\]

\[
= \frac{1}{1.05} D_{MAC} = \frac{2.85446713}{1.05} = 2.71854012
\]

However, convexity is nasty to calculate.

\[
Convexity = \frac{1}{p} \frac{1}{(1+5\%)^2} \left[ \frac{1}{4} \sum_{t=1}^{6} t^2 \cdot CF(t) \cdot v^t + \frac{1}{2} \sum_{t=1}^{6} t \cdot CF(t) \cdot v^t \right]
\]
The tricky part is to evaluate
\[
\sum_{t=1}^{6} t^2 \ CF(t) \ v' = 2 \sum_{t=1}^{6} t^2 \ v' + 6^2 (100) v^6
\]

Though Kellison gives us a formula for evaluating \( \sum_{t=1}^{n} t^2 \ v' \) (his formula 9.23), such a formula is unwieldy and not worth memorizing – so do NOT memorize it.

Because here we have \( n = 6 \) (not too big), we simply calculate \( \sum_{t=1}^{6} t^2 \ v' \) directly without using any formulas.

\[
\sum_{t=1}^{6} t^2 \ CF(t) \ v' = 1^2 (2) \left( 1.05^\frac{1}{2} \right) + 2^2 (2) \left( 1.05^\frac{2}{2} \right) + 3^2 (2) \left( 1.05^\frac{3}{2} \right) \\
+ 4^2 (2) \left( 1.05^\frac{4}{2} \right) + 5^2 (2) \left( 1.05^\frac{5}{2} \right) + 6^2 (102) \left( 1.05^\frac{6}{2} \right)
\]

\( \Rightarrow \sum_{t=1}^{6} t^2 \ CF(t) \ v' = 3,271.595675 \)

We already know that
\[
\sum_{t=1}^{6} t \ CF(t) \ v' = 556.1144433
\]
\[
P = \sum_{t=1}^{6} CF(t) \ v' = 97.41125361
\]

\( \Rightarrow \) Convexity = \[
\frac{1}{P \ (1+5\%)} \left[ \frac{1}{4} \sum_{t=1}^{6} t^2 \ CF(t) \ v' + \frac{1}{2} \sum_{t=1}^{6} t \ CF(t) \ v' \right] \\
= \frac{1}{97.41125361 (1+5\%)} \left[ \frac{1}{4} (3,271.595675) + \frac{1}{2} (556.1144433) \right] \\
= 10.2048222
\]
Alternatively,

\[
Convexity = \frac{1}{(1+r)^2} \left[ \frac{1}{p} \sum_{t=1}^{n} t^2 \frac{CF(t)}{v^t} + \text{Duration} \right]
\]

Unit time is one year

\[
= \frac{1}{(1+r)^2} \left[ \frac{1}{p} \sum_{t=1}^{6} \frac{1}{4} t^2 \frac{CF(t)}{v^t} + \text{Duration} \right]
\]

Unit time = half a year

\[
= \frac{1}{(1+5\%)^2} \left[ \frac{1}{97.41125361} \frac{1}{4} (3.271.595675 + 2.85446713) \right] = 10.2048222
\]

I think it's unlikely that SOA will ask you to calculate the convexity of a coupon bond with a long maturity; the calculation is too intensive. However, SOA can ask you to calculate the convexity of a simple bond such as

- a zero bond
- 1-year or 2-year bond, with coupons payable annually or semiannually

Make sure you know how to calculate.

In addition, you need to be able to quickly calculate the duration of a regular bond. Duration is always easy to calculate.

**Problem 6**

<table>
<thead>
<tr>
<th>Bond face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>6% semiannual</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>4 years</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>8% annual effective.</td>
</tr>
<tr>
<td>Redemption</td>
<td>$105</td>
</tr>
</tbody>
</table>

Calculate the bond’s duration using the following method:

- BA II Plus/BA II Plus Professional Cash Flow Worksheet
- BA II Plus Professional Bond Worksheet
- Formula-driven approach

I’ll let you solve the problem. The correct answer is: 3.61512007
Problem 7

<table>
<thead>
<tr>
<th>Bond face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>6% semiannual</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>4 years</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>8% annual effective.</td>
</tr>
<tr>
<td>Redemption</td>
<td>$92</td>
</tr>
</tbody>
</table>

Calculate the bond’s duration using the following method:
- BA II Plus/BA II Plus Professional Cash Flow Worksheet
- BA II Plus Professional Bond Worksheet
- Formula-driven approach

I’ll let you solve the problem. The correct answer is: 3.573273749

Problem 8

<table>
<thead>
<tr>
<th>Bond face</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupon</td>
<td>4% semiannual</td>
</tr>
<tr>
<td>Term to maturity</td>
<td>3 years</td>
</tr>
<tr>
<td>Yield to maturity</td>
<td>5% annual effective.</td>
</tr>
<tr>
<td>Duration</td>
<td>2.854</td>
</tr>
<tr>
<td>Convexity</td>
<td>10.205</td>
</tr>
<tr>
<td>Bond price</td>
<td>97.41</td>
</tr>
</tbody>
</table>

Calculate the bond updated price if the yield to maturity changes to 6% annual effective.

Solution

We’ll calculate the new price twice. First time, we’ll use the duration and convexity. The second time, we’ll directly calculate the price without using the duration and convexity. We’ll compare the two results.

Find the new price using the duration and convexity:

\[
\frac{\Delta P}{P} \approx 1 \frac{dP}{dr} \Delta r + \frac{1}{2} \frac{d^2P}{dr^2}(\Delta r)^2 = -\frac{1}{1+r} D_{MAC} \Delta r + \frac{1}{2} (Convexity)(\Delta r)^2
\]

We have:

\[ r = 5\%, \; \Delta r = 1\%, \; P = 97.41, \; D_{MAC} = 2.854, \; Convexity = 10.205 \]
\[
\frac{\Delta P}{P} \approx - \frac{1}{1 + r} D_{MAC} \Delta r + \frac{1}{2} \left(\text{Convexity}\right) \left(\Delta r\right)^2
\]

\[
\frac{\Delta P}{P} \approx - \frac{1}{1 + 5\%} (2.854)1\% + \frac{1}{2} (10.205)(1\%)^2 = -2.667\%
\]

\[
\Rightarrow \quad \Delta P \approx P(-2.667\%)
\]

\[
\Rightarrow \quad P = P + \Delta P \approx P(1 - 2.667\%) = 97.41(1 - 2.667\%) = 94.81
\]

If we ignore the convexity, we’ll have:

\[
\frac{\Delta P}{P} \approx - \frac{1}{1 + 5\%} D_{MAC} \Delta r = - \frac{1}{1 + 5\%} (2.854)1\% = -2.718\%
\]

\[
\Rightarrow \quad P = P + \Delta P \approx P(1 - 2.718\%) = 97.41(1 - 2.718\%) = 94.76
\]

Next, we directly calculate the new price:

\[
P' = 2a_{6|} + 100i^6 \quad \text{ @ } i = \sqrt{1 + 6\%} - 1 = 2.9563\%
\]

\[
\Rightarrow \quad P = 2a_{6|} + 100i^6 = 94.81
\]

We see that for a small change of the yield, the duration and convexity are good at predicting the change of the bond price.

**Problem 9  (SOA May 2003 Course 6 #5 simplified)**

You are given the following with respect to a five-year bond:
- annual coupons of \((2+t)\%\) are payable at the end of each year
- par value of $1,000
- yield-to-maturity \((y)\) of 5.5%

Calculate the Macaulay duration.

**Solution**

\[
D_{MAC} = \frac{\sum_{t=1}^{5} t \, CF(t) \, v^t}{\sum_{t=1}^{5} CF(t) \, v^t}
\]
Method #1 – Use Cash Flow Worksheet:

Cash flow table:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$CF(t)$</th>
<th>$t CF(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2+1)%1,000=30</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>(2+2)%1,000=40</td>
<td>80</td>
</tr>
<tr>
<td>3</td>
<td>(2+3)%1,000=50</td>
<td>150</td>
</tr>
<tr>
<td>4</td>
<td>(2+4)%1,000=60</td>
<td>240</td>
</tr>
<tr>
<td>5</td>
<td>(2+5)%1,000+1,000=1,070</td>
<td>5,350</td>
</tr>
</tbody>
</table>

First, we’ll find the bond’s price. We enter $CF(t)$ into Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$CF(t)$</th>
<th>Cash flows</th>
<th>Cash flow frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$30$</td>
<td>CF1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$40$</td>
<td>CF2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$50$</td>
<td>CF3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$60$</td>
<td>CF4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>$1,070$</td>
<td>CF5</td>
<td>1</td>
</tr>
</tbody>
</table>

Interest Rate per coupon period $i = 5.5\%$

Calculate NPV $\sum_{t=1}^{5} CF(t) v^t = 974.0815620$

$\Rightarrow \sum_{t=1}^{5} t CF(t) v^t = 974.0815620$

To calculate $\sum_{t=1}^{5} t CF(t) v^t$, we enter $t CF(t)$ (bold numbers below) into Cash Flow Worksheet:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$CF(t)$</th>
<th>$t CF(t)$</th>
<th>Cash flows</th>
<th>Cash flow frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2+1)%1,000=30</td>
<td>$30$</td>
<td>CF1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(2+2)%1,000=40</td>
<td>$80$</td>
<td>CF2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>(2+3)%1,000=50</td>
<td>$150$</td>
<td>CF3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>(2+4)%1,000=60</td>
<td>$240$</td>
<td>CF4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>(2+5)%1,000+1,000=1,070</td>
<td>$5,350$</td>
<td>CF5</td>
<td>1</td>
</tr>
</tbody>
</table>

Interest Rate per coupon period $i = 5.5\%$

Calculate NPV $4,515.255073$
\[
\Rightarrow \sum_{t=1}^{5} t \cdot CF(t) \cdot v^t = 4,515.255073
\]

\[
\Rightarrow D_{MAC} = \frac{\sum_{t=1}^{5} t \cdot CF(t) \cdot v^t}{\sum_{t=1}^{5} CF(t) \cdot v^t} = \frac{4,515.255073}{974.0815620} = 4.63539733
\]

**Method #2 – directly calculate**

\[
D_{MAC} = \frac{\sum_{t=1}^{5} t \cdot CF(t) \cdot v^t}{\sum_{t=1}^{5} CF(t) \cdot v^t}
\]

\[
\sum_{t=1}^{5} t \cdot CF(t) \cdot v^t = \frac{30}{1.055} + \frac{40}{1.055^2} + \frac{50}{1.055^3} + \frac{60}{1.055^4} + \frac{1070}{1.055^5} = 974.0815620
\]

\[
\sum_{t=1}^{5} CF(t) \cdot v^t = \frac{1(30)}{1.055} + \frac{2(40)}{1.055^2} + \frac{3(50)}{1.055^3} + \frac{4(60)}{1.055^4} + \frac{5(1070)}{1.055^5} = 4,515.255073
\]

\[
\Rightarrow D_{MAC} = \frac{\sum_{t=1}^{5} t \cdot CF(t) \cdot v^t}{\sum_{t=1}^{5} CF(t) \cdot v^t} = \frac{4,515.255073}{974.0815620} = 4.63539733
\]

Please note that for this problem we can NOT use the bond worksheet in BA II Plus Professional to calculate the duration; we don’t have a standard bond here.

**Problem 10**

What’s the duration of a 5 year zero coupon bond?

[A] 4 years
[B] 5 years
[C] 6 years
[D] 7 years
[E] 8 years
Solution

The duration of a zero coupon bond is always equal to its maturity. The correct answer is [B]

Problem 11

What’s the modified duration of a 20 year zero coupon bond, if the yield to maturity is 10% annual effective?

Solution

\[
D_{MOD}^{Textbook} = \frac{1}{1 + \text{annual effective yield}} D_{MAC} = \frac{1}{1.1} D_{MAC} = \frac{20}{1.1} = 18.18 \text{ (years)}
\]

Since the coupons (which happen to be zero) are paid annually, the nominal yield to maturity compounding annually is the same as the annual effective yield. Consequently, the modified duration under the Wall Street method is the same as the one under the textbook method.

\[
D_{MOD}^{Wall Street} = \frac{1}{1 + \text{annual nominal yield}} D_{MAC} = \frac{1}{1.1} D_{MAC} = \frac{20}{1.1} = 18.18 \text{ (years)}
\]

Problem 12

8 years ago, Mark bought a 15 zero coupon bond. Today, while the bond still has 7 years to maturity, Mark sold it to John.

\[X = \text{the duration of Mark's bond.}\]
\[Y = \text{the duration of John's bond.}\]

Calculate \(X - Y\).

Solution

The duration of a zero coupon bond is always its maturity. So \(X = 15\) years and \(Y = 7\) years. \(X - Y = 8\).
Problem 13

<table>
<thead>
<tr>
<th>Cash flows</th>
<th>perpetual annuity immediate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest rate</td>
<td>5% annual effective</td>
</tr>
</tbody>
</table>

Calculate the duration of the cash flows.

Solution

The key formula is

\[
D_{MAC} = -\frac{dp}{p} \frac{d(1+r)}{1+r} = -(1+r)\left[\frac{1}{p} \frac{dp}{dr}\right]
\]

\[
P = \frac{1}{r}
\]

(This is the price of a perpetual annuity immediate)

\[
\Rightarrow \frac{dP}{dr} = \frac{d}{dr} \left(\frac{1}{r}\right) = -\frac{1}{r^2}
\]

\[
\Rightarrow D_{MAC} = -(1+r)\left[\frac{1}{p} \frac{dp}{dr}\right] = -(1+r)\left[\frac{1}{r} \left(\frac{1}{r^2}\right)\right] = \frac{1+r}{r} = 1 + \frac{1}{r}
\]

\[
\Rightarrow D_{MAC} = 1 + \frac{1}{r} = 1 + \frac{1}{5\%} = 21
\]

Problem 14

Calculate the duration and the convexity of the liability, given:

- The liability has a continuous payment stream of $5,000 per year over the next ten years
- The interest rate is 6% annual effective
Solution

For a continuous payment stream,

\[
\text{Duration} = \frac{\int_0^n t v'(CF(t))dt}{\int_0^n v'(CF(t))dt} = \frac{\int_0^n t v'(CF(t))dt}{P}
\]

\[
\text{Convexity} = \frac{1}{p} \frac{1}{(1+r)^2} \int_0^n t(t+1) CF(t) v'(dr) = \frac{1}{p} \frac{1}{(1+r)^2} \left[ \int_0^n t^2 CF(t) v'(dr) + \int_0^n t CF(t) v'(dr) \right]
\]

If \( CF(t) = c \) where \( c \) is constant for any \( t \), then

\[
\text{Duration} = \frac{c \int_0^n t v'(dt)}{c \int_0^n v'(dt)} = \frac{(\bar{a})_{\overline{n|}}}{\bar{a}_{\overline{n|}}} = \frac{(\bar{a})_{\overline{n|}}}{\bar{a}_{\overline{n|}}}
\]

\[
(\bar{a})_{\overline{10|}} = \frac{\bar{a}_{\overline{10|}} - 10v^{10}}{\delta}
\]

\[
\bar{a}_{\overline{10|}} = \frac{i}{\delta} a_{\overline{10|}} = \frac{6\%}{\ln 1.06} (7.36008705) = 7.57874546
\]

In the above, \( a_{\overline{10|}} \) is calculated using BA II Plus/BA II Plus Professional TVM.

\[
(\bar{a})_{\overline{10|}} = \frac{\bar{a}_{\overline{10|}} - 10v^{10}}{\delta} = \frac{7.57874546 - 10(1.06^{10})}{\ln 1.06} = 34.23434140
\]

\[
\text{Duration} = \frac{(\bar{a})_{\overline{10|}}}{\bar{a}_{\overline{10|}}} = \frac{34.23434140}{7.57874546} = 4.51715150
\]

Next, we’ll calculate the convexity.
Convexity = \frac{1}{(1+r)^2} \frac{1}{\bar{a}_m} \left[ \int_0^n t^2 \, CF(t) \, v^t \, dr + \int_0^n t \, CF(t) \, v^t \, dr \right]

Because the cash flow is constant, we have:

Convexity = \frac{1}{(1+r)^2} \frac{1}{\bar{a}_m} \left[ \int_0^n t^2 \, v^t \, dr + \int_0^n t \, v^t \, dr \right]

\Rightarrow \quad \int_0^n t^2 \, v^t \, dt = \frac{2(\bar{a}_m) - n^2 \, v^n}{\delta}

\Rightarrow \quad \int_0^{10} t^2 \, v^t \, dt = \frac{2(\bar{a}_m) - 10^2 \, v^n}{\delta} = \frac{2(34.23434140) - 10^2 \cdot 1.06^{-10}}{\ln 1.06} = 216.7400337

\Rightarrow \quad Convexity = \frac{1}{(1+r)^2} \frac{1}{\bar{a}_m} \left[ \int_0^n t^2 \, v^t \, dr + (\bar{a}_m) \right]

\Rightarrow \quad Convexity = \frac{1}{(1+r)^2} \frac{1}{\bar{a}_m} \left[ 216.7400337 + 34.23434140 \right]

Alternatively,

\Rightarrow \quad Convexity = \frac{1}{(1+r)^2} \left[ \frac{1}{P} \sum \frac{1}{n} \int_0^n t^2 \, CF(t) \, v^t \, dr + Duration \right]

\Rightarrow \quad Convexity = \frac{1}{1.06^2} \left[ \frac{216.7400337}{7.57874546} + 4.51715150 \right] = 29.47272758
Problem 15 (#6 May 2005 FM)

John purchased three bonds to form a portfolio as follows:

- Bond A has semi-annual coupons at 4%, a duration of 21.46 years, and was purchased for 980.
- Bond B is a 15-year bond with a duration of 12.35 years and was purchased for 1015.
- Bond C has a duration of 16.67 years and was purchased for 1000.

Calculate the duration of the portfolio at the time of purchase.

(A) 16.62 years  
(B) 16.67 years  
(C) 16.72 years  
(D) 16.77 years  
(E) 16.82 years

Solution

As a general rule, the duration (or convexity) of a portfolio is the weighted average duration (or convexity) of the assets, with weight being the present value of each asset.

\[
\text{portfolio duration} = \frac{\sum_{k=1}^{n} PV_k \times \text{Duration}_k}{\sum_{k=1}^{n} PV_k} \\
\text{portfolio convexity} = \frac{\sum_{k=1}^{n} PV_k \times \text{Convexity}_k}{\sum_{k=1}^{n} PV_k}
\]

Let’s prove this. To make our proof simple, let’s assume that a portfolio consists of two assets, \( A \) and \( B \). The proof is the same if we have more than two assets.
The duration of the portfolio consisting of two assets is:

\[ D = -(1+i) \left( \frac{1}{PV} \frac{d}{di} PV \right) \]

\[ PV = PV_A + PV_B \]

\[ D = -(1+i) \frac{1}{PV_A + PV_B} \frac{d}{di} (PV_A + PV_B) = -(1+i) \frac{d}{di} \left( \frac{PV_A + PV_B}{PV_A + PV_B} \right) \]

\[ = (1+i) PV_A \left( -\frac{1}{PV_A} \frac{d}{di} PV_A \right) + (1+i) PV_B \left( -\frac{1}{PV_B} \frac{d}{di} PV_B \right) \]

\[ = (1+i) \frac{PV_A}{PV_A + PV_B} - (1+i) \frac{PV_B}{PV_A + PV_B} \]

But \( -(1+i) \frac{d}{di} \frac{PV_A}{PV_A + PV_B} = D_A \), \( -(1+i) \frac{d}{di} \frac{PV_B}{PV_A + PV_B} = D_B \)

\[ \Rightarrow \quad D = \frac{PV_A D_A + PV_B D_B}{PV_A + PV_B} = \frac{PV_A}{PV_A + PV_B} D_A + \frac{PV_B}{PV_A + PV_B} D_B \]

Similarly, we can show that the convexity of the portfolio is the weighted average convexity with weights being the present value of each asset.

\[ C = -\frac{1}{PV} \frac{d^2}{di^2} PV \]

\[ PV = PV_A + PV_B \]

\[ C = -\frac{1}{PV_A + PV_B} \frac{d^2}{di^2} (PV_A + PV_B) = -\frac{d^2}{di^2} \left( \frac{PV_A + PV_B}{PV_A + PV_B} \right) \]

\[ = \frac{PV_A}{PV_A + PV_B} \left( -\frac{1}{PV_A} \frac{d^2}{di^2} PV_A \right) + PV_B \left( -\frac{1}{PV_B} \frac{d^2}{di^2} PV_B \right) \]

But \( -\frac{d^2}{di^2} \frac{PV_A}{PV_A} = C_A \), \( -\frac{1}{PV_B} \frac{d^2}{di^2} PV_B = C_B \)
\[ C = \frac{PV_A C_A + PV_B C_B}{PV_A + PV_B} = C_A + \frac{PV_B}{PV_A + PV_B} C_B \]

Come back to the problem.

<table>
<thead>
<tr>
<th>Asset</th>
<th>PV</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>980</td>
<td>21.46</td>
</tr>
<tr>
<td>#2</td>
<td>1,015</td>
<td>12.35</td>
</tr>
<tr>
<td>#3</td>
<td>1,000</td>
<td>16.67</td>
</tr>
</tbody>
</table>

The weighted average duration is:

\[
\frac{980(21.46) + 1,015(12.35) + 1,000(16.67)}{980 + 1,015 + 1,000} = 16.7733
\]

So the answer is [D]

**Problem #16**

A portfolio consists of the following three assets:

<table>
<thead>
<tr>
<th>Asset</th>
<th>PV</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>1,000</td>
<td>20</td>
</tr>
<tr>
<td>#2</td>
<td>1,500</td>
<td>32</td>
</tr>
<tr>
<td>#3</td>
<td>2,000</td>
<td>15</td>
</tr>
</tbody>
</table>

Calculate the portfolio’s convexity.

**Solution**

The weighted average convexity is:

\[
\frac{1,000(20) + 1,500(32) + 2,000(15)}{1,000 + 1,500 + 2,000} = 21.78
\]

So the portfolio’s convexity is 21.78.

**Problem #17 (SOA May 2000 EA-1 #8)**

\[ \delta = 0.07 \]
The modified duration of a 20-year bond with 7% annual coupons with a maturity and par value of $1,000 is $y$.

Calculate $y$.

**Solution**

\[ e^i = 1 + i, \quad i = e^i - 1 = e^{0.07} - 1 = 7.25\% \text{ (annual effective interest rate)} \]

\[ D_{MOD} = \frac{1}{1+i} D_{MAC} \]

Let’s first calculate $D_{MAC}$.

<table>
<thead>
<tr>
<th>time $t$ (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>…</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CF(t)$</td>
<td>$70$</td>
<td>$70$</td>
<td>$70$</td>
<td>$70$</td>
<td>$1,070$</td>
<td></td>
</tr>
</tbody>
</table>

\[
D_{MAC} = \frac{\sum_{t=1}^{20} t CF(t) v^t}{\sum_{t=1}^{20} CF(t) v^t} = \frac{70 (1a_{\overline{20}}) + (20)1,000v^{20}}{70 a_{\overline{20}} + 1,000v^{20}} = \frac{70(85.67388) + 4,931.93928}{973.9385} = 11.22
\]

\[
D_{MOD} = \frac{1}{1+i} D_{MAC} = 11.22 e^{-0.07} = 10.46
\]

**Problem #18 (SOA May 2001 EA-1 #13)**

Purchase date of a perpetuity: 1/1/2001

Date of the 1st payment: 12/31/2001

Frequency of payments: Annual

Amount of each payment: $1

Interest rate: 6% per year, compounded annually

Calculate the absolute value of the difference between the modified duration of the perpetuity and the present value of the perpetuity.

**Solution**
The present value of perpetuity immediate is \( P = \frac{1}{i} \).

The Macaulay duration of perpetuity immediate is:

\[
D^{MAC} = -(1 + i) \frac{dP}{di} = -(1 + i) i \left( \frac{1}{i} \right) = -(1 + i) \left( -\frac{1}{i^2} \right) = \frac{1 + i}{i}
\]

The modified duration is:

\[
D^{MOD} = \frac{D^{MAC}}{1 + i} = \frac{1}{1 + i} \times \frac{1 + i}{i} = \frac{1}{i}
\]

\[\Rightarrow \quad |P - D^{MOD}| = \left| \frac{1 - \frac{1}{i}}{i} \right| = 0\]
Chapter 17  Immunization

The need for immunization

- Insurance companies often have to pay prescheduled payments in the future.

- For example, some insurance companies offer CD like investment products called GIC (guaranteed investment contracts). In GIC, investor deposits cash to the GIC account, which earns a guaranteed interest rate for a specific period of time. For example, an investor deposited $10,000 today. The GIC offers a guaranteed interest rate of 8% for 5 years. Then at the end of Year 5, the insurance company must pay the investor $10,000(1+8%)^5=\$14,693.28.

- In this GIC, the insurance company promises to pay the investor 8% annual effective. So the insurance company needs to earn at least 8% annual effective to break even.

- How can the insurance company invest its collected deposit of $10,000 wisely so it can earn at least 8% for 5 years?

- At first glance, we might suggest that the insurance company buys, at time zero, a 5-year bond with 8% annual coupon and $10,000 face amount. Assume the current market interest rate is 8%. Then this bond costs exactly $10,000 at \(t=0\). And it generates exactly $14,693.28 at \(t=5\).

- Let’s calculate the total cash flow at \(t=5\) if we bought the bond at \(t=0\). Assume the market interest rate is 8%. Then we can reinvest each coupon of $800 at 8%.

- The total cash flow at \(t=5\): $800\sum_{t=0}^{5} v^t + 10,000 = 14,693.28$

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$800</td>
<td>$10,800</td>
<td></td>
</tr>
</tbody>
</table>

- After more analysis, however, we realize that buying this bond may not work. What if that immediately after we bought the bond, the
interest drops to 7% and stays at 7%? If this happens, our total accumulated cash flow at \( t = 5 \) is:

\[
800\times\left(\frac{1}{1.07}\right)^5 + 10,000 = 14,600.59 < 14,693.28
\]

- The accumulated value is lower because now we have to reinvest the coupons at 7% (instead of 8%).

- A general question arises. In many occasions, an insurance company has to pay its prescheduled payments in the future. However, the returns the insurance company gets from its investment are volatile, depending on the current market interest rate. If the interest rates are too volatile, the insurance company won’t have sufficient investment income to pay its fixed liabilities.

- How can the insurance company invest its collected premiums in such a way that no matter how the interest rate changes, it always has enough money to pay its prescheduled liabilities in the future?

- This led many actuaries to search for an investment strategy that will produce a guaranteed level of cash flows no matter what happens to the interest rate. One strategy is immunization.

**Basic ideas behind immunization**

- First, we don’t worry that the interest rate may change. At time zero, we simply hold assets that exactly offset liabilities. We make this happen by forcing \( PV \) of Assets = \( PV \) of Liabilities. This ensures that we have enough assets to pay our liabilities if the interest rate does not change.

- Next, we’ll consider the possibility that the interest rate may change. We can’t stop the interest rate from going up or down, but we can make our assets and liabilities equally sensitive to the change of the interest rate. If assets and liabilities can increase and decrease by roughly the same amount for a given change in the interest rate, then the market value of the asset and the market value of the liability will offset each other. As a result, our assets will be enough to pay our liabilities.

- To make our assets and liabilities equally sensitive to the change of the interest rate, we first force our assets and liabilities to have an equal duration. Duration is the 1st derivative of the interest rate. As a result, we want assets and liabilities to match duration.

- Next, we force our assets and liabilities have an equal second derivative (convexity) as to the interest rate. To be safe, we force
our assets to have a slightly bigger second derivative than our liabilities does. This ensures that for a given change in the interest rate, our assets change by a slightly bigger amount than our liabilities.

- We can follow this line of thinking and force assets and liabilities match 3rd or 4th derivatives to the interest rate. However, matching assets and liabilities in their 3rd or 4th derivatives is more difficult and more expensive than matching the 1st and 2nd derivatives.

**Sample problems and solutions**

**Problem 1**
You have a leaky roof. Every time it rains, water drips down from the ceiling. In the past every time you decided to have the roof fixed, something came up; you used up all the money you had and had no money left to repair the roof. Finally, you had enough. You vowed to repair the leaky roof.

These are the facts:
- According to the weather forecast, there will not be any rainfall in the next five years in your town. However, there will be a big rain 5 years and 10 days from today.
- You decide to have the roof fixed 5 years from now (i.e. at \( t = 5 \)).
- The cost of having the roof fixed at \( t = 5 \) is $10,000.
- Your only investment opportunity is in your local bank, which offers only two products -- saving accounts and CD (certified deposit). Besides your local bank, there are no other investment opportunities for you.
- A saving account in your local bank earns an interest rate adjustable once every 6 months. Every 6 months, the board of directors of your local bank sets an interest rate according to their whims. For example, on January 1 last year, the board of directors felt the economy was good and decided that all saving accounts should earn a 10% annual effective interest rate for the next 6 months. However, on July 1, the board of directors felt that 10% was too high and declared that all saving accounts should earn a 1% annual effective interest rate per year for the next 6 months.
CD offered by your local bank has a 5 year term and earns a guaranteed annual effective interest rate of 3%.

Not to repeat your past failures, you decide to immunize your cost of repairing the roof at $t=5$.

Explain what immunization means in this case. Design an immunization strategy.

**Solution**

Immunization in this case means that you invest money somewhere at $t=0$ to generate exactly $10,000 at $t=5$, no matter how the interest rate changes during the next five years. If you can have exactly $10,000 at $t=5$ no matter how low the interest can be, you have secured your future payment of $10,000 at $t=5$.

You have only 2 investment options – investing in a saving account or in a 5 year CD.

Investing in a saving account will not work. The interest rate you earn in a saving account is unpredictable for the next 5 years. Unless you deposit $10,000 at $t=0$, there’s no way to guarantee that you will have $10,000 at $t=5$.

You can invest your money in the 5 year CD. Because the 5 year CD offers a guaranteed interest rate of 3%, your initial deposit at $t=0$ should be: $10,000\left(1+\frac{3}{100}\right)^{-5} = 8,626.09$

If you invest $8,626.09 in the 5 year CD, you are guaranteed to have $10,000 at $t=5$. You have immunized your payment of $10,000 at $t=5$ against any adverse interest rate changes. In other words, you have locked in a guaranteed interest rate of 3% no matter how low the market interest rate can be.

**Comment one:**

If a saving account in your local bank offers a guaranteed interest rate, then you can invest in a saving account and immunize your future payment at $t=5$. For example, if the saving account offers a guaranteed a 2% annual effective interest rate, then your initial deposit at $t=0$ should be: $10,000\left(1+\frac{2}{100}\right)^{-5} = 9,057.31$

If you invest $9,057.31 in a saving account today, you are 100% certain to get $10,000 at $t=5$. You have immunized your payment of $10,000 at
\( t = 5 \). However, this immunization strategy is less favorable than if you invest in a 5 year CD.

**Comment Two:**
Immunization typically means investing money in bonds, not in a saving account or CD, because the interest rate generated by a saving account or CD is too low. In this problem, we use a saving account and CD as simple examples to illustrate the essence of immunization.

**Essence of immunization**

| When the interest is volatile, we need to find ways to lock in a guaranteed return and accumulate just enough money to exactly pay our future liability. |

**Problem 2**
You have a leaky roof. Every time it rains, water drips down from the ceiling. In the past every time you decided to have the roof fixed, something came up; you used up all the money you had and had no money left to repair the roof. Finally, you had enough. You vowed to repair the leaky roof.

These are the facts:

- According to the weather forecast, there will not be any rainfall in the next five years in your town. However, there will be a big rain 5 years and 10 days from today.

- You decide to have the roof fixed 5 years from now (i.e. at \( t = 5 \)).

- The cost of having the roof fixed at \( t = 5 \) is $10,000.

- The bond market offers a 5 year zero-coupon bond with $10,000 face amount yielding 6% annual effective.

Not to repeat your past failures, you decide to immunize your cost of repairing the roof at \( t = 5 \). Design an immunization strategy.

**Solution**
Your goal is to invest money somewhere today to accumulate exactly $10,000 at \( t = 5 \). If you buy this 5 year zero bond with $10,000 par value today, you are guaranteed to have $10,000 at \( t = 5 \).

The market price of the bond is: \( 10,000(1+6\%)^{-5} = 7,472.58 \)
To immunize your future payment of $10,000 at $t=5$, you should buy this 5 year zero bond at $7,472.58$ today. By doing so, you have locked in a 6% interest rate, no matter how low the market interest can be.

**Problem 3**

You have a leaky roof. Every time it rains, water drips down from the ceiling. In the past every time you decided to have the roof fixed, something came up: you used up all the money you had and had no money left to repair the roof. Finally, you had enough. You vowed to repair the leaky roof.

These are the facts:
- According to the weather forecast, there will not be any rainfall in the next five years in your town. However, there will be a big rain 5 years and 10 days from today.
- You decide to have the roof fixed 5 years from now (i.e. at $t=5$).
- The cost of having the roof fixed at $t=0$ is $10,000$. Due to inflation and the shortage of roof repair skills, the cost of labor and materials increases by 8% per year.
- To fund your repair cost, at $t=0$ you bought a 5 year $10,000 par value bond that offers annual coupons of 8%. The current market interest rate is also 8%.

Analyze whether your bond will generate sufficient fund to pay the repair cost $t=5$ under the following scenarios:

1. the market interest rate stays at 8% forever.
2. immediately after you have bought the bond, the market interest rate drops to 7.5% and stays at 7.5%.
3. immediately after you have bought the bond, the market interest rate rises to 8.5% and stays at 8.5%.

**Solution**

**Scenario 1 – the market interest rate stays at 8%**.

Let’s first calculate your repair cost at $t=5$. If you repair your roof now, you pay $10,000. If you repair it at $t=5$, you’ll pay

$$10,000(1+8\%)^5 = 14,693.28$$

Next, let’s see how much money your bond can accumulate at $t=5$. The bond pays you 5 annual coupons of $800 each. You can reinvest these
coupons at the market rate of 8% and accumulate to $800s_{\overline{5}\mid 8\%}$ at $t = 5$. In addition, you’ll get a payment of $10,000 at $t = 5$. Your total money at $t = 5$ is: 

$$800s_{\overline{5}\mid 8\%} + 10,000 = 14,693.28$$

Your bond will accumulate just enough money to pay your repair cost.

**Scenario 2 – immediately after you bought the bond, the market interest rate drops to 7.5% and stays at 7.5%.**

You still get 5 annual coupons each worth $800, but this time you can reinvest them only at 7.5%. Your accumulated value at $t = 5$ is:

$$800s_{\overline{5}\mid 7.5\%} + 10,000 = 14,646.71$$

Your repair cost at $t = 5$ is still $14,693.28$. Your shortfall: $14,693.28 - 14,647.71 = 45.57$

Why this time don’t you have enough money to pay your repair cost at $t = 5$? Because when the interest rate drops to 7.5%, you can no longer reinvest your coupons at 8%. However, to come up with $14,693.28, you must be able to reinvest your coupons at least at 8%.

**Scenario 3 – the market interest rate rises to 8.5% immediately after you bought the bond and stays at 8.5%.**

This time, you can reinvest your coupons at 8.5%. Your accumulated value at $t = 5$ is:

$$800s_{\overline{5}\mid 8.5\%} + 10,000 = 14,740.30$$

You’ll have more than enough to pay the repair cost at $t = 5$. Your surplus at $t = 5$ is

$$14,740.30 - 14,693.28 = 47.02$$

**Moral of this problem:**

When you first bought your asset (5 year 8% annual coupon with par $10,000), at the then market interest rate of 8%, your asset will generate the exact amount of money to pay your repair cost at $t = 5$. However, if the interest rate goes up or down after you bought your asset, your asset will accumulate more or less money than your payment at $t = 5$. While you are happy if you end up with more than you need at $t = 5$, you’ll be sad if you incur a loss at $t = 5$. 
Now imagine a bank or an insurance company that must pay $10,000,000,000 five years from now. If the company does not diligently manage the volatility of the interest rate, it may lose millions of dollars. This is why immunization is important.

**Problem 4**

You want to invest your money now to have \(10,000(1+8\%)^5=14,693.28\) at \(t=5\) to fix your leaky roof. Originally, you were thinking of buying a 5 year bond with $10,000 par and 8% annual coupons. Then you did some math (like in Problem 3). You realized that if the interest rate drops immediately after you bought the bond, you won’t be able to have $14,693.28 at \(t=5\).

Then an immunization wizard advises you to buy a 6 year bond with $10,000 par and 8% annual coupons. He says that if you buy this bond, even if the interest rate rises or drops a bit, you will always be able to accumulate $14,693.28 at \(t=5\).

Test the validity of the wizard’s advice under the following scenarios:

- Immediately after you have bought the bond, the market interest rate drops to 7.5% and stays at 7.5%.
- Immediately after you have bought the bond, the market interest rate rises to 8.5% and stays at 8.5%.

Explain your findings.

**Solution**

Let’s calculate the accumulated value at \(t=5\) under the two scenarios:

**Scenario 1 - Immediately after you have bought the bond, the market interest rate drops to 7.5% and stays at 7.5%.

Your money at \(t=5\) comes from two sources. First, you can reinvest your annual coupons at 7.5% and let them accumulate to \(t=5\). Second, you can sell your bond at \(t=5\). At \(t=5\), your bond still has 1 year term left with an incoming cash flow of $10,800 at \(t=6\). You can sell this cash flow at 7.5% interest rate.

Your total money at \(t=5\) is:

\[
800s_{\overline{5}|7.5\%} + \frac{10,800}{1.075} = 14,693.22 \approx 14,693.28
\]
**Scenario 2 - Immediately after you have bought the bond, the market interest rate drops to 8.5% and stays at 8.5%.**

Your total money at $t = 5$ is:

$$800s_{\overline{5}|8.5\%} + \frac{10,800}{1.085} = 14,694.22 \approx 14,693.28$$

Let’s analyze why you can always accumulate $10,000(1+8\%)^5 = 14,693.28$ at $t = 5$ no matter the interest rate rises or falls.

<table>
<thead>
<tr>
<th>You bought a 6 year bond with $10,000 par and 8% annual coupons</th>
<th>The accumulated value at $t = 5$ by reinvesting coupons at the market interest rate.</th>
<th>The sales price of the bond’s remaining cash flow (a zero coupon bond with cash flow of $10,800 one year from now)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scenario #1</strong></td>
<td>$800s_{\overline{5}</td>
<td>8%} = 4,693.28$</td>
<td>$10,800/1.08 = 10,000$ We sell the remaining zero bond at par.</td>
</tr>
<tr>
<td>The market interest rate stays at 8% (this should exactly accumulate the needed amount)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Scenario #2</strong></td>
<td>$800s_{\overline{5}</td>
<td>7.5%} = 4,646.71$</td>
<td>$10,800/1.075 = 10,046.51$ We sell the remaining zero bond at a premium of $46.51.</td>
</tr>
<tr>
<td>Immediately after you bought the bond, the market interest rate dropped to 7.5% and stayed at 7.5%</td>
<td><strong>Decrease</strong> $= 4,693.28 - 4,646.71 = 46.57$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Scenario #3</strong></td>
<td>$800s_{\overline{5}</td>
<td>8.5%} = 4,740.30$</td>
<td>$10,800/1.085 = 9,953.92$ We sell the remaining zero bond at a discount of $47.02.</td>
</tr>
<tr>
<td>Immediately after you bought the bond, the market interest rate rose to 8.5% and stayed at 8.5%</td>
<td><strong>Increase</strong> $= 4,740.30 - 4,693.28 = 47.02$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Moral of this problem:**

When you buy a bond and sell it before its maturity, you have two sources of incomes: the accumulated value of coupons reinvested at the market interest rate and the sales price of the bond’s remaining cash flows discounted at the market interest rate. These are two opposing forces.

If the market interest rate goes down and stays low immediately after you buy a bond, you reinvest your coupons at a lower interest rate and incur a loss. At the same time, however, you can sell the bond’s remaining cash flows discounted at a lower interest rate and have a gain.

If the market interest rate goes up and stays high immediately after you buy a bond, you can reinvest your coupons at a higher interest rate and have a gain. At the same time, however, you sell the bond’s remaining cash flows discounted at a higher interest rate and incur a loss.

If you can find the right bond and hold it for the right amount of time, you can make these two opposing forces exactly offset each other, accumulating a guaranteed amount of money no matter the interest rate goes up or down. This is how immunization works.

The key is to find the right bond and hold it for the right period of time. How? This is our next problem.

**Problem 5**

State the 3 requirements of immunization. Verify that the 3 requirements are satisfied in the following immunization arrangement:

<table>
<thead>
<tr>
<th>Asset</th>
<th>6 year bond with $10,000 par and 8% annual coupons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability</td>
<td>$10,000(1+8%)^5 = 14,693.28 \text{ at } t = 5.</td>
</tr>
<tr>
<td>Interest rate</td>
<td>8%</td>
</tr>
</tbody>
</table>

**Solution**

The 3 requirements of immunization:

- PV Assets = PV Liabilities
- Duration of Assets = Duration of Liabilities
- Convexity of Asset > Convexity of Liabilities
If these 3 conditions are met, then the assets will generate sufficient cash flows to pay the future known liabilities, even though the interest rate might uniformly increase or uniformly decrease by a small amount.

Next, let’s verify that the 3 immunization requirements are met.

(1) PV Assets = PV of 6 year bond with $10,000 par and 8% annual coupons discounted at 8%.

Without doing any calculations (with or without a calculator), you should immediately know that PV of a 6 year bond with $10,000 par and 8% annual coupons discounted at 8% is $10,000. As a general rule, for any annual coupon bond, if the discount rate is equal to the annual coupon rate, the PV of the bond is the par value.

PV Liabilities = $10,000.

⇒ PV Assets = PV Liabilities

(2) Next, we check whether Duration of Assets = Duration of Liabilities. We’ll use the general formula:

\[
\text{Duration} = \frac{\sum_{t=1}^{n} TVFCF(t)}{\sum_{t=1}^{n} v^t CF(t)}, \text{ where } CF(t) \text{ stands for a cash flow at } t.
\]

<table>
<thead>
<tr>
<th>Asset</th>
<th>Time t</th>
<th>CF(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>800</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>800</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>800</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>800</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10,800</td>
<td></td>
</tr>
</tbody>
</table>

Duration of Assets:

\[
\frac{\sum_{t=1}^{6} TVFCF(t)}{\sum_{t=1}^{6} v^t CF(t)} = \frac{(1)800v + (2)800v^2 + (3)800v^3 + (4)800v^4 + (5)10,800v^5}{800v + 800v^2 + 800v^3 + 800v^4 + 10,800v^5}
\]

\[
\approx \frac{49,927}{10,000} = 4.9927 \approx 5
\]
Duration of liability (only one cash flow of $10,000(1+8\%)^3 = 14,693.28$ at $t = 5$):

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>$CF(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10,000(1+8%)^3</td>
</tr>
</tbody>
</table>

$$\text{Duration} = \frac{\sum_{t=5}^5 t^4 CF(t)}{\sum_{t=5}^5 t^4 CF(t)} = \frac{(5)10,000(1+8\%)^5 (1+8\%)^{-5}}{10,000(1+8\%)^5 (1+8\%)^{-5}} = 5$$

Generally, the duration of a single cash flow (such as a zero-coupon bond) at time $t$ is $t$.

$\Rightarrow$ Duration of Assets = Duration of Liabilities.

(3) Let check whether Convexity of Assets > Convexity of Liabilities

$$\text{Convexity} = v^2 \left[ -\frac{1}{P} \sum_{t=1}^n t^2 CF(t) v^t + \text{Duration} \right] = v^2 \left[ \frac{\sum_{t=1}^n t^2 CF(t)}{\sum_{t=1}^n v^t CF(t)} + \text{Duration} \right]$$

Because $\text{Duration}^{\text{Asset}} = \text{Duration}^{\text{Liability}}$ (we already checked this), to make $\text{Convexity}^{\text{Asset}} > \text{Convexity}^{\text{Liability}}$, we just need to

$$\frac{\sum_{t=1}^n t^2 \text{Asset}CF(t)}{\sum_{t=1}^n v^t \text{Asset}CF(t)} > \frac{\sum_{t=1}^m t^2 \text{Liability}CF(t)}{\sum_{t=1}^m v^t \text{Liability}CF(t)}$$

In the above equation, $n$ is the term to maturity of the asset and $m$ is the term to maturity of the liability.

In other words, if assets and liabilities already have an equal duration, then by making
\[
\frac{\sum_{t=1}^{n} t^2 \text{AssetCF}(t)}{\sum_{t=1}^{n} v^t \text{AssetCF}(t)} > \frac{\sum_{t=1}^{m} t^2 \text{LiabilityCF}(t)}{\sum_{t=1}^{m} v^t \text{LiabilityCF}(t)}
\]

We’ll surely have

\[\text{Convexity}^{\text{Asset}} > \text{Convexity}^{\text{Liability}}\]

\[
\frac{\sum_{t=1}^{n} t^2 C(t)}{\sum_{t=1}^{n} v^t C(t)}
\]
is called the Macaulay convexity.

Let’s summarize the above discussion.

**Original conditions for immunization**

- PV asset = PV of liability
- Duration of asset = Duration of liability
- Convexity of asset > Convexity of liability

**Revised conditions for immunization**

- PV asset = PV of liability
- Duration of asset = Duration of liability
- Macaulay convexity of asset > Macaulay convexity of liability

So in the future, we will use the revised conditions to check whether the immunization stands true; it’s easier to calculate the Macaulay convexity than to calculate the convexity. Let’s check:

For assets:

\[
\text{Macaulay Convexity}^{\text{Asset}} = \frac{\sum_{t=1}^{6} t^2 v^t \text{CF}(t)}{\sum_{t=1}^{6} v^t \text{CF}(t)}
\]

\[
= \frac{(1^2)800v + (2^2)800v^2 + (3^2)800v^3 + (4^2)800v^4 + (5^2)10,800v^5}{800v + 800v^2 + 800v^3 + 800v^4 + 10,800v^5}
\]

\[
\approx \frac{277,229.8141}{10,000} \approx 27.72
\]
For liability:

\[
Macaulay Convexity_{\text{Liability}} = \frac{\sum t^2 v^t CF(t)}{\sum v^t CF(t)} = 5^2 = 25
\]

(because liability has only one cash flow at \(t=5\))

\[\Rightarrow \quad Macaulay Convexity_{\text{Assets}} > Macaulay Convexity_{\text{Liability}}\]

\[\Rightarrow \quad Convexity_{\text{Assets}} > Convexity_{\text{Liabilities}}\]

So Convexity of Assets > Convexity of Liabilities.

As a shortcut, to check whether the convexity of assets exceeds the convexity of liabilities, you often don’t need to calculate the convexity of assets and the convexity of liabilities. You can simply draw the cash flows of assets and the cash flows of liabilities. If you can visually see that the cash flows of assets are more spread out than the cash flows of liabilities, then the convexity of assets will exceeds the convexity of liabilities.
You can see that asset cash flows are more spread out than liabilities cash flows. As a result, the convexity of assets is greater than the convexity of the liabilities.

Why so? The answer lies in our reinterpretation of the meaning of duration and convexity.
$$\textit{Duration} = \frac{\sum_{t=1}^{n} t v^t CF(t)}{\sum_{t=1}^{n} v^t CF(t)} = \sum_{t=1}^{n} t \left[ \frac{v^t CF(t)}{\sum_{t=1}^{n} v^t CF(t)} \right] = E(t)$$

$$\textit{Macaulay Convexity} = \frac{\sum_{t=1}^{n} t^2 v^t CF(t)}{\sum_{t=1}^{n} v^t CF(t)} = \sum_{t=1}^{n} t^2 \left[ \frac{v^t CF(t)}{\sum_{t=1}^{n} v^t CF(t)} \right] = E(t^2)$$

$$\textit{Spread out Varance} = \variance(t) = E(t^2) - E^2(t)$$

In other words, duration is the weighted average time of the cash flows, with weights being the present value of each cash flow. Macaulay convexity is the weighted average time squared of the cash flows, with weights being the present value of each cash flow. How cash flows spread out can be measured by variance of the time of the cash flows.

Then we can see that if the duration of asset is equal to the duration of liability and the asset cash flows are more spread out (i.e. having a bigger variance) than the liabilities cash flows, the convexity of the asset is greater than the convexity of the liability.

Mathematically, this is:

If \( E_{\text{Asset}}(t) = E_{\text{Liabililty}}(t) \),

If \( \variance_{\text{Asset}}(t) > \variance_{\text{Liabililty}}(t) \)

Then \( E_{\text{Asset}}(t^2) > E_{\text{Liabililty}}(t^2) \)

Because \( E(t^2) = E^2(t) + \variance(t) \)

Then \( \text{Convexity}_{\text{Asset}} > \text{Convexity}_{\text{Liabililty}} \)
**Problem 6**

Explain why if the following 3 conditions are met:
- PV Assets = PV Liabilities
- Duration of Assets = Duration of Liabilities
- Convexity of Asset > Convexity of Liabilities

then even if the interest rate uniformly increases or uniformly decreases by a small amount, the present value of the asset will be greater than the present value of the liability (i.e. you have immunized against a small change of the interest rate).

**Solution**

One easy way to understand this is to use Taylor series. Assume the current interest rate is \( r \). Immediately after you purchase an asset (such as a bond), the interest rate changes by \( \Delta r \). In other words, the new interest rate is \( r + \Delta r \) immediately after you purchase the asset.

\[
P(r) = \sum_{t=1}^{n} CF(t) v^t = \sum_{t=1}^{n} \frac{CF(i)}{(1+r)^t}
\]

\[
\Rightarrow \Delta P \approx \frac{dP}{dr} \Delta r + \frac{1}{2} \frac{d^2P}{dr^2} (\Delta r)^2 \quad \text{(Taylor series)}
\]

\[
\Rightarrow \frac{\Delta P}{P} \approx \frac{1}{P} \frac{dP}{dr} \Delta r + \frac{1}{2} \frac{1}{P} \frac{d^2P}{dr^2} (\Delta r)^2 = -\frac{1}{1+r} D_{MAC} \Delta r + \frac{1}{2} (\text{Convexity})(\Delta r)^2
\]

\[
\Rightarrow \frac{\Delta P_{\text{Asset}}}{P_{\text{Asset}}} \approx -\frac{1}{1+r} (\text{Asset Duration}) \Delta r + \frac{1}{2} (\text{Asset Convexity})(\Delta r)^2
\]

\[
\Rightarrow \frac{\Delta P_{\text{Liability}}}{P_{\text{Liability}}} \approx -\frac{1}{1+r} (\text{Liability Duration}) \Delta r + \frac{1}{2} (\text{Liability Convexity})(\Delta r)^2
\]

if the 3 conditions of immunization are satisfied, we have:

\[
P_{\text{Asset}} = P_{\text{Liab}}
\]

Asset Duration = Liability Duration
Asset Convexity > Liability Convexity

\[
\Rightarrow \frac{\Delta P_{\text{Asset}}}{P_{\text{Asset}}} > \frac{\Delta P_{\text{Liability}}}{P_{\text{Liability}}}
\]

\[
\Rightarrow \Delta P_{\text{Asset}} > \Delta P_{\text{Liability}}
\]
\[
\Rightarrow P'_{\text{Asset}} = P^{\text{Asset}} + \Delta P^{\text{Asset}} \quad > \quad P'_{\text{Liability}} = P^{\text{Liability}} + \Delta P^{\text{Liability}}
\]

So if the interest rate changes, the PV of the asset is greater than the PV of the liability.

**Problem 7**

Besides satisfying the 3 conditions

- PV Assets = PV Liabilities
- Duration of Assets = Duration of Liabilities
- Convexity of Asset > Convexity of Liabilities

immunization implicitly assumes that the following standards are met:

- both the timing and the dollar amount of each liability cash flows must be 100% known in advance
- the interest rate change must be small
- the interest rate change takes place immediately after the asset is purchased
- the interest rate change must be uniform (i.e. the same increase is applied to any time \( t \), resulting a constant discount rate \( i_0 + \Delta i \)).

Explain why immunization implicitly assumes so.

**Solution**

If either the timing or the dollar amount of a liability cash flow is not 100% certain at \( t=0 \), then we don’t know when we need to pay our bill. Hence, we don’t have a clear goal to start immunization.

For example, if you don’t know when you are going to fix your roof (timing unknown), or if you don’t know how much it will cost you to fix the roof at \( t=5 \) (amount unknown), you really don’t know how much money you need to invest now. Immunization can’t be used.

All the other implicit assumptions must be met for the Taylor series to stand. For example, if \( \Delta i \) is not small (for example, if \( \Delta i =500\% \)), or if \( \Delta i \) is not a constant, or if \( \Delta i \) doesn’t take place immediately after \( t = 0 \), the Taylor series will not stand.
Chapter 18  Cash flow matching

Why to match liability cash flows?
- An alternative strategy to immunization
  - Like immunization, cash flow matching shields us from adverse interest rate changes so we have enough money to pay bills.
  - If assets have exactly the same cash flows as do the liabilities, no matter how the interest rate changes in the future, we are guaranteed to have just enough cash to pay our liabilities.
  - If assets’ cash flows perfectly match liability cash flows, we are 100% safe against any interest rate changes. In contrast, immunization shields us against only a small parallel interest rate shift.
  - However, perfectly matching liability cash flows is (a) impossible -- bonds exceeding 30 years maturity are hard to find; (b) or prohibitively expensive

Matching procedures (working backwards, starting from the final liability cash flow to the earliest liability cash flow)
1. Match the final liability cash flow by buying the following bond
   - The bond matures at the same time when the final liability cash flow is due.
   - The bond’s final coupon plus the redemption value is equal to the final liability cash flow.
2. Remove the matching asset’s cash flows (periodic coupons plus a final redemption value) from the liability cash flows. Throw away our first matching asset. It has done its share and won’t be needed any more. Now the # of available assets for us to match the remaining liability cash flows is reduced by one.
3. After the 1st matching asset’s cash flows are removed, the final liability cash flow becomes zero; the next-to-final liability cash flow pops up and becomes the final liability cash flow. Then, we apply Step 1 and Step 2 to this new final liability cash flow.
4. Repeat Step 3 until the earliest liability cash flow is matched.
Sample problems and solutions

Problem 1 (Sample FM #51, #52, and #53)
The following information applies to questions 51 through 53.

Joe must pay liabilities of 1,000 due 6 months from now and another 1,000 due one year from now.

There are two available assets:
- A 6-month bond with face amount of 1,000, 8% nominal annual coupon rate convertible semiannually, and a 6% nominal annual yield rate convertible semiannually;
- A 1-year bond with face amount of 1,000, a 5% nominal annual coupon rate convertible semiannually, and a 7% nominal annual yield rate convertible semiannually.

Q #51 - How much of each bond should Joe purchase in order to exactly (absolutely) match the liabilities?

Q #52 - What’s Joe’s total cost of purchasing the bonds required to exactly (absolutely) match the liabilities?

Q #53 - What’s the annual effective yield rate for investing the bonds required to exactly (absolutely) match the liabilities?

Solution
Q #51 – buy bonds to match the liability cash flows.

As always, we draw cash flow diagrams for our liability and two bonds.

<table>
<thead>
<tr>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t</td>
</tr>
<tr>
<td>Cash flow</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bond #1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time t</td>
</tr>
<tr>
<td>Cash flow</td>
</tr>
</tbody>
</table>

\[ PV = 1,040 \left(1 + \frac{6\%}{2}\right)^{-1} = 1,009.7087 \quad YTM = i^{(2)} = 6\% \]
Let's combine the three diagrams into a table:

<table>
<thead>
<tr>
<th></th>
<th>Selling price t=0</th>
<th>Cash flow at t=0.5</th>
<th>Cash flow at t=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
<tr>
<td>Bond #1</td>
<td>$1,009.7087</td>
<td>$1,040</td>
<td>$0</td>
</tr>
<tr>
<td>Bond #2</td>
<td>$981.0031</td>
<td>$25</td>
<td>$1,025</td>
</tr>
</tbody>
</table>

Procedure to match liability cash flows:

Step 1 – Match the final liability cash flow.
The final liability cash flow is $1,000 at \( t=1 \). Because Bond #1 doesn’t have any cash flows at \( t=1 \), we can’t use it to match the final cash flow. So we are left with Bond #2, which produces $1,025 at \( t=1 \).

Assume we buy \( x \) units of Bond #2. To match the final liability cash flow, we need to have:

\[
x \times 1,025 = 1,000 \quad \Rightarrow \quad x = \frac{1,000}{1,025} = 0.97561
\]

<table>
<thead>
<tr>
<th></th>
<th>Selling price t=0</th>
<th>Cash flow at t=0.5</th>
<th>Cash flow at t=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
<tr>
<td>Bond #1</td>
<td>$1,009.7087</td>
<td>$1,040</td>
<td>$0</td>
</tr>
<tr>
<td>( x ) (Bond #2)</td>
<td>( 981.0031 ) ( \frac{1,000}{1,025} )</td>
<td>( 25 ) ( \frac{1,000}{1,025} )</td>
<td>( 1,025 ) ( \frac{1,000}{1,025} )</td>
</tr>
<tr>
<td>( x ) (Bond #2)</td>
<td>( 957.0762 ) ( \frac{1,000}{1,025} )</td>
<td>( 24.3902 ) ( \frac{1,000}{1,025} )</td>
<td>( 1,000 ) ( \frac{1,000}{1,025} )</td>
</tr>
</tbody>
</table>
Step 2 – Remove the cash flows of the matched asset from the liability cash flows. Throw away Bond #2. Now we have only Bond #1 left.

<table>
<thead>
<tr>
<th></th>
<th>Selling price t=0</th>
<th>Cash flow at t=0.5</th>
<th>Cash flow at t=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability – X(Bond 2)</td>
<td></td>
<td>$1,000-24.3902</td>
<td>$1,000-$1,000</td>
</tr>
<tr>
<td>Bond #1</td>
<td>$1,009.7087</td>
<td>$1,040</td>
<td>$0</td>
</tr>
</tbody>
</table>

Now, the liability cash flow at t=0.5 becomes the final liability cash flow.

Step 3 – Match the current final cash flow of $975.6098 at t=0.5.

Assume we buy Y units of Bond #1. Then to match the final liability cash flow, we need to have:

\[ Y(1,040) = 975.6098 \quad \Rightarrow Y = 0.9381 \]

<table>
<thead>
<tr>
<th></th>
<th>Selling price t=0</th>
<th>Cash flow at t=0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability – X(Bond 2) - Y(Bond #1)</td>
<td>$1,009.7087(0.9381)</td>
<td>$1,040(0.9381)</td>
</tr>
<tr>
<td></td>
<td>$975.6098</td>
<td>$947.2077</td>
</tr>
</tbody>
</table>

After matching the liability cash flow at t=0.5, we need to throw away Bond #1. Now we have no assets left. Fortunately, we have no liability cash flows left either. All of the liability cash flows are matched.

To match our liability cash flows, we need to buy \( X = 0.9756 \) units of Bond #2 and \( Y = 0.9381 \) units of bond #1; we assume that we can buy a fractional bond.

Next, we’ll find the cost of the matching assets. This should be easy.

\[ X (PV \text{ Bond #2}) + Y (PV \text{ Bond #1}) = 957.0762 + 947.2077 = 1,904.2839 \]
Finally, we'll calculate annual effective yield for investing $X$ units of Bond #2 and $Y$ units of Bond #1. We'll draw a cash flow diagram:

Matching Asset = $X (\text{Bond #2}) + Y (\text{Bond #1})$

<table>
<thead>
<tr>
<th>Time t</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matching asset's cash flows</td>
<td></td>
<td>$1,000$</td>
<td>$1,000$</td>
</tr>
</tbody>
</table>

$PV$ of the matching asset = 1,904.2839

We need to solve the following equation:

$$1,904.2839 = 1,000a_{\frac{1}{2}}i \Rightarrow i = 3.33269524\%$$

Next, we'll convert $i = 3.33269524\%$ (interest rate per 6-month) into an annual effective rate:

$$(1 + i)^2 - 1 = (1 + 3.33269524\%)^2 - 1 = 6.77645906\%$$

**Problem 2 (SOA Course 6, #9, May 2001)**

You are given the following information:

Projected liability cash flows

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>43</td>
<td>123</td>
<td>214</td>
<td>25</td>
<td>275</td>
</tr>
</tbody>
</table>

Available assets for investment:
- 2-year bond with annual coupon of 5%
- 3-year bond with annual coupon of 8%
- 5-year bond with annual coupon of 10%

Face amount of the bond: 100
Current market yield curve: 7% for all durations

Calculate the initial cost to cash-flow match the projected liability cash flows utilizing the assets listed above.

**Solution**
We can use the standard backward matching method to solve this problem. We first match the liability cash flow at Year 5. Then we move on to Year 4, 3, 2, and 1. This is the method used in the SOA official solution. Please download the SOA official solution. Make sure you can recreate the solution.

However, we’ll use a quicker method to solve this problem. Notice that the three bonds have an identical yield of 7% at all durations. As a result, we can discount all the cash flows of our constructed matching asset (which is a mixture of bond #1, #2, and #3) at 7%.

The constructed matching asset will have the same cash flows as does the liability. So we really don’t need to know how to construct the matching asset. No matter how we mix the three bonds, the resulting matching asset will surely have the following cash flows:

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>43</td>
<td>123</td>
<td>214</td>
<td>25</td>
<td>275</td>
</tr>
</tbody>
</table>

As a result, the cost of the matching asset is simply the above cash flows discounted at 7%:

$$43(1.07^{-1}) + 123(1.07^{-2}) + 214(1.07^{-3}) + 25(1.07^{-4}) + 275(1.07^{-5}) = 537.4512$$

We don’t need to manually calculate the above result. We can simply enter the cash flows and the interest rate into BA II Plus/BA II Plus Professional Cash Flow Worksheet. The calculator will generate the result for us.

Please note that this method works only when all the assets have the same yield to maturity. Otherwise, we have to use the standard backward matching method.

**Problem 3 (FM #10, Nov 2005)**

A company must pay liabilities of 1000 and 2000 at the end of years 1 and 2, respectively. The only investments available to the company are the following two zero-coupon bonds:

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Effective annual yield</th>
<th>Par</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10%</td>
<td>1000</td>
</tr>
<tr>
<td>2</td>
<td>12%</td>
<td>1000</td>
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Determine the cost to the company today to match its liabilities exactly.
**Solution**

SOA made this problem easy because there aren’t any coupons. To match the liability, you just need to buy a 1-year zero bond with face amount of $1,000 and buy two 2-year zero bonds with face amount of $1,000 each.

Your total cost of the matching assets is:

\[
\frac{1,000}{1.1} + \frac{2,000}{1.12^2} \approx 2,503
\]

**Key point to remember**

Cash flow matching problems are not hard, but it’s very easy for candidates to make silly mistakes. To eliminate errors, use a systematic approach.
Value of this PDF study manual

1. Don’t pay the shipping fee (can cost $5 to $10 for U.S. shipping and over $30 for international shipping). Big saving for Canadian candidates and other international exam takers.

2. Don’t wait a week for the manual to arrive. You download the study manual instantly from the web and begin studying right away.

3. Load the PDF in your laptop. Study as you go. Or if you prefer a printed copy, you can print the manual yourself.

4. Use the study manual as flash cards. Click on bookmarks to choose a chapter and quiz yourself.

5. Search any topic by keywords. From the Adobe Acrobat reader toolbar, click Edit ->Search or Edit ->Find. Then type in a key word.
About the author

Yufeng Guo was born in central China. After receiving his Bachelor’s degree in physics at Zhengzhou University, he attended Beijing Law School and received his Masters of law. He was an attorney and law school lecturer in China before immigrating to the United States. He received his Masters of accounting at Indiana University. He has pursued a life actuarial career and passed exams 1, 2, 3, 4, 5, 6, and 7 in rapid succession after discovering a successful study strategy.

Mr. Guo’s exam records are as follows:

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Mr. Guo currently teaches an online course for Exam P, FM, MLC, and MFE. For more information, visit http://actuary88.com.

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Deeper Understanding Exam FM Part II: Derivatives Markets

Yufeng Guo
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Chapter 0

Introduction

0.1 Recommended study method

The following are my recommendations on how to prepare for Derivatives Markets:

1. Download the FM syllabus from the SOA website. Know precisely what chapters of Derivatives Markets are the required readings for Exam FM.

2. Buy Derivatives Markets (2nd Edition). Derivatives Markets is expensive, but you’d better buy it. No study guides can thoroughly explain every detail of the textbook. Or if it does, it’ll be more expensive than the textbook. Besides, if you buy Derivatives Markets, you can use it for Exam MF, MFE, and C.

3. Read all of the required readings of Derivatives Markets. Work through all of the examples in Derivatives Markets.

4. Work through all of the problems in Derivatives Markets at the end of the required chapter. Use the solution manual to check your answers.

5. Download load the sample problems for Derivatives Markets from the SOA website. Work through these problems.

0.2 Types of questions to be tested

SOA can write two types of questions on Derivatives Markets:

1. Numerical calculations. For example, SOA can give you some facts about an interest rate swap and asks you to calculate the fixed swap rate $R$. Many students find that solving a numerical question is easier because they can use a formula to calculate the answer.

2. Essay-type multiple choice questions. This is an example: “Which of the following statements is true about the interest rate swap (or Eurodollars futures contracts)?” This is another example: “All the followings are differences between a forward contract and a futures contract EXCEPT...” Essay-type questions are generally harder than numerical calculations because you don’t have a formula to produce the answer. To answer an
essay type multiple choice question correctly, you’ll probably need to memorize lot of facts.

When reading Derivatives Markets, be prepared to learn formulas and memorize facts. Memorizing facts is no fun, but you’ll probably have to do it to answer essay-type questions. Many people find that reading the textbook multiple times and using flash cards help them memorize facts. You can try these two methods.

0.3 How to use this study manual

This manual is written under the following philosophy: “If the textbook explains it well, I’ll skip it or explain it briefly; if the textbook doesn’t explain it well, I’ll try to explain it well.” It’s best that you read this manual together with the textbook. This manual is not a replacement of Derivatives Markets.
Chapter 1

Introduction to derivatives

1.1 What is a derivative?

Today, a U.S. company (called Mr. US) signed a purchase contract with a manufacturer in Britain (called Mr. UK). According to the contract, Mr. US will buy an expensive machine from Mr. UK. The price of the machine is 1,000,000 pounds. Mr. UK will deliver the machine to Mr. US in exactly 6 months from today. Upon receiving the machine, Mr. US will pay Mr. UK exactly 1,000,000 pounds.

Mr. US faces the currency risk. The exchange rate between US dollars and British pounds fluctuates. If 6 months from today the value of British pounds goes up and the value of American dollars goes down, Mr. US will need to spend more dollars to convert to 1 million British pound, incurring a loss.

For example, the exchange rate today is 1 British Pound=1.898 US Dollars. If Mr. US pays Mr. UK today, Mr. US will spend $1,898,000 to buy 1 million pounds. If 6 months from today the exchange rate is 1 British Pound=2 US Dollars, Mr. US will need to spend $2,000,000 to buy 1 million pounds.

How can Mr. US reduce its currency risk?

• One simple approach is for Mr. US to buy 1 million pound today at the price of 1,898,000 US dollars. This eliminates the currency risk, but it requires Mr. US to spend $1,898,000 today.

• Another approach is for Mr. US to pre-order 1 million pounds at a fixed exchange rate from a third party. All Mr. US has to do is to find a third party willing to sell, in 6 months, 1 million pounds at a fixed exchange rate. When the pre-order contract is signed, no money exchanges hands. Mr. US is not required to take out any capital. After 6 months has passed, Mr. US will buy 1 million British pound from the third party at the pre-agreed exchange rate (say 1 British pound=1.9 US dollars), regardless of the market exchange rate between pounds and US dollars 6 months from today.

The pre-order contract here is called the forward contract. A forward contract is an agreement today to transact in the future. Under the currency forward contract, Mr. US (buyer) agrees to buy a specific amount of goods or services (1 million British pound) from the seller (the third party) at a pre-agreed price (1 British pound=1.9 US dollars) in a specific future date (6 months from today). And the seller (the third party) agrees to deliver
the specific amount of goods or services (1 million pounds) at a pre-agreed price (1 British pound=1.9 US dollars) in a specific future date (6 months from today). When the forward is signed, no money or goods exchanges hands between Mr. US and Mr. UK. When the specified future date (6 months from today) arrives, Mr. US and Mr. UK each fulfill their promises. Then the forward contract terminates.

The currency forward contract is just one example of financial derivatives.

### 1.1.1 Definition of derivatives

Derivatives are financial contracts designed to create pure price exposure to an underlying commodity, asset, rate, index, or event. In general they do not involve the exchange or transfer of principal or title. Rather, their purpose is to capture, in the form of price changes, some underlying price change or event. ¹

You can think of derivatives as children and the underlying assets as parents. Parents give birth to children; the underlying gives birth to derivatives.

Two key characteristics of a derivative:

1. A derivative is a contract between two parties. One party is a seller and the other buyer.

2. A derivative is a conditional asset whose value depends on something else (called the underlying asset or the underlying for short). In the currency forward contract between Mr. US and Mr. UK, the forward contract by itself doesn’t have any value. Its value depends on the currency exchange rate; the currency exchange rate is the underlying asset. If, 6 months from today, the British pound becomes more valuable thus more US dollars are required to buy one British pound, Mr. US has a gain in the currency forward and Mr. UK has a loss. If on the other hand the British pound becomes less valuable (thus fewer US dollars are required to buy one British pound), Mr. US has a loss and Mr. UK has a gain.

### 1.1.2 Major types of derivatives

There are 2 major types of derivatives:

- **Option type derivatives.** An option is a contract between a buyer (i.e. the owner of the option) and seller (one who sold the option) that gives the option owner the right, not the obligation, to buy or sell an asset at a pre-determined price in a specified future date.

  Two major types of options:

  - **Call.** A call option gives the option owner the right to buy an asset for a specified price (the exercise price or strike price) on or before a specified expiration date.
  
  - **Put.** A put option gives the option owner the right to sell an asset for a specified price (the exercise price or strike price) on or before a specified expiration date.

¹Definition by [http://www.financialpolicy.org/dsprimer.htm](http://www.financialpolicy.org/dsprimer.htm)
• **Forward type derivatives.** In forward-type derivatives, buyers and sellers agree to do business in a specified future date at a pre-determined price. The major use of forward type derivatives is pricing-locking. Forward type derivatives include:

- **Forward.** A forward contract is a commitment to buy/sell in a future date. The difference between a forward and an option is that an option represents the right, not obligation, to buy or sell in the future (hence the word option) but a forward contract represents the obligation to buy or sell in the future.

- **Futures.** A futures contract is also a commitment between a buyer and a seller to transact in a future date at a price agreed upon today. One key difference between a forward contract and a futures contract is that a futures contract is a standardized contract traded over the exchange whereas a forward contract is a privately negotiated contract traded over the counter.

- **Swap.** In a swap, two parties agree to exchange cash flows in the future. One common swap is an interest rate swap, where one party pays floating (i.e. variable) interest rate and receives fixed interest rate and the other party pays a fixed interest rate and receives a floating interest rate.\(^2\)

### 1.1.3 Basic vocabulary

The following terms are used throughout the textbook. They look odd to newcomers. However, after a while, these terms will become your second nature.

1. **Non-arbitrage.** Financial derivatives are generally priced using the non-arbitrage principle. The non-arbitrage principle means that “there’s no free lunch.” If two things have identical cash flows, they must sell at the same price. Otherwise, anyone can become an instant millionaire by “buy low, sell high.”

2. **Long and short.** Long means to own or buy something. Short means to sell something. If you are long in a stock, then you have bought a stock or you own a stock. If you are short on a stock, then you have sold a stock. Like credits and debits in accounting, long and short were invented by professionals probably to scare novice.

3. **Short selling.** If you borrow something from someone else and sell you what you have borrowed, you are selling short.

4. **Spot price, futures price, and forward price.** For virtually every commodity, there are two prices: spot price and futures price (or forward price). Spot price is the price you pay to get something immediately (spot=immediate delivery). Futures price or forward price is the price you pay in the future date in order to get something in the same future date (futures and forward mean future delivery). Most of us pay the spot price (like shopping in a department store). Only a small portion of people pay forward prices or futures prices.

5. **Spot market and futures (or forward) market.** The market associated with the spot price is the spot market. The market associated with the futures (or forward) prices is the futures market.

\(^2\)For more information, visit [http://www.answers.com/topic/interest-rate-swap](http://www.answers.com/topic/interest-rate-swap)
price is the futures (forward) market. Most of us trade in the spot market. The spot market is the largest. The forward market is the smallest. The futures market is in between.

6. **In the money, at the money, out of the money.** Suppose you own a call or put option.

- If exercising your call or put option brings you some profit, then you are in the money or the option is in the money.
- If exercising your call or put option brings you a loss, then you are out of the money or the option is out of the money.
- If after the option is exercised your wealth stays unchanged, then you are at the money or your option is at the money.

For example, you own a call option on one IBM stock. The call option allows you to buy one share of IBM stock for $80.

- If IBM stocks sell for $85 per share, then by exercising your call option you’ll make $5 profit. Then you are in the money.
- If IBM stocks sell for $75 per share, then exercising the call option brings you $5 loss. Then you are out of the money.
- If IBM stocks sell for $80 per share, then you gain zero profit if exercising the call option. You are at the money.

### 1.1.4 Uses of Derivatives

The textbook lists four major uses of derivatives:

- **Risk management.** For example, companies can use forwards and futures to lock in fixed prices of raw materials and exchange rates and protect themselves against fluctuations of raw materials and foreign currency exchange rates.

- **Speculation.** You can use derivatives to earn some profits (or incur losses).

- **Reducing transaction costs.** For example, by entering an interest rate swap, a firm can reduce its borrowing cost.

- **Regulatory arbitrage.** For example, you can use derivatives to produce temporary losses to lower your tax.

These are common sense uses of derivatives and should be pretty easy to remember.

### 1.1.5 Perspectives on Derivatives

- End-user perspective.

- Market-maker perspective.

- Economic observer

Note: There isn’t much meat in this section. Feel free to skip it.
1.1.6 Financial Engineering and Security Design

First, let’s think about the term financial engineering. Engineering means to build. Mechanical engineering is about building mechanical devices (engines for example). Electrical engineering is about building electrical devices. Similarly, financial engineering is about building financial devices (ie. financial products).

Let’s look at an example of a new financial product called “index linked CD (certified deposit).” While a conventional CD guarantees a fixed interest rate, a market index linked CD offers a variable interest rate linked to performance of a market index (such as S&P 500). If the market index goes up, the CD owner earns a higher interest rate. If the market index goes down, however, the CD owner is guaranteed a minimal interest rate.

However, there’s a difference between financial engineering and mechanical engineering. While a new model of cars can be built better than the old model, newly built financial products are not necessarily better than an older product. A newly built product might give an investor a higher returns but it also exposes him to higher risks.³

Next, the textbook says that financial engineering has four implications:

1. The fact that new financial products with desired payoffs can be built using basic ingredients (such as CD’s, stocks, bonds, calls, puts, forwards, futures, swaps) makes it feasible for banks and other financial institutions to produce new financial products and hedge the risks in the newly built financial products.

2. Sellers can build a special product for a special customer.

3. Sellers can refine their process for building new financial products (since building a new financial product is no longer a mysterious process any more).

4. A company can evade tax or circumvent regulations by building a new financial product that behaves the same ways as a security that’s taxed or regulated (so the company can say it doesn’t directly own the security and thus is not subject to the tax code or regulations otherwise applicable).

1.2 The role of financial markets

1.2.1 Financial Markets and Averages

Many financial risks are split and parceled out to others. Risk sharing is an important function of financial markets.

1.2.2 Risk-sharing

Risk is an inevitable; individuals and companies all face risks. Naturally, we want to set up risk-sharing mechanism where the lucky shares with the unlucky (similar to life insurance where the healthy help pay the death benefits of the unhealthy).

Some risks are diversifiable; others are non-diversifiable. Risk-sharing benefits everyone.

³For drawbacks of the market index linked CD, visit http://www.usatoday.com/money/perfi/columnist/block/2006-05-01-cds_x.htm
1.3 Derivatives in practice

The use and types of derivatives have grown a lot in the past 30 years. Derivatives based on government statistics, industrial production, retail sales, and consumer price indexes were invented.

1.3.1 Growth in derivatives trading

When the price of an asset fluctuates a lot, derivatives on this asset are often invented to help manage the price risk.

Millions of contracts are traded annually at the Chicago Board of Trade (CBT), Chicago Mercantile Exchange (CME), and the New York Mercantile Exchange (NYMEX), the three largest exchanges in the U.S..

The use of futures contracts has grown significantly over the last 30 years.

1.3.2 How are derivatives used?

Little is known about how companies use derivatives to manage risks.

1.4 Buying and short-selling financial assets

When calculating the cost of buying an asset, remember commission and the bid-ask spread.

How to memorize bid-ask spread

\[
\begin{align*}
\text{Bid price} &= \text{dealer’s Buying price} = \text{what you get if you sell your security to a dealer} \\
\text{Ask price} &= \text{dealer’s Selling price} = \text{what you pay if you buy a security from a dealer}
\end{align*}
\]

Short Selling. Short selling XYZ stocks means that you first borrow XYZ stocks and then selling them.

3 reasons to short sell:

- To speculate.
- To borrow money.
- To hedge a risk.

1.4.1 The lease rate of an asset

If you borrow an asset from the asset owner, you may have to pay a fee to the owner. This fee is called the least rate of an asset.

If you short sell a stock and the stock pays dividend before you return the stock to the lender, you need to pay the dividend to the lender.
1.4.2 Risk and scarcity in short-selling

- Credit risk is the risk that the short-seller won’t return the borrowed asset. To address the credit risk, the lender keeps as collateral the proceeds generated from the short sale.

- Since the lender keeps the proceeds from the short sale, the lender should give the proceeds back to the short seller after the short seller returns the borrowed asset. In addition, the lender should pay the short seller interest for temporarily holding on the proceeds. The amount of interest on collateral depends on supply and demand. If lot of people want to short sell an item, the lender may pay a small interest. The interest rate paid on collateral is called the repo rate in bond market and the short rebate in the stock market. The difference between this rate and the market interest rate is another cost of borrowing.

1.5 Chapter summary

Derivatives are conditional assets whose values depend on something else. The three major derivatives are forwards (or futures), options, and swaps.

1.6 Review questions

Problem 1

You are the owner of an airline company. Fuel cost is a significant portion of your annual operating expense. You heard the news that the rising fuel cost brought several airlines to bankruptcy. Outline ways you can manage you fuel cost using financial derivatives.

Solution

One way to lock in fuel cost is for you to enter an energy swap with a fuel supplier. In this swap, you pay fixed cost each year to the supplier over a number of years. In return, the supplier gives you certain amount of fuel each year while the swap is effective. Now your fuel cost is fixed during the duration of the energy swap. A swap generally doesn’t require any initial cash outlay.

You can also buy a certain number of call options on fuel price. If fuel price goes up, then you can exercise your call option and buy fuel at the predetermined strike price. However, unlike an energy swap, buying call options requires you to pay premiums upfront.

You can also enter a futures or forward contract to order certain amount of fuel to be delivered to you at specified future dates with prices determined today.

Problem 2

Why is it reasonable for us to assume that financial markets are free of arbitrage?

Solution
There are well-informed buyers and sellers in the market. If there are arbitrage opportunities, intelligent buyers and sellers will rush to the opportunities and soon the opportunities will vanish.
Chapter 2

An Introduction to Forwards and Options

2.1 Forward contracts

2.1.1 Definition

Buyer of a forward contract

- Has an obligation to buy an asset (the underlying)
- At a specific future date (maturity/expiration date)
- In an amount (contract size)
- At a priced agreed upon today (the forward price)

Seller of a forward contract

- Has an obligation to deliver an asset (the underlying)
- At a specific future date (maturity/expiration date)
- In an amount (contract size)
- At a priced agreed upon today (the forward price)

A prepaid forward contract is a forward contract except that the payment is made at $t = 0$ when the contract is signed.

2.1.2 The payoff on a forward contract

If you buy something, you have a long position (long=having more of something).

If you sell something, you have a short position (short=having less of something).

In a forward contract, the buyer’s payoff=$A - B$, where
• A = what the buyer has to pay to get the asset at $T$ assuming he doesn’t have a forward contract

• B = what the buyer has to pay to get the asset at $T$ if he has a forward contract

Clearly, $A =$ the spot price at $T$ (without a forward contract, the buyer has to buy the asset from the market at $T$); $B =$ the forward price at $T$.

So the buyer’s payoff at $T =$ Spot price at $T$ - forward price at $T$

Since the buyer of a forward contract has a long position in the forward contract, so payoff to long forward at $T =$ Spot price at $T$ - forward price at $T$

The seller’s payoff = C - D

1. C = What the seller has to sell at $T$ because he has a forward contract

2. D = What the seller can sell at $T$ if he doesn’t have a forward contract

Clearly, $C =$ forward price at $T$; $D =$ Spot price at $T$

Since the seller in a forward contract has a short position, payoff to short forward at $T =$ Forward price at $T$ - Spot price at $T$

payoff to long forward + payoff to short forward = 0

A forward is a zero sum game. If one party gains, the other party must lose.

2.1.3 Graphing the payoff on a forward contract

If you transfer the data from Table 2.1 to a graph (putting S&H Index in 6 months as X and Payoff as Y), you’ll get Figure 2.2. Don’t make it overly-complex. Just draw dots and connect the dots.

To draw the payoff diagram for the long forward position, draw the following points: (900, -120), (950, -70), (1000, -20), (1020, 0), (1050, 30), (1100, 80). Then connect these dots.

To draw the payoff diagram for the short forward position, draw the following points: (900, 120), (950, 70), (1000, 20), (1020, 0), (1050, -30), (1100, -80). Then connect these dots.

2.1.4 Comparing a forward and outright purchase

What’s the difference between buying an asset at $t = 0$ as opposed to entering into a forward contract and getting the same thing at $T$?

If you buy an asset outright at $t = 0$:

• What’s good: You get what you need right away. You don’t need to enter into a forward contract. It takes time to find a seller who wants to sign a forward contract with you. You may have to hire an attorney to review the complex clauses of a forward contract. Finally, buying an asset outright eliminates the risk that the seller might not deliver the asset at $T$. 
• What’s bad: Outright purchase ties up your capital. In addition, if you don’t need the asset right away, you have to store the asset. Storing the asset may cost you money.

If you get the asset at $T$ through a forward contract

• What’s good: Just in time delivery saves you storage cost. In addition, you don’t have to pay anything at $t = 0$.

• What’s bad: You need to take the time to find a willing seller who has what you need and is willing to enter a forward contract with you. In addition, the seller in a forward contract may break his promise (credit risk).

Make sure you can reproduce Figure 2.3.

Make sure you memorize the conclusion of this section. Since it costs zero to store an index (index is not a physical asset such as wheat or corn), assuming there’s zero credit risk (assuming both parties in a forward contract keep their promises), then a forward contract and a cash index are equivalent investments, differing only in the timing of the cash flows. Neither form of investing has an advantage over the other.

Make sure you understand the difference between payoff and profit; between a payoff diagram and a profit diagram.

### 2.1.5 Zero-coupon bonds in payoff and profit diagrams

The textbook says that “buying the physical index is like entering into the forward contract and simultaneously investing $1000 in a zero-coupon bond.” Please note that “investing $1000 in a zero-coupon bond” simply means that you put $1000 in a savings account. This way, at the expiration date $T$, your initial $1000 will grow into $1000 \times 1.02 = 1200$, which is exactly what you need to pay the seller to buy the index.

### 2.1.6 Cash settlement vs. delivery

The idea is simple. If you owe me an orange (worth $2) and I owe you an apple (worth $5), one way to settle out debts is that you give me an orange and I give you an apple. This is called the physical delivery.

A simpler approach is that I give you $3. This is called the cash settlement.

### 2.1.7 Credit risk

This is the risk that one party in a contract breaks his promise and doesn’t do what he is supposed to do according to the contract.

### 2.2 Call options

#### 2.2.1 Option terminology

Make sure you understand the following terms: strike price, exercise, expiration, exercise style (American and European). The textbook explains well. Just follow the textbook.
2.2.2 Payoff and profit for a purchased call option

Just memorize the key formulas:

- Purchased call payoff = \[ \max(0, S_T - K) \]
- Purchased call profit = Purchased call payoff - FV of Option Premium
- Purchased call profit = \[ \max(0, S_T - K) - \text{FV of Option Premium} \]

where \( S_T \) is the stock price at expiration date \( T \) and \( K \) is the strike price.

2.2.3 Payoff and profit for a written call option

- Written call payoff + Purchased call payoff = 0 (zero sum game)
- Written call payoff = \[ -\max(0, S_T - K) \]
- Written call profit + Purchased call profit = 0 (zero sum game)
- Written call profit = FV of Option Premium \[ -\max(0, S_T - K) \]

2.3 Put options

2.3.1 Payoff and profit for a purchased put option

- Purchased put payoff = \[ \max(0, K - S_T) \]
- Purchased put profit = Purchased put payoff - FV of Option Premium
- Purchased put profit = \[ \max(0, K - S_T) - \text{FV of Option Premium} \]

2.3.2 Payoff and profit for a written put option

- Written put payoff + Purchased put payoff = 0 (zero sum game)
- Written put payoff = \[ -\max(0, K - S_T) \]
- Written put profit + Purchased put profit = 0 (zero sum game)
- Written put profit = FV of Option Premium \[ -\max(0, K - S_T) \]

2.4 Summary of forward and option positions

Make sure you understand Table 2.4. Make sure you are comfortable with the following terms:

1. Long Positions
   (a) Long forward.
   (b) Purchased call.
   (c) Written call.
2. (a) Short forward.
   (b) Written call.
   (c) Purchased call.

2.5 Options are insurance

This is the basic idea. First, options are insurance. This shouldn’t surprise you. A call option is an insurance policy against the risk that the price of a stock may go up in the future. Similarly, a put option is an insurance policy against the risk that the price of a stock may go down in the future.

Next, a homeowner insurance policy is a put option. This is a simple idea too. If you have a put option on a stock, you are guaranteed to sell your stock at a fixed price no matter how low the stock price has become. Similarly, if you have a homeowner insurance policy, no matter how low your house has become, you are guaranteed to sell your house to the insurance company at a fixed price (i.e. the insurance company will give you a fixed amount of money regardless of how low your house has become).

Make sure you can recreate Figure 2.13. To recreate Figure 2.13, just draw a few critical points and then connect these points. There are 2 critical points:

(House price, Profit) = (0, 160,000), (200,000, −15,000).

This is how to get the first critical point (0, 160,000). If the house is blown away by big wind, the house price is zero. Then the insurer pays the full replacement value of the house $200,000 subject to a deductible of $25,000. So the insurance company will pay you $200,000 − $25,000 = $175,000. However, you pay premium $15,000 at t = 0. If we ignore the lost interest of your premium (had you put your premium in a savings account, you would have earned some interest), then your profit is $175,000 − $15,000 = $160,000.

This is how to get the second critical point (200,000, −15,000). If after you bought the insurance policy, nothing bad happens to your house, then the payoff of your insurance is zero. You lost your premium $15,000; you pay $15,000 for nothing. Your loss is $15,000.

You can verify for yourself that if nothing bad happens to your house, even if the value of your house goes up, you’ll always lose $15,000 from the insurance policy. Now you should get Figure 2.13.

2.6 Example: equity-linked CD

SOA can easily test Formula (2.11). Memorize it.

\[ V_T = V_0 \times \left( 1 + k \times \max \left[ 0, \frac{S_T}{S_0} - 1 \right] \right) \]

- \( V_T \) is the CD value at time \( T \)
- \( V_0 \) is the CD value at time 0
- \( S_T \) is the index value at time \( T \)
- \( S_0 \) is the index value at time 0
• $k$ is the participation rate

Make sure you can recreate Figure 2.14 and Table 2.5.

◊
Chapter 3
Insurance, Collars, and Other Strategies

Unfortunately, this is a messy chapter with lot of new phrases such as bull spread and collars. Just follow the text and understand these new phrases.

The most important topic in this chapter is the put-call parity. Most likely SOA will test this topic. Make sure you understand the put-call parity.

I’ll highlight the key points in this chapter.

3.1 Basic insurance strategies

3.1.1 Insuring a long position: floors

A put option gives you a price floor. If you have a put option on an asset, then you don’t need to worry that the asset price drops below the strike price. If indeed the asset’s price drops below the strike price, you can exercise the put and sell the asset at the strike price.

Make sure you can recreate Table 3.1 and Figure 3.1

The 6-month interest rate is 2%. A put with 1000-strike price with 6-month to expiration sells for $74.201. A call with 1000-strike price with 6-month to expiration sells for $93.809.

Payoff at expiration $T$ of Put + Stock is $max(S_T, K)$. If you have a put and a stock, then the value of Put + Stock at the expiration date $T$ is the greater of the strike price $K$ and the stock price $S_T$. If, at $T$, $S_T < K$, then you can exercise your put and sell your stock for $K$. If $S_T \geq K$, then you just let your put expire worthless and your payoff is $S_T$.

So the payoff of Put + Stock at expiration $= max(S_T, K)$.

When recreating Table 3.1, please note that Profit = Payoff - (Cost + Interest)

To recreate Figure 3.1 (a), (b), (c), and (d), just plug in the data from Table 3.1. For example, to recreate Figure 3.1(d), enter the following points of (S&R index, Profit):

$(900, -95.68), (950, -95.68), (1000, -95.68), (1050, -45.68), (1100, 4.32), (1150, 54.32), and (1200, 104.32)$

Connect these points and you should get Figure 3.1(d).

Make sure you can recreate Figure 3.2. Figure 3.2 is continuation of Figure 2.13. This is how to draw the profit diagram for the uninsured house. If the house is washed away by flood (ie. the price of the house becomes zero), then you loose $200,000. If the house appreciates and...
the new price is $250,000, then you gain $250,000 − 200,000 = 50,000. So you have two data points:

(Price of the house, Profit)=(0, −200,000), (250,000, 50,000).

Connecting these two points, you’ll get the profit line for the uninsured house.

This is how to draw the diagram for the insured house. The profit is −40,000 when the house price is from 0 to 175,000. If the damage to your house is equal to or greater than the deductible $25,000 (i.e. the house price is 200,000 − 25,000 = 175,000 or less), you’ll lose both the deductible and the premium. So your total loss is 25,000 + 15,000 = 40,000. This is a straight line from (0, −40,000) to (175,000, −40,000).

If after you bought the insurance, nothing bad happens to your house and the price of your house goes up to say $250,000, then you gain $50,000 in the value of your house but you lose your premium. So your profit is 50,000 − 15,000 = 35,000. This gives us the point (250,000, 35,000).

If you connect (175,000, −40,000), (200,000, −15,000), and (250,000, 35,000), you’ll get the profit diagram when the house price is 175,000 or greater.

The conclusion drawn from Figure 3.2 is that an insured house has a profit diagram that looks like a call option.

### 3.1.2 Insuring a short position: caps

A call option gives you a price ceiling (or cap). If you have a call option on an asset, then you don’t need to worry that the asset price may go up. If indeed the price of the asset goes above the strike price, you can exercise the call and buy the asset at the strike price.

Make sure you can reproduce Table 3.2 and Figure 3.3.

### 3.1.3 Selling insurance

If you sell a call on a stock, you’d better have a stock on hand. This way, if the stock price goes up and the call is exercised, you already have a stock to deliver to the owner of the call.

If you buy a stock and simultaneously sell a call option, you are selling a **covered call**. A covered call limits your loss in case the stock price goes up (because you already have a stock on hand).

If you sell a call option but don’t have a stock on hand, you are selling a **naked call**. Writing a naked call is a big risk if the stock price goes up, in which case the call writer has to buy a stock from the market.

**Covered put.** If you sell a put, you have two ways to cover it. One is to short sell a stock: when you sell a put, you simultaneously short sell a stock. If the stock happens to be worthless and the buyer of the put sells the worthless stock to you at the guaranteed price (i.e. strike price), you turn around and give the worthless stock to the broker who lent you a stock for short selling.

Another way to cover your put is to set aside the present value of the strike price at \( t = 0 \). If the owner of the put exercises the put, you already have the strike price in your pocket to pay the owner of the put.

If you don’t cover your put and the stock happens to be worthless, then you may suddenly find that you don’t have the money to pay the strike price.
3.2 Synthetic forwards

Buying a call and selling a put on the same underlying with each option having the same strike price and time to expiration produces a synthetic forward.

Difference between a synthetic long forward and an actual forward:

1. The actual forward contract has zero premium, while a synthetic forward requires a net option premium.
2. With the forward contract, we pay the forward price; with the synthetic forward, we pay the strike price.

3.2.1 Put-call parity

The net cost of buying the index using options must equal the net cost of buying the index using a forward contract.

\[ \text{Call}(K,T) - \text{Put}(K,T) = \text{PV}(F_{0,T} - K) \]

To prove the parity equation, let’s rewrite it as \( \text{Call}(K,T) + \text{PV}(K) = \text{Put}(K,T) + \text{PV}(F_{0,T}) \).

Proof. Assume that at \( t = 0 \), you hold two portfolios A and B. Portfolio A consists of two assets: (1) a call option with strike price \( K \) and maturity \( T \), and (2) the present value of the strike price \( \text{PV}(K) \). Please note that at time \( T \), \( \text{PV}(K) \) will become \( K \) and you’ll have \( K \) in your pocket.

Portfolio B consists of two assets: (1) a put option with strike price \( K \) and maturity \( T \), and (2) the present value of the forward price \( \text{PV}(F_{0,T}) \). Please note that at time \( T \), \( \text{PV}(F_{0,T}) \) will accumulate to \( (F_{0,T}) \), which is the price you pay to buy one stock from the seller in the forward contract.

Let’s consider 3 situations.

1. If at \( T \), \( S_T > K \). For Portfolio A, you exercise the call option and buy one share of the stock at price \( K \); Portfolio A is worth \( S_T \). For Portfolio B, you let your put option expire worthless. Portfolio B is also worth \( S_T \).

2. If at \( T \), \( S_T < K \). For Portfolio A, you let your call option expire worthless. Portfolio A is worth \( K \). For Portfolio B, you exercise the put option, sell your stock, and get \( K \). So Portfolio B is also worth \( K \).

3. If at \( T \), \( S_T = K \). Both the call option and the put option expire worthless. Portfolio A and B are both worth \( S_T = K \).

Under any situation, Portfolio A and B have equal values. Hence

\[ \text{Call}(K,T) + \text{PV}(K) = \text{Put}(K,T) + \text{PV}(F_{0,T}) \]

\[ \implies \text{Call}(K,T) - \text{Put}(K,T) = \text{PV}(F_{0,T} - K) \]
3.3 Spreads and collars

3.3.1 Bull and bear spreads

Suppose you expect that a stock price will go up. To make some money, you can do one of the following three things today

- buy a share (or a forward contract)
- buy a call
- sell a put

Buying a stock is expensive and has the risk that the stock price may drop in the future. Buying a call requires paying upfront premium. Selling a put receives money upfront but has a large downside risk that the stock price may go down.

Instead of the above 3 strategies, you can use a bull spread.

**Bull spread.** Two ways to build a bull spread.

- Buy one call at a lower strike price $K_1$ and sell another call at a higher strike price $K_2 (> K_1)$, with two calls on the same stock and having the same maturity. This strategy costs you some money upfront. You get some premiums for selling another call at a higher strike price $K_2 (> K_1)$, but by doing so you give up some upside potentials. If the stock price exceeds $K_2$, your selling price is capped at $K_2$.

- Buy one put with a lower strike price $K_1$ and sell another put with a higher strike price $K_2 (> K_1)$, with two puts on the same stock and having the same maturity. This strategy gives you positive cash flows upfront but has zero or negative payoff.

No matter you use calls or puts, in a bull spread, you always buy low and sell high.

The word ”spread” means buying and selling the same type of option. The word ”bull” means that you can possibly make money if the stock price goes up.

**Bear spread.** Two ways to build a bear spread.

- Buy one call at a higher strike price and sell another call at a lower strike price, with two calls on the same stock and having the same maturity. (This strategy gives you positive cash flows upfront)

- Buy one put with a higher strike price and sell another put with a lower strike price, with two puts on the same stock and having the same maturity. (This strategy costs you some money upfront)

The word ”bear” means that you can possibly make money if the stock price goes down. No matter you use calls or puts, in a bear spread, you always buy high and sell low.
3.3.2 Box spreads

A box spread is accomplished by using options to create a synthetic long forward at one price and a synthetic short forward at another price.

A box spread is a means of borrowing or lending money: it has no stock price risk.

Consider the textbook example. You can do the following two transactions at the same time:

1. Buy a $40-strike call and sell a $40-strike put.
2. Sell a $45-strike call and buy a $45-strike put.

If you buy a $40-strike call and sell a $40-strike put, you are really buying a stock for $40. If you sell a $45-strike call and buy a $45-strike put, you are really selling a stock for $45. By doing these two things, you buy a stock for $40 and sell it for $45. This produces a guaranteed profit of $5.

Of course, there’s no free lunch. The $5 profit to be earned at $t = 0$ must be discounted by the risk free interest rate. So the box spread must sell at $t = 0$ for

$$5 \times 1.0833^{-0.25} = 4.9$$

As the textbook points out, $4.9 is exactly the net premium of the 4 options in the box spread.

3.3.3 Ratio spreads

**Ratio spread.** An options strategy in which an investor simultaneously holds an unequal number of long and short positions. A commonly used ratio is two short options for every option purchased.

For example, a ratio spread can be achieved by buying one call option with a strike price of $45 and selling two call options with a strike price of $50. This allows the investor to capture a gain on a small upward move in the underlying stock’s price. However, any move past the higher strike price ($50) of the sold options will cause this position to lose value. Theoretically, an extremely large increase in the underlying stock’s price can cause an unlimited loss to the investor due to the extra short call.

3.3.4 Collars

**Collars.** A spread involves either a call or a put but not both. However, in a collar, you use both a call and a put.

A collar is to buy a put with one strike price $K_1$ and sell a call with a higher strike price $K_2 > K_1$, with both options having the same underlying asset and having the same expiration date.

The collar width is $K_2 - K_1$.

Say you have lot of stocks on Google. Say Google stocks are currently selling $200 per share. You are worrying that Google stocks may go down. You can put a lower bound on Google stock price by buying a put with strike price $180. This way, if Google’s stock price goes to $100, you can sell your Google stocks for $180.
However, buying a put option requires money up front. To raise money to buy a put, you can sell a call option with strike price equal to $220. If the money you get from selling a call offsets the cost of buying a put, you have a zero-cost collar.

Selling a call limits the stock’s upside potential. If Google’s stock price goes above $220, you won’t realize the gain because your stock will be called away at $220.

### 3.4 Speculating on volatility

Here the investor thinks that the stock price may change by a substantial amount but is unsure about the direction of the change (i.e. not sure whether the price will go up or down). The investor’s goal is to make some money whether the stock price will go up or down as long as it is within a range. The investor is speculating on the stock’s volatility.

#### 3.4.1 Straddle

**Straddle.** In a straddle, you buy both a call and a put on the same stock with the same strike price and the same expiration date.

Suppose that a court will issue its ruling on a company. The ruling may make or break the company so the company’s stock price may double or be cut in half. Unsure of the outcome, you can buy both a call and a put. If the stock price goes up, you exercise the call; if the stock price goes down, you exercise the put.

The disadvantage of a straddle is the high premium cost. The investor has to pay both the call premium and the put premium.

**Strangle.** A strangle is similar to a straddle except that now you buy an out-of-money call and an out-of-money put. A strangle is cheaper than a straddle.

**Written straddle v.s. purchased straddle.** If you think the stock volatility is higher than the market’s assessment (i.e. you think that the option is underpriced), you buy a straddle. If you think that the stock volatility is lower than expected (i.e. you think the option is over priced), you write (i.e. sell) a straddle.

#### 3.4.2 Butterfly spread

A butterfly spread insures against large losses on a straddle. ¹

#### 3.4.3 Asymmetric butterfly spreads

Pay attention to the following formulas

\[
\lambda = \frac{K_3 - K_2}{K_3 - K_1}
\]

\[
K_2 = \lambda K_1 + (1 - \lambda)K_3
\]

3.5 Example: another equity-linked note

The main point of this example is that debt can be designed like an option. Marshall & Ilsley Corp paid its debt by making annual coupons 6.5% before maturity and by paying shares of its stocks at maturity. The number of shares paid by Marshall & Ilsley Corp at the maturity of the debt was calculated using the formula in Table 3.7.

According to Exam FM syllabus, candidates need to know definitions of key terms of financial economics at an introductory level. This section seems to be beyond the introductory level. So don’t spend lot of time on convertible bond.
Chapter 4

Introduction to risk management

Firms convert inputs into goods and services. A firm is profitable if the cost of what it produces exceeds the cost of the inputs.

A firm that actively uses derivatives and other techniques to alter its risk and protect its profitability is engaging in risk management.

4.1 Basic risk management: the producer’s perspective

This is a case study on Golddiggers. Golddiggers has to pay its fixed cost. As long as the gold price is higher than the variable cost, Golddiggers should continue its production.

Golddiggers sells gold and has an inherent long position in gold. It needs to manage the risk that the price of gold may go down.

To lock in price of gold, Golddiggers can sell forwards, buy puts, and buy collars.

4.1.1 Hedging with a forward contract

A producer can use a short forward contract to lock in a price for his output. Golddiggers can enter into a short forward contract, agreeing to sell gold at a price of $420/oz. in 1 year.

Make sure you can reproduce Table 4.2 and Figure 4.1.

4.1.2 Insurance: guaranteeing a minimum price with a put option

A downside of locking-in gold price with a short forward contract is that gold price may go up. If gold price goes up, a firm can’t profit from the rise of the price because it has locked in fixed price during the life of the forward contract.

To gain from the rise of the product price and still have a floor on the price, a producer can buy a put.

Make sure you understand Table 4.3 and Figure 4.2. Sometimes a short forward gives a producer more profit; other times buying a put yield more profit.

No hedging strategy always outperforms all other strategies.

4.1.3 Insuring by selling a call

Golddiggers can sell a call. Selling a call earns premiums. However, if the gold price goes up beyond the call strike price, Golddiggers has to sell gold at the strike price.
Make sure you understand Figure 4.4.

4.1.4 Adjusting the amount of insurance

Buying a put is like buying insurance. Insurance is expensive. There are at least 2 ways to reduce insurance premium:

- Reduce the insured amount by lowering the strike price of the put. This permits some additional losses.
- Sell some of the gain and put a cap on the potential gain.

4.2 Basic risk management: the buyer’s perspective

A buyer faces price risk on an input and has an inherent short position in the commodity. If the price of the input goes up, the buyer’s profit goes down.

4.2.1 Hedging with a forward contract

A long forward contract lets a buyer lock in a price for his input. For example, Auric can lock in gold price at $420 per ounce.

Make sure you understand Table 4.4 and Figure 4.6.

4.2.2 Insurance: guaranteeing a maximum price with a call option

Auric might want to put a cap on gold price but pay the market price if the price of gold falls. Auric can buy a call option.

Make sure you understand Table 4.5.

4.3 Why do firms manage risk?

In a world with fairly priced derivatives, no transaction costs, and no other market imperfections such as taxes, derivatives don’t increase the value of cash flow; they change the distribution of cash flows.

Using derivatives, Goldiggers shifts dollars from high gold prices states to low gold price states. Hedging is beneficial for a firm when an extra dollar of income received in times of high profits is worth less than an extra dollar of income received in times of low profits.

4.3.1 An example where hedging adds value

Make sure you can reproduce the calculation that the expected profit before tax is $0.1.

Make sure you know why the expected profit after tax is -$0.14. On the after tax basis, there’s 50% chance that the profit is $0.72 and 50% chance that the profit is -$1. So the expected after-tax profit is:

\[0.5 \times (0.72 - 1) = -0.14\]

Make sure you can reproduce Figure 4.8. Point A and B are based on Table 4.6. Point C is based on the fact that if the unit price is $10.1, then the after-tax profit is $0.06. The
product cost is $10. So the price $10.1 gives us $0.1 profit before tax, which is $0.06 after tax.

Point D is the expected after-tax profit of -$0.14 if the unit price is either $9 or $11.2 with equal probability. It’s calculated as follows: $0.5 \times (0.72 - 1) = -0.14$

Line ACB is concave. A concave is an upside down U shape. When profits are concave, the expected value of profits is increased by reducing uncertainty.

### 4.3.2 Reasons to hedge

Concave profit patterns can arise from:

- **Tax.** If you can fully deduct your loss as it occurs, then you might not need to hedge every loss; you can just incur loss and deduct your loss from your gain to reduce your taxable income. However, if you can’t fully deduct your loss or if you can deduct this year loss the next year, then losses are no good. You might want to use derivatives to hedge against potential losses.

- **Bankruptcy and distress costs.** A large loss can threaten the survival of a firm. A firm may be unable to meet its fixed obligations (such as debt payments and wages). Customers may be less willing to purchase goods of a firm in distress. Hedging allows a firm to reduce the probability of bankruptcy or financial distress.

- **Costly external financing.** If you pay losses, you have less money for money-making projects. You might have to borrow money to fund your projects. Raising funds externally can be costly. There are explicit costs (such as bank and underwriting fees) and implicit costs due to asymmetric information. So you might want to use derivatives to hedge against losses. Costly external financing can lead a firm to forego investment projects it would have taken had cash been available to use for financing. Hedging can safeguard cash reserves and reduce the need of raising funds externally.

- **Increase debt capacity.** The amount that a firm can borrow is its debt capacity. When raising funds, a firm may prefer debt to equity because interest expense is tax-deductible. However, lenders may be unwilling to lend to a firm that has a high level of debt; high-debt companies have higher probability of bankruptcy. Hedging allows a firm to credibly reduce the riskiness of its cash flows, and thus increase its debt capacity.

- **Managerial risk aversion.** Managers have incentives to reduce uncertainty through hedging.

- **Nonfinancial risk management.** If you start a new business, you may need to decide on where to set up the plan and choose between leasing and buying equipment. You need to think through various risks associated with your decision and find ways to manage your risk. Risk management is not a simple matter of hedging or not hedging using financial derivatives, but rather a series of decisions that start when the business is first conceived.

### 4.3.3 Reasons not to hedge

Reasons why firms may elect not to hedge:
• High transaction cost in using derivatives (commissions and the bid-ask spread)
• A firm must assess costs and benefits of a given strategy; this requires costly expertise. (You need to hire experts to do hedging for you. Experts aren’t cheap!)
• The firm must set up control procedures to monitor transactions and prevent unauthorized trading.
• If you use derivatives to hedge, be prepared for headaches in doing accounting and filing tax returns. Accounting and tax for derivatives are complex.
• A firm can face collateral requirements if their derivatives lose money.

4.3.4 Empirical evidence on hedging
• About half of nonfinancial firms report using derivatives
• Big firms are more likely to use derivatives than small firms.
• Among firms that do use derivatives, less than 25% of perceived risk is hedged
• Firms are more likely to hedge short term risks than long term risks.
• Firms with more investment opportunities are more likely to hedge.
• Firms that use derivatives have a higher market value and more leverages.

4.4 Golddiggers revisited

This section discusses additional hedging strategies for Golddiggers. The textbook has clear explanations. Follow the textbook.

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Chapter 5

Financial forwards and futures

5.1 Alternative ways to buy a stock

Make sure you understand Table 5.1 from the textbook.

- Outright purchase. This is like buying things from Wal-Mart. After paying the cashier at the checkout, you immediately own the items bought.

- Fully leveraged purchase. This is like buying a car with a car loan. You go to a bank and borrow the full price of the car. Then you pay the car dealer and bring the car home right away.

- Prepaid forward contract. This is pre-ordering an item. You pay the full price at \( t = 0 \). The product is delivered to you at time \( T > 0 \).

- Forward contract. This is another way of pre-ordering an item. You don’t pay anything at \( t = 0 \). The product is delivered to you at time \( T > 0 \). You’ll pay the full price at \( T \).

5.2 Tailing

Next, let’s learn a phrase called tailing. Tailing answers the following questions:

*My goal is to possess one share of stock at time \( T \). How many shares of stock do I need to buy at \( t = 0 \) so I’ll have one share of stock at time \( T \)?*

If the stock doesn’t pay dividend, you’ll need to buy one share of stock at \( t = 0 \) in order to have one share at time \( T \). However, if the stock pays dividend at a continuous rate of \( \delta \), you can reinvest your dividend continuously and keep buying additional fractional shares of stocks. This way, you need to buy only a fractional share of stock at \( t = 0 \) to have one share of stock at time \( T \).

The textbook shows you that if you buy one share at \( t = 0 \), after dividends are continuously reinvested, your initial one share will become \( e^{\delta T} \) shares at time \( T \). Consequently, you’ll need to buy \( e^{-\delta T} \) share of stock at \( t = 0 \) to have one share of stock at \( T \).
Tailing Diagram: $e^{-\delta T}$ share at $t = 0$ becomes one share at time $T$. $\delta$ is the continuously compounded dividend rate.

### 5.3 Pricing a forward contract

Price of a prepaid forward contract is:

- $F^P_{0,T} = S_0$ - PV of future dividends (if dividends are discrete)
- $F^P_{0,T} = S_0 e^{-\delta T}$ (if dividends are continuous)

Price of a forward contract is:

- $F_{0,T} = S_0 e^{rt}$ - FV of dividends at time $T$ (if dividends are discrete)
- $F_{0,T} = S_0 e^{(r-\delta)T}$ (if dividends are continuous)

The textbook derives these formulas using 3 approaches: by analogy, by discounting cash flows, and by the no-arbitrage principle. The easiest way to derive and memorize these formulas, however, is to use the cost-of-carry concept.

**Cost of carry: The forward price $F_{0,T}$ must equal the spot price at $t = 0$ plus the cost of carrying the spot price asset from $t = 0$ to $T$ for delivery.**

Suppose you and I sign a forward contract at $t = 0$. This contract requires me to deliver a stock at time $T$ to you and requires you to pay me $F_{0,T}$ at time $T$. The essence of this forward contract is that I need to sell a stock to you at time $T$ for a preset price $F_{0,T}$.

Let’s calculate how much it costs me to have a stock ready for delivery at time $T$. If the stock pays continuous dividend at the rate of $\delta$, to have one share of stock at time $T$, I just need to buy $e^{-\delta T}$ share at $t = 0$. Then the cost of having a stock ready for delivery at $T$ is:

Spot price at time zero + Cost-of-carry during $[0,T]$

Spot price at time zero = Cost of buying $e^{-\delta T}$ share of stock at time zero=$S_0 e^{-\delta T}$
Cost-of-carry during \([0, T]\) = Foregone interest of tying my capital during \([0, T]\)

The foregone interest of tying my capital during \([0, T]\) is \(S_0 e^{-\delta T} (e^{rT} - 1)\), where \(r\) is the continuously compounded risk-free interest rate. Had I put \(S_0 e^{-\delta T}\) in a bank account (instead of buying stocks) at \(t = 0\), my fund balance at \(T\) would be \(S_0 e^{-\delta T} e^{rT}\) and I would have earned \(S_0 e^{-\delta T} e^{rT} - S_0 e^{-\delta T} = S_0 e^{-\delta T} (e^{rT} - 1)\) interest during \([0, T]\).

So the cost of having a stock ready for delivery at \(T\) is:

\[
S_0 e^{-\delta T} + S_0 e^{-\delta T} (e^{rT} - 1) = S_0 e^{-\delta T} e^{rT} = S_0 e^{(r-\delta)T}
\]

The forward price is: \(F_{0,T} = S_0 e^{-\delta T} e^{rT} = S_0 e^{(r-\delta)T}\)

If the stock doesn’t pay any dividend, to find the forward price, just set \(\delta = 0\): \(F_{0,T} = S_0 e^{rT}\)

What if the stock pays discrete dividends? Easy. To have a stock ready for delivery at \(T\), I can buy one share of stock at time zero and pay \(S_0\). The foregone interest of tying up my capital \(S_0\) during \([0, T]\) is \(S_0 e^{rT} - S_0\). The dividend I receive will reduce my cost of carry. Then the forward price is:

\[
S_0 + S_0 e^{rT} - S_0 = \text{FV of all the dividends received} = S_0 e^{rT} - \text{FV of all the dividends received}
\]

Next, let’s apply the cost-of-carry concept to a prepaid forward contract. With a prepaid forward contract, the buyer pays me right away and I don’t tie up my capital any more; my foregone interest is zero. The PV of the dividends I will receive during \([0, T]\) will reduce of the cost of carrying the asset from time zero to \(T\). Now we have:

\[
F_{0,T}^P = \text{Spot price + Cost of carry} = S_0 - \text{PV of future dividends (If dividends are discrete)}
\]

However, if the dividend is continuous, to have one stock ready for delivery at \(T\), I just need to buy \(e^{-\delta T}\) share of stock at time zero. So the prepaid forward price is:

\[
F_{0,T}^P = S_0 e^{-\delta T}
\]

Now you shouldn’t have trouble memorizing the pricing formulas for a prepaid forward and a typical (post-paid) forward contract.

### 5.4 Forward premium

The forward premium is the ratio of the forward price to the spot price:

Forward Premium = \(\frac{F_{0,T}}{S_0}\)

The annualized forward premium (AFP) is defined as:

\[
S_0 e^{AFP \times T} = F_{0,T} \implies AFP = \frac{1}{T} \ln \left( \frac{F_{0,T}}{S_0} \right)
\]
5.5 Creating a synthetic forward contract

The seller in a forward contract faces the risk that the security’s spot price at time $T$ (that is $S_T$) may exceed the forward price $F_{0,T}$ by a large amount. For example, you and I sign a forward contract at $t=0$. Under the contract, I need to give you one share of ABC stock for $20 at time $T$. If one share of ABC stock will sell for $100 at $T$, I will lose $80. (If I sell one share of ABC stock in the market at $T$, I can earn $100. However, I have to sell one share of ABC stock to you at for $20. So I lose $80 and you gain $80.)

At the expiration date $T$:

- The payoff earned by the buyer in the forward contract is $S_T - F_{0,T}$
- The payoff earned by the seller in the forward contract is $F_{0,T} - S_T$
- The payoff earned by the seller + the payoff earned by the buyer = 0

To hedge the risk that the spot price $S_T$ may rise, the seller in a forward contract can buy $e^{-\delta T}$ share of stock at $t=0$ by paying a fixed cost of $S_0e^{-\delta T}$ and hold it to $T$ for delivery. This way, he’ll already have one share of the stock and doesn’t have to worry how high $S_T$ may become. (If he doesn’t have one share of stock ready at $T$ for delivery and $S_T$ happens to be very high, the seller is forced to buy the stock at in the open market and will suffer a huge loss).

To avoid tying up his own money, the seller can go to a bank and borrow $S_0e^{-\delta T}$. When the delivery time $T$ arrives, the seller just needs to pay back the principal and the interest to the bank. The sum of the principal and the interest due at $T$ is $S_0e^{-\delta T}e^{rT}$. This strategy costs the seller nothing to set up but gives the seller $S_T - S_0e^{-\delta T}e^{rT}$ at time $T$.

Please note that $S_T - S_0e^{-\delta T}e^{rT}$ is exactly the net payment at $T$ by the seller to the buyer in a fairly priced forward contract. A fairly priced forward contract on one share of stock to be delivered at $T$ is priced at $F_{0,T} = S_0e^{-\delta T}e^{rT}$. Then at time $T$, the buyer needs to pay the seller $F_{0,T} = S_0e^{-\delta T}e^{rT}$; the seller needs to deliver a stock worth $S_T$. Instead of the buyer writing the seller a check in the amount of $F_{0,T} = S_0e^{-\delta T}e^{rT}$ and the seller delivering a stock worth $S_T$ to the buyer, the forward contract can be settled by having the seller send the buyer a check in the amount of $S_T - S_0e^{-\delta T}e^{rT}$.

These transactions are summarized in Table 5.3.

The textbook gives you the formula “Forward = Stock - Zero-coupon bond.” To understand what this means, notice that borrowing money from a bank is the same as issuing a zero coupon bond; lending money to someone is just like buying a zero-coupon bond.

Say you borrow $100 from a bank at time zero. You plan to pay off your loan at time one. The interest rate is 8%. Your total loan payment (interest plus principal) to the bank at time one is $100 \times 1.08 = 108$. Conceptually, this is the same as you issuing (ie. selling) a 1-year zero coupon bond with $100 par value and 8% annual coupon to the bank. At time zero, you collect $100 from a bank. At time one, you pay the bank face amount plus coupon in the total amount of $100 + 8=108$.

So conceptually, borrowing from a bank is selling a zero coupon bond to the bank. Lending your money to bank (ie. putting money in savings account) is the same as buying a zero coupon bond.
“Forward = Stock - Zero-coupon bond” means that borrowing money and buying some stock can synthetically create a forward contract. As explained earlier, if at time zero you borrow $S_0 e^{-\delta T}$ and buy $e^{-\delta T}$ share of stock, then at time $T$ you’ll have $S_T - S_0 e^{-\delta T} e^{rT}$, which is exactly the payoff earned by the buyer in the forward contract.

To help memorize the equation “Forward = Stock - Zero-coupon bond”, notice that negative zero-coupon bond means borrowing money (ie issuing a zero coupon bond).

Make sure you understand Table 5.4 and Table 5.5. Table 5.4 has two transaction. First, you enter a forward contract to buy one share of stock at $T$ (i.e. having a long position in a forward contract). To make sure you indeed can pay the forward price $F_{0,T} = S_0 e^{-\delta T} e^{rT}$ at time $T$, at time zero you deposit the present value of this forward price $F_{0,T} e^{-rt} = S_0 e^{-\delta T} e^{rT} e^{-rt} = S_0 e^{-\delta T} e^{rT}$, which you give to the seller and receive one share of stock. So your initial cost at time zero is $S_0 e^{-\delta T}$. Your wealth at time $T$ is $S_T$ (ie. one share of stock).

The formula “Stock=Forward + Zero-coupon bond” means that buying a forward contract and a zero coupon bond is the same as owning a stock. Please note a positive sign before zero-coupon bond means that you are lending more (ie buying a zero-coupon bond).

Table 5.5 shows how to synthetically create a savings account. At time zero, you sell one forward contract and simultaneously buy $e^{-\delta T}$ share of stock. Then at $T$, you have exactly one share of stock, which you deliver to the buyer and get $F_{0,T}$. So your initial cost at time zero is $S_0 e^{-\delta T}$; your wealth at time $T$ is $F_{0,T}$. This is like you putting $S_0 e^{-\delta T}$ at time zero into a savings account and getting $F_{0,T}$ at time $T$.

The formula “Zero-coupon bond = Stock - Forward” means that buying a stock and selling is the same as lending more (ie. buying a zero-coupon bond).

5.5.1 Synthetic forwards in market-making and arbitrage

The seller in a forward contract can hedge his risk. The seller’s biggest risk is this: If at time $T$ the stock price is $S_T = \infty$ and he doesn’t already have one share of stock ready for delivery at $T$, then he’s forced to buy a share of stock from the open market at time $T$ and pay $S_T = \infty$; he’ll be bankrupt.

To hedge this risk, the seller needs to have one share of stock ready ahead of time. The seller’s hedging transactions are listed at Table 5.6.

The buyer in a forward contract can also hedge his risk. The buyer’s biggest risk is this: if at time $T$ the stock price is $S_T = 0$, he has to pay $F_{0,T}$ and buy a worthless stock. To hedge this risk, the buyer should somehow get rid of this worthless stock immediately after receiving it. This can be achieved by shorting selling $e^{-\delta T}$ share of stock at time zero (then at time $T$ one share of stock needs to be returned to the broker who facilitates the short selling). These hedging transactions are listed in Table 5.7.

How to arbitrage if the forward price is too high or too low. If the forward price is too high, you can use cash-and-carry to arbitrage: you buy stocks at time zero and carry it forward to $T$ for delivery. Cash-and-carry transactions are listed in Table 5.6.

If, on the other hand, the forward price is too low, you use reverse cash-and-carry: at time zero, you short sell stocks and simultaneously buy a forward. Reverse cash-and-carry
transactions are listed in Table 5.7.

5.5.2 No-arbitrage bounds with transaction costs

How to derive the no-arbitrage upper bond. If the forward price $F_{0,T}$ is too high, this is how to make some money:

- (1) Buy low: At $t = 0$ pay ask price and buy one stock. Incur transaction cost $k$. Total cash outgo: $S^a_0 + k$.
- (2) Sell high: At $t = 0$ sell a forward contract. Incur transaction cost $k$. Total cash outgo: $k$.
- (3) At $t = 0$, borrow $S^a_0 + 2k$ to pay for (1) and (2)
- (4) At time $T$, deliver the stock to the buyer. Get $F_{0,T}$.
- (5) At time $T$, pay $(S^a_0 + 2k)e^{rT}$

Your initial cash outgo after (1) through (5) is zero. Your payoff at time $T$ is

$$F_{0,T} - (S^a_0 + 2k)e^{rT}$$

Arbitrage is possible if:

$$F_{0,T} - (S^a_0 + 2k)e^{rT} > 0$$

$$\Rightarrow F_{0,T} > F^+ = (S^a_0 + 2k)e^{rT}$$

How to derive the no-arbitrage lower bond. If the forward price $F_{0,T}$ is too low, this is how to make some money:

- (1) Buy low: At $t = 0$ enter a forward contract to buy one stock. Incur transaction cost $k$. Total cash outgo: $k$.
- (2) Sell high: At $t = 0$ sell a stock short and get bid price $S^b_0$. Incur transaction cost $k$. Total cash inflow: $S^b_0 - k$.
- (3) After (1) and (2), the net cash inflow is $S^b_0 - 2k$. At time zero, lend $S^b_0 - 2k$ at the lending rate $r^l$ (i.e. depositing $S^b_0 - 2k$ into a savings account). At time $T$, your account should be $(S^b_0 - 2k)e^{r^lT}$.
- (4) At time $T$, pay $F_{0,T}$ and buy one stock from the seller in the forward contract. Return the stock to the broker who facilitated the short sale.
- (5) At time $T$, take out $(S^b_0 - 2k)e^{r^lT}$ from the savings account.

Your initial cash outgo after (1) through (5) is zero. Your payoff at time $T$ is

$$(S^b_0 - 2k)e^{r^lT} - F_{0,T}$$
Arbitrage is possible if:

\[(S_b^b - 2k)e^{r^T} - F_{0,T} > 0\]

\[\Rightarrow F_{0,T} < F^- = (S_b^b - 2k)e^{r^T}\]

To avoid arbitrage, we need to have:

\[(S_b^b - 2k)e^{r^T} = F^- \leq F_{0,T} \leq F^+ = (S_a^a + 2k)e^{r^T}\]

How to memorize

\[(S_b^b - 2k)e^{r^T} = F^- \leq F_{0,T} \leq F^+ = (S_a^a + 2k)e^{r^T}\]

Please note that \(S_b^b < S_a^a\) due to the bid-ask spread.

In addition, \(r^l < r^b\); when you put money in a savings account (i.e. lending money to a bank), you get a lower interest rate; when you borrow money from a bank, you pay a higher interest rate.

Your lower bound should be small; your higher bound should be big. So it makes sense to have

\[(S_b^b - 2k)e^{r^T} = F^- \leq F_{0,T} \leq F^+ = (S_a^a + 2k)e^{r^T}\]

Please note that there may be complex structure for the transaction cost where the above non-arbitrage formulas can’t be blindly used. Under a complex transaction cost structure, you’ll want to derive the non-arbitrage bound using the approach presented above. Please refer to my solution to the textbook problem 5.15 on how to handle complex transaction cost.

### 5.5.3 Quasi-arbitrage

In a full arbitrage, you either borrow money from a bank to buy a stock (if it’s priced low relative to a forward) or short sell a stock (if it is priced high relative to a forward). If you don’t need to borrow money or short sell a stock yet you can still earn free money, you are doing a quasi-arbitrage.

**Quasi-arbitrage Example 1 (textbook example)**

The corporation borrows at 8.5% and lends at 7.5%. Knowing the formula “Zero-coupon bond = Stock - Forward,” it can build (synthetically create) a zero coupon bond using stocks and forward. In this zero coupon bond, the corporation lends money at time zero and gets principal and interest paid at time \(T\). If it earns more than 7.5% (such as 8%) in this synthetically created zero-coupon bond, the corporate can stop lending at 7.5% and keep building lots of zero-coupon bonds to earn an 8% interest rate.

This is a quasi-arbitrage. Here the corporation doesn’t need to borrow money from a bank to make free money (because it already has borrowed money at 8.5% and it’s now trying to pay off its debt). If the corporation doesn’t already have a 8.5% debt to begin with, it will be silly for the corporation to borrow at 8.5%, buy a bunch of stocks, and sell bunch of forwards so it can use the formula “Zero-coupon bond = Stock - Forward” to earn an 8% interest rate.

**Quasi-arbitrage Example 2**

Suppose the forward price on a stock is too low and you already own a stock. To arbitrage, you can sell your stock (instead of selling a stock short) and buy a forward contract. You
can deposit the sales proceeds in a savings account and earn interest. Then on the expiration
date $T$ of the forward contract, you pay a low forward price $F_{0,T}$ and buy a stock. The net
effect is that you temporarily give up your ownership of a stock during $[0, T]$ but regain the
ownership of a stock at $T$. Such a transaction will give you some free money if $F_{0,T}$ is too
low.

### 5.5.4 Does the forward price predict the future spot price?

Though the author says that the forward price $F_{0,T}$ doesn’t reflect the future spot price $S_T$, 
this is a controversial topic. Experts disagree on this issue. Some argue that $F_{0,T}$ reflects $S_T$. 
For example, if you think $S_T$ will be low, you don’t want to pay a high forward price $F_{0,T}$. 
However, since SOA chose Derivatives Markets as the textbook, you might want to accept 
the view that $F_{0,T}$ doesn’t reflect the expected future spot price $S_T$ at least when you are 
taking the exam.

### 5.6 Futures contracts

#### 5.6.1 Role of the clearing house

The clearinghouse guarantees the performance of every buyer and every seller of a futures
contract. It stands as a buyer to every seller and a seller to every buyer. The buyer and the 
seller don’t have to worry that the counterparty may default.

Instead of having the seller and the buyer in a forward contract deal with each other 
directly, the clearinghouse sits between them serving as a counterparty to each of them. To 
the buyer, the clearinghouse is the seller; the clearinghouse is obligated to deliver the asset 
to the buyer at the agreed upon price. To the seller, the clearinghouse is the buyer; the 
clearinghouse is obligated to buy the asset at the expiration date with the agreed upon price.

![Diagram of futures contract](image)

With a clearinghouse as the counterparty, the buyer/seller in a futures contract can easily 
liquidate his position prior to the expiration date. If the buyer or seller wants to back out 
from a futures contract before the contract matures, he can do a reverse trade and close out 
his position.

#### 5.6.2 S&P 500 futures contract

The most important concept for the purpose of passing the exam is the value of one S&P 500 
futures contract. The value of one S&P 500 futures contract is calculated by multiplying the
5.6.3 Difference between a forward contract and a futures contract

This is a very important topic. The textbook has a nice comparison of forwards and futures. Make sure you memorize the differences.

- Forward contracts are settled at expiration; futures contracts are settled daily through marking to market.
- Forward contracts are less liquid; futures contracts are more liquid.
- Forward contracts are customized to fit the special needs of the buyer and the seller (hence less liquid); futures contracts are standardized (hence more liquid).
- Forward contracts have credit risk (any party may break its promises); futures contracts have minimal credit risks.
- Futures markets have daily price limits while forward contracts don’t.

Please note that these are general differences. Some forward contracts (such as in foreign exchanges) are standardized and liquid; some futures contracts give the buyer and the seller greater flexibility in contractual obligations.

Final point. Forward contracts are typically simpler than futures contracts. Forwards are settled at the expiration date; futures contracts are settled daily and have margins requirement. As a result, forward contracts require less management control and are good for unsophisticated and infrequent users.

5.6.4 Margins and markings to market

Open interest. Open interest is the total number of outstanding contracts held by market participants at the end of each day. For each seller of a futures contract there is a buyer. A seller and a buyer together are counted as only one contract. (The clearinghouse is excluded in the calculation of the open interest since its net position is zero).

To determine the total open interest for any given market we need only to know the totals from one side or the other, buyers or sellers, not the sum of both.

If two parties have just signed a futures contract, the open interest will increase by one contract. If the buyer and the seller have closed out a futures contract, the open interest will decrease by one contract. If one party passes off his position to a new party (one old buyer sells to one new buyer), then the open interest will not change.

Initial margin, Maintenance margin, Margin call. In the futures market, margin is the initial “good faith” deposit made into an account in order to enter into a futures contract.

When you open a futures contract, the futures exchange will ask you to deposit a minimum amount of money (cash or cash equivalent such as T-bills) into your account. This original
deposit is called the initial margin. At initial execution of a trade, both the seller and the buyer must establish a margin account.

When your position is closed out, you will be refunded the initial margin plus or minus any gains or losses that occur over the span of the futures contract.

The initial margin is the minimum deposit one needs to put in the margin account before he can enter into a new futures contract. The maintenance margin is the lowest amount an account can reach before needing to be replenished. If the margin account drops to a certain level due to daily losses, the buyer or the seller will get a margin call asking him to deposit more money into the account to bring the margin back up.

**Marking to market.** On any date when futures contracts are traded, futures prices may go up or down. Instead of waiting until the maturity date for the buyer and the seller to realize all gains and losses, the clearinghouse requires that the buyer and the seller recognize their gain and loss at the end of each day.

The best way to explain marking to market is through an example.

Example 1. Assume the current futures price for gold for delivery 6 days form today is $495 per ounce. Contract size is 100 ounces. The initial margin is 5% of the contract value. The maintenance margin is $2,000.

You are given the following prices of gold futures.

<table>
<thead>
<tr>
<th>Day</th>
<th>Futures Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (Today)</td>
<td>$495</td>
</tr>
<tr>
<td>1</td>
<td>$496</td>
</tr>
<tr>
<td>2</td>
<td>$487</td>
</tr>
<tr>
<td>3</td>
<td>$493</td>
</tr>
<tr>
<td>4</td>
<td>$480</td>
</tr>
<tr>
<td>5</td>
<td>$490</td>
</tr>
<tr>
<td>6 (Delivery Date)</td>
<td>$485</td>
</tr>
</tbody>
</table>

Calculate the daily account value of the buyer’s margin account. Determine when the buyer gets the margin call.

**Solution**

<table>
<thead>
<tr>
<th>Day</th>
<th>Futures price</th>
<th>price change</th>
<th>Gain/loss</th>
<th>Margin Bal</th>
<th>Margin call</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$495</td>
<td></td>
<td></td>
<td>$2,475</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$496</td>
<td>$1</td>
<td>$100</td>
<td>$2,575</td>
<td>$0</td>
</tr>
<tr>
<td>2</td>
<td>$487</td>
<td>-$9</td>
<td>-$900</td>
<td>$1,675</td>
<td>$325</td>
</tr>
<tr>
<td>3</td>
<td>$493</td>
<td>$6</td>
<td>$600</td>
<td>$2,600</td>
<td>$0</td>
</tr>
<tr>
<td>4</td>
<td>$480</td>
<td>-$13</td>
<td>-$1,300</td>
<td>$1,300</td>
<td>$700</td>
</tr>
<tr>
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<td>$10</td>
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<tr>
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<td>$485</td>
<td>-$5</td>
<td>-$500</td>
<td>$2,500</td>
<td>$0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Total</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total</td>
<td></td>
<td>-$1,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Explanations:
• The futures prices $495, $496, $487, $493, $480, $490, and $485 all have Day 6 as the delivery time. The Day 6 futures price $485 is the spot price on Day 6.

• The initial margin is $2,475 = 100 \times 495 \times 5\%$. Both the buyer and the seller must deposit at least this amount to their respective margin account to initiate a futures contract.

• $2,575 = 2,475 + 100$. Each day the clearinghouse updates the delivery price using the current day’s futures price. So on Day 1, the clearinghouse (who acts as the seller to the buyer) assumes that it will delivery 100 ounces of gold to the buyer at the delivery price of $496 per ounce. The buyer gain $1 per ounce (it locked in the $495 delivery price, while the current delivery price is $496). $100 gains are immediately added to the buyer’s margin account.

• $1,675 = 2,575 – 900$. The buyer loses $9 per ounce. Yesterday, the clearinghouse could delivery gold at the price of $496 per ounce. However, today the clearinghouse can deliver it at a cheaper price of $487 per ounce. The total loss $900 was immediately deducted from the buyer’s margin account.

• $325 = 2,000 – 1,675$. The margin balance $1,675 is below the maintenance margin of $2,000. So the buyer must deposit $325 to bring his margin account to the maintenance margin level.

• $2,600 = 1,675 + 325 + 600$. The buyer gains $6 per ounce.

The seller’s margin account:

<table>
<thead>
<tr>
<th>Day</th>
<th>Futures price</th>
<th>price change</th>
<th>Gain/loss</th>
<th>Margin Bal</th>
<th>Margin call</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$495</td>
<td></td>
<td></td>
<td>$2,475</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$496</td>
<td>$1</td>
<td>-$100</td>
<td>$2,375</td>
<td>$0</td>
</tr>
<tr>
<td>2</td>
<td>$487</td>
<td>-$9</td>
<td>$900</td>
<td>$3,275</td>
<td>$0</td>
</tr>
<tr>
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<td>$6</td>
<td>-$600</td>
<td>$2,675</td>
<td>$0</td>
</tr>
<tr>
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<td>$480</td>
<td>-$13</td>
<td>$1,300</td>
<td>$3,975</td>
<td>$0</td>
</tr>
<tr>
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<td>$490</td>
<td>$10</td>
<td>-$1,000</td>
<td>$2,975</td>
<td>$0</td>
</tr>
<tr>
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<td>$485</td>
<td>-$5</td>
<td>$500</td>
<td>$3,475</td>
<td>$0</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td></td>
<td>$1,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 2 Reproduce Table 5.8 in *Derivatives Markets*.

Solution

Facts:

• The notional value of one contract = 250 \times 1,100

• The number of contracts is 8

• Initial margin = 10%

• Margin account earns 6\% continuously compounding
<table>
<thead>
<tr>
<th>week</th>
<th>multiplier</th>
<th>Futures Price</th>
<th>price change</th>
<th>interest earned</th>
<th>Loss</th>
<th>Margin balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2,000.00</td>
<td>$1,100.00</td>
<td>$0.00</td>
<td>$253.99</td>
<td>$220,000.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2,000.00</td>
<td>$1,027.99</td>
<td>-$72.01</td>
<td>$88.01</td>
<td>$19,780</td>
<td>$253.99</td>
</tr>
<tr>
<td>2</td>
<td>2,000.00</td>
<td>$1,037.88</td>
<td>$9.89</td>
<td>$110.95</td>
<td>$70,700</td>
<td>$118,205.66</td>
</tr>
<tr>
<td>3</td>
<td>2,000.00</td>
<td>$1,073.23</td>
<td>$35.35</td>
<td>$192.70</td>
<td>$73,080</td>
<td>$234,894.67</td>
</tr>
<tr>
<td>4</td>
<td>2,000.00</td>
<td>$1,048.78</td>
<td>-$24.45</td>
<td>$16.26</td>
<td>$8,080</td>
<td>$44,990.57</td>
</tr>
<tr>
<td>5</td>
<td>2,000.00</td>
<td>$1,090.32</td>
<td>$41.54</td>
<td>$83,080</td>
<td>$201,422.13</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2,000.00</td>
<td>$1,106.94</td>
<td>$16.26</td>
<td>$33,240</td>
<td>$234,894.67</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2,000.00</td>
<td>$1,110.98</td>
<td>$4.04</td>
<td>$8,080</td>
<td>$243,245.86</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2,000.00</td>
<td>$1,024.74</td>
<td>-$86.24</td>
<td>$8,080</td>
<td>$234,245.86</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>2,000.00</td>
<td>$1,007.30</td>
<td>-$17.44</td>
<td>$36,248.72</td>
<td>$36,248.72</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2,000.00</td>
<td>$1,011.65</td>
<td>$4.35</td>
<td>$8,700</td>
<td>$44,990.57</td>
<td></td>
</tr>
</tbody>
</table>

Explanations

- The margin account balance at the end of Week 0 is

$$220,000 = 8 \times 250 \times 1,100 \times 10\%$$

- The interest earned in Week 1 is $253.99 = 220,000(e^{0.06/52} - 1)$

- The loss incurred in Week 1 is $-144,020 = 8 \times 250 \times (-72.01)$

- The margin account balance at the end of Week 1 is $76,233.99 = 220,000 + 253.99 - 144,020$

- The interest earned in Week 2 is $88.01 = 76,233.99(e^{0.06/52} - 1)$

5.6.5 Comparing futures and forward prices

At expiration $T$, forward price and futures price are both equal to the spot price $S_T$.

At $T-1$ (one day before expiration), a futures contracts will be marked to market the very next day $T$ and a forward contract will also be marked to market at the very next day $T$ (a forward contract is marked to market only at $T$). As a result, one day prior to expiration, a forward contracts and a futures contracts have identical cash flows and should have the same price.

The price of a futures contract two or more days before expiration is more complex. However, the following general conclusions can be drawn:

- If interest rates are positively correlated with futures prices (i.e. interest rates go up if futures prices go up), then the investor holding a long position (i.e. the buyer) will prefer a futures contract to a forward contract. When futures prices go up, the buyer realizes his gain through daily marking-to-market. This gain earns a higher interest rate. On the other hand, if futures prices decline, interest rates also decline. The buyer suffer losses through marking to market but losses are incurred when the opportunity cost is lower (because interest rates are lower). On average, a long futures contract will outperform a long forward contract.
• If interest rates are negatively correlated with futures prices (i.e. interest rates go down if futures prices go up), then the investor holding a long position (i.e. the buyer) will prefer a forward contract to a futures contract. When futures prices go up, the buyer’s gain through marking to market is reinvested at a lower interest rate. When future prices go down, the buyer’s losses are incurred at a higher interest rate. On average, a long futures contract will perform worse than a long forward contract.

5.6.6 Arbitrage in practice: S&P 500 index arbitrage

Multiple theoretically correct prices may exist for futures contracts. Reasons include:

• Future dividends are uncertain.

• Transaction costs are not zero and hence a range of fair prices may exist.

• Assumptions (such as margin requirement, credit risk, daily settlement, clearing house guarantee, etc) in futures pricing may be different, producing difference prices for the same futures contract.

• Arbitrageurs usually buy a subsect of 500-stock index. As such, futures contracts and the offsetting position in stock may not move together.
5.6.7 Appendix 5.B: Equating forwards and futures

The most important thing about Appendix 5.B is Table 5.13. Before taking Exam FM, make sure can reproduce Table 5.13.

To understand Table 5.13, let’s look at three different positions.

**Position 1** You enter into 8 long forward contracts on S&P 500 Index at time zero. The delivery date is 10 weeks from today. The forward price is 1100. Your initial margin for 8 long forwards is 217,727.21. The margin account earns a continuously compounded annual interest rate 6%. What’s your profit from your long forward position?

Additional information is:

<table>
<thead>
<tr>
<th>Week</th>
<th>Forward price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1,100.00</td>
</tr>
<tr>
<td>1</td>
<td>$1,027.99</td>
</tr>
<tr>
<td>2</td>
<td>$1,037.88</td>
</tr>
<tr>
<td>3</td>
<td>$1,073.23</td>
</tr>
<tr>
<td>4</td>
<td>$1,048.78</td>
</tr>
<tr>
<td>5</td>
<td>$1,090.32</td>
</tr>
<tr>
<td>6</td>
<td>$1,106.94</td>
</tr>
<tr>
<td>7</td>
<td>$1,110.98</td>
</tr>
<tr>
<td>8</td>
<td>$1,024.74</td>
</tr>
<tr>
<td>9</td>
<td>$1,007.30</td>
</tr>
<tr>
<td>10</td>
<td>$1,011.65</td>
</tr>
</tbody>
</table>

In the above Table, $1,100.00 is the forward price you’ll pay at Week 10. $1,027.99 is the new forward price one week later (i.e if you enter into a long S&P 500 forward at Week 1 to receive the index at Week 10, you’ll pay $1,027.99 at Week 10 to receive the index). Similarly, $1,037.88 is the forward price at Week 2 to receive the index at Week 10. So on and so forth. The final price $1,011.65 is the spot price at Week 10.

Since a forward contract is only marked to market at the expiration date, the forward prices at Week 1, 2, 3, ..., and 9 are not needed for calculating the profit. Only the forward price at Week 1 and the forward price at Week 10 matter. Please note one S&P 500 forward contract is worth 250 times the forward price. Since you long 8 contracts, your the profit is

\[ 217,727.21 e^{0.06 \times \frac{10}{52}} + 250 \times 8 \times (1,011.65 - 1,100) = 43,554.997 = 43,554.00 \]

Your profit is:

\[ 43,554.00 - 217,727.21 e^{0.06 \times \frac{10}{52}} = -176,700 \]

Alternative calculation of the profit:

\[ 250 \times 8 \times (1,011.65 - 1,100) = -176,700 \]
Position 2 You enter into 8 long futures contracts on S&P 500 Index at time zero. The
delivery date is 10 weeks from today. The futures price is 1100. Your initial margin for 8 long
futures is 217,727.21. The margin account earns a continuously compounded annual interest
rate 6%. Assuming weekly marking to market. What’s your profit from your long futures
position?

Additional information is:

<table>
<thead>
<tr>
<th>Week</th>
<th>Futures price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1,100.00</td>
</tr>
<tr>
<td>1</td>
<td>$1,027.99</td>
</tr>
<tr>
<td>2</td>
<td>$1,037.88</td>
</tr>
<tr>
<td>3</td>
<td>$1,073.23</td>
</tr>
<tr>
<td>4</td>
<td>$1,048.78</td>
</tr>
<tr>
<td>5</td>
<td>$1,090.32</td>
</tr>
<tr>
<td>6</td>
<td>$1,106.94</td>
</tr>
<tr>
<td>7</td>
<td>$1,110.98</td>
</tr>
<tr>
<td>8</td>
<td>$1,024.74</td>
</tr>
<tr>
<td>9</td>
<td>$1,007.30</td>
</tr>
<tr>
<td>10</td>
<td>$1,011.65</td>
</tr>
</tbody>
</table>

The solution is

<table>
<thead>
<tr>
<th>Week</th>
<th>Multiplier</th>
<th>Futures Price</th>
<th>Price Change</th>
<th>Margin Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2,000</td>
<td>$1,100.00</td>
<td></td>
<td>$217,727.21</td>
</tr>
<tr>
<td>1</td>
<td>2,000</td>
<td>$1,027.99</td>
<td>-$72.01</td>
<td>$73,958.58</td>
</tr>
<tr>
<td>2</td>
<td>2,000</td>
<td>$1,037.88</td>
<td>$9.89</td>
<td>$93,823.96</td>
</tr>
<tr>
<td>3</td>
<td>2,000</td>
<td>$1,073.23</td>
<td>$35.35</td>
<td>$164,632.29</td>
</tr>
<tr>
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<td>2,000</td>
<td>$1,048.78</td>
<td>-$24.45</td>
<td>$115,922.36</td>
</tr>
<tr>
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<td>2,000</td>
<td>$1,090.32</td>
<td>$41.54</td>
<td>$199,136.19</td>
</tr>
<tr>
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<td>2,000</td>
<td>$1,106.94</td>
<td>$16.62</td>
<td>$232,606.09</td>
</tr>
<tr>
<td>7</td>
<td>2,000</td>
<td>$1,110.98</td>
<td>$4.04</td>
<td>$240,954.64</td>
</tr>
<tr>
<td>8</td>
<td>2,000</td>
<td>$1,024.74</td>
<td>-$86.24</td>
<td>$68,752.83</td>
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<tr>
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<td>2,000</td>
<td>$1,007.30</td>
<td>-$17.44</td>
<td>$33,952.20</td>
</tr>
<tr>
<td>10</td>
<td>2,000</td>
<td>$1,011.65</td>
<td>$4.35</td>
<td>$42,691.40</td>
</tr>
</tbody>
</table>

Explanation of the above table.

\[
250 \times 8 = 2,000
\]

\[
217,727.21e^{0.06\times \frac{1}{52}} + 2,000 \times (-72.01) = 73,958.58
\]

\[
73,958.58e^{0.06\times \frac{1}{52}} + 2,000 \times 9.89 = 93,823.96
\]

Your profit is:

\[
42,691.40 - 217,727.21e^{0.06\times \frac{10}{52}} = -177,562.60
\]

If you compare Position 1 with Position 2, you’ll see that your profit on 8 long futures
contracts is lower than your profit on 8 long forward contracts. This is due to futures marking
to market.
**Position 3** Your enter into long futures contracts S&P 500 Index at time zero. Your goal is for Position 3 to produce exactly the same profit as in Position 1. The futures prices are as follows:

<table>
<thead>
<tr>
<th>Week</th>
<th>Futures price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1,100.00</td>
</tr>
<tr>
<td>1</td>
<td>$1,027.99</td>
</tr>
<tr>
<td>2</td>
<td>$1,037.88</td>
</tr>
<tr>
<td>3</td>
<td>$1,073.23</td>
</tr>
<tr>
<td>4</td>
<td>$1,048.78</td>
</tr>
<tr>
<td>5</td>
<td>$1,090.32</td>
</tr>
<tr>
<td>6</td>
<td>$1,106.94</td>
</tr>
<tr>
<td>7</td>
<td>$1,110.98</td>
</tr>
<tr>
<td>8</td>
<td>$1,024.74</td>
</tr>
<tr>
<td>9</td>
<td>$1,007.30</td>
</tr>
<tr>
<td>10</td>
<td>$1,011.65</td>
</tr>
</tbody>
</table>

How can you make Position 3 and Position 1 have the same profit?

Solution
<table>
<thead>
<tr>
<th>Week</th>
<th>Multiplier</th>
<th>Futures Price</th>
<th>Price Change</th>
<th>Margin Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1,979.34</td>
<td>$1,100.00</td>
<td></td>
<td>$217,727.21</td>
</tr>
<tr>
<td>1</td>
<td>1,981.62</td>
<td>$1,027.99</td>
<td>-$72.01</td>
<td>$75,446.43</td>
</tr>
<tr>
<td>2</td>
<td>1,983.91</td>
<td>$1,037.88</td>
<td>$9.89</td>
<td>$95,131.79</td>
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<tr>
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<td>$1,073.23</td>
<td>$35.35</td>
<td>$165,372.88</td>
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<td>-$24.45</td>
<td>$117,001.18</td>
</tr>
<tr>
<td>5</td>
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<td>$1,090.32</td>
<td>$41.54</td>
<td>$199,738.33</td>
</tr>
<tr>
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<td>1,993.09</td>
<td>$1,106.94</td>
<td>$16.62</td>
<td>$233,055.87</td>
</tr>
<tr>
<td>7</td>
<td>1,995.39</td>
<td>$1,110.98</td>
<td>$4.04</td>
<td>$241,377.01</td>
</tr>
<tr>
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<td>$1,024.74</td>
<td>-$86.24</td>
<td>$69,573.26</td>
</tr>
<tr>
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<td>$1,007.30</td>
<td>-$17.44</td>
<td>$34,813.80</td>
</tr>
<tr>
<td>10</td>
<td>$1,011.65</td>
<td></td>
<td></td>
<td>$43,554.00</td>
</tr>
</tbody>
</table>

Explanation. Since marking to market magnifies gains and losses, you’ll need to hold fewer than 8 long futures contracts at $t = 0$. To determine how many contracts to hold at $t = 0$, notice that if Week 1 futures price is one dollar more than Week 0 futures price, you’ll realize one dollar gain. This one dollar will be deposited into your margin account at the end of Week 1 and accumulate with interest to the end of Week 10 (i.e. earning 9-month interest). To undo the effect of marking to market, you should hold $8e^{-0.06\times\frac{9}{52}} = 7.91735295$ contracts.

The contract size is $250 \times 7.91735295 = 1,979.34$, which is less than 2,000.

What if the Week 1 futures price is one dollar less than Week 0 futures price? In this case, you’ll realize one dollar loss. This one dollar will be deducted from your margin account at the end of Week 1, reducing the future value of your margin account. Once again, you need to hold $8e^{-0.06\times\frac{9}{52}} = 7.91735295$ contracts.

Similarly, you need to hold $8e^{-0.06\times\frac{3}{52}} = 1,981.62$ contracts at the end of Week 1 (every dollar gain recognized at the end of Week 2 will accumulate 8 months interest to Week 10). And you need to hold $8e^{-0.06\times\frac{7}{52}} = 7.93564486$ at the end of Week 3. So on and so forth.

After marking to market at the end of Week 9, the futures contract becomes a forward contract (since both contracts will be marked to market at the final week). At the end of Week 9, you should hold exactly 8 futures contracts.

Explanations of other calculations.

$217,727.21e^{0.06\times\frac{9}{52}} + 1,979.34 \times (-72.01) = 75,446.43$

$75,446.43e^{0.06\times\frac{9}{52}} + 1,981.62 \times (9.89) = 95,131.79$

You ending margin account value at the end of Week 10 is

$34,813.80e^{0.06\times\frac{9}{52}} + 2,000 \times (4.35) = 43,554.00$

Your profit is:

$43,554.00 - 217,727.21e^{0.06\times\frac{9}{52}} = -176,700.00$

Now you see that Position 1 and Position 3 have the same ending margin account and the same profit.
**Practice problem** You enter into 8 long futures contracts on S&P 500 Index at time zero. The delivery date is 10 weeks from today. The initial margin for the 8 contracts is 300,000. The margin account earns a continuously compounded annual interest rate 6%. Calculate your profit from the long position under the following 3 scenarios:

1. Marking to market is done only at the expiration date.
2. Marking to market is done weekly and you always hold 8 contracts.
3. Marking to market is done weekly. You re-balance your holdings weekly to produce the same profit as Scenario 1.

Futures prices are:

<table>
<thead>
<tr>
<th>Week</th>
<th>Futures Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1,000</td>
</tr>
<tr>
<td>1</td>
<td>$1,020</td>
</tr>
<tr>
<td>2</td>
<td>$1,015</td>
</tr>
<tr>
<td>3</td>
<td>$1,026</td>
</tr>
<tr>
<td>4</td>
<td>$1,110</td>
</tr>
<tr>
<td>5</td>
<td>$1,065</td>
</tr>
<tr>
<td>6</td>
<td>$1,105</td>
</tr>
<tr>
<td>7</td>
<td>$1,085</td>
</tr>
<tr>
<td>8</td>
<td>$1,093</td>
</tr>
<tr>
<td>9</td>
<td>$1,008</td>
</tr>
<tr>
<td>10</td>
<td>$1,004</td>
</tr>
</tbody>
</table>

Solution

Scenario 1

The ending balance of the margin account is

\[ 300,000e^{0.06 \times \frac{10}{52}} + 8 \times 250 \times (1,004 - 1,000) = 311,481.59 \]

The profit is:

\[ 311,481.59 - 300,000e^{0.06 \times \frac{10}{52}} \]

The profit is:

\[ 8 \times 250 \times (1,004 - 1,000) = 8,000 \]
Scenario 2

<table>
<thead>
<tr>
<th>Week</th>
<th>Multiplier</th>
<th>Futures Price</th>
<th>Price Change</th>
<th>Margin Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2,000</td>
<td>$1,000</td>
<td></td>
<td>$300,000.00</td>
</tr>
<tr>
<td>1</td>
<td>2,000</td>
<td>$1,020</td>
<td>$20</td>
<td>$340,346.35</td>
</tr>
<tr>
<td>2</td>
<td>2,000</td>
<td>$1,015</td>
<td>-$5</td>
<td>$330,739.29</td>
</tr>
<tr>
<td>3</td>
<td>2,000</td>
<td>$1,026</td>
<td>$11</td>
<td>$353,121.13</td>
</tr>
<tr>
<td>4</td>
<td>2,000</td>
<td>$1,110</td>
<td>$84</td>
<td>$521,528.81</td>
</tr>
<tr>
<td>5</td>
<td>2,000</td>
<td>$1,065</td>
<td>-$45</td>
<td>$432,130.92</td>
</tr>
<tr>
<td>6</td>
<td>2,000</td>
<td>$1,105</td>
<td>$40</td>
<td>$512,629.82</td>
</tr>
<tr>
<td>7</td>
<td>2,000</td>
<td>$1,085</td>
<td>-$20</td>
<td>$473,221.66</td>
</tr>
<tr>
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<td>2,000</td>
<td>$1,093</td>
<td>$8</td>
<td>$489,768.00</td>
</tr>
<tr>
<td>9</td>
<td>2,000</td>
<td>$1,008</td>
<td>-$85</td>
<td>$320,333.44</td>
</tr>
<tr>
<td>10</td>
<td>2,000</td>
<td>$1,004</td>
<td>-$4</td>
<td>$312,703.27</td>
</tr>
</tbody>
</table>

The ending balance at the end of Week 10 is $312,703.27. The profit is: $312,703.27 - $300,000e^{0.06\frac{4}{12}} = 9,221.69

Sample calculations:

\[340,346.35 = 300,000e^{0.06\frac{4}{12}} + 2000 \times 20\]
\[330,739.29 = 340,346.35e^{0.06\frac{4}{12}} + 2000 \times (-5)\]
Scenario 3

<table>
<thead>
<tr>
<th>Week</th>
<th>Multiplier</th>
<th>Futures Price</th>
<th>Price Change</th>
<th>Margin Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1,979.34</td>
<td>$1,000</td>
<td></td>
<td>$300,000.00</td>
</tr>
<tr>
<td>1</td>
<td>1,981.62</td>
<td>$1,020</td>
<td>$20</td>
<td>$339,933.12</td>
</tr>
<tr>
<td>2</td>
<td>1,983.91</td>
<td>$1,015</td>
<td>-$5</td>
<td>$330,417.46</td>
</tr>
<tr>
<td>3</td>
<td>1,986.20</td>
<td>$1,026</td>
<td>$11</td>
<td>$352,621.95</td>
</tr>
<tr>
<td>4</td>
<td>1,988.49</td>
<td>$1,110</td>
<td>$84</td>
<td>$519,870.00</td>
</tr>
<tr>
<td>5</td>
<td>1,990.79</td>
<td>$1,065</td>
<td>-$45</td>
<td>$430,987.93</td>
</tr>
<tr>
<td>6</td>
<td>1,993.09</td>
<td>$1,105</td>
<td>$40</td>
<td>$511,117.13</td>
</tr>
<tr>
<td>7</td>
<td>1,995.39</td>
<td>$1,085</td>
<td>-$20</td>
<td>$471,845.44</td>
</tr>
<tr>
<td>8</td>
<td>1,997.69</td>
<td>$1,093</td>
<td>$8</td>
<td>$488,353.32</td>
</tr>
<tr>
<td>9</td>
<td>2,000.00</td>
<td>$1,008</td>
<td>-$85</td>
<td>$319,113.17</td>
</tr>
<tr>
<td>10</td>
<td>$1,004</td>
<td></td>
<td>-$4</td>
<td>$311,481.59</td>
</tr>
</tbody>
</table>

The ending balance at the end of Week 10 is $311,481.59. The profit is $311,481.59 - $300,000e^{0.06\times\frac{1}{52}} = 8,000.

Sample calculation:

\[ 1,979.34 = 8 \times 250e^{-0.06\times\frac{0}{52}} \]
\[ 1,981.62 = 8 \times 250e^{-0.06\times\frac{1}{52}} \]
\[ 1,997.69 = 8 \times 250e^{-0.06\times\frac{1}{52}} \]
\[ 2,000 = 8 \times 250e^{-0.06\times\frac{0}{52}} \]
\[ 339,933.12 = 300,000e^{0.06\times\frac{1}{52}} + 1,979.34 \times 20 \]
\[ 330,417.46 = 339,933.12e^{0.06\times\frac{1}{52}} + 1,981.62 \times (-5) \]

◊
Chapter 8

Swaps

Chapter 8 of Derivative Markets is a difficult but important chapter. The most important concept is pricing of swaps. SOA can easily come up questions on this topic. Make sure you know how to calculate the price of a swap.

8.1 An example of a commodity swap

First, let’s understand what a swap is. Let’s not bother memorizing the textbook definition of a swap but focus on the essence of a swap.

A swap is multiple forward contracts combined into one big contract. For example, you are an owner of an airline company. Your two biggest operating expenses will be labor and fuel. Labor cost can be pretty much controlled by adding or reducing staff. However, the fuel cost is subject to a wild fluctuation. From time to time, we all complain about how expensive gas is, but wait till you own multiple airplanes and see how quickly your gas bill goes to the roof.

One way to control the fuel cost is to use a series of forward contracts. For example, at $t = 0$, you sign up 3 separate forward contracts with the same supplier to lock in the fuel cost for the next 3 years.

The first forward contract requires the supplier to deliver 100 barrels of oil at $t = 1$ (end of Year 1) at $20$ per barrel; the second contract requires the supplier to deliver 100 barrels of oil at $t = 2$ at $25$ per barrel; and the third contract requires the supplier to deliver 100 barrels of oil at $30$ per barrel.

Accordingly, you’ll make 3 separate payments to the supplier. Your first payment is $100 \times 20 = 2,000$ at $t = 1$; your second payment is $100 \times 25 = 2,500$ at $t = 2$; and your third payment is $100 \times 30 = 3,000$ at $t = 3$.

As a busy CEO, you quickly realize that signing up 3 separate contracts and keeping track of 3 separate payments is a bit of work. ”Why don’t I sign one big contract and make 3 level payments to the supplier?” you ask. Indeed, combing 3 separate forward contracts into one and making level payments is a good idea.

In the combined contract, the supplier is required to deliver 100 barrels of oil at $20$ per barrel at $t = 1$, 100 barrels of oil at $25$ per barrel at $t = 2$, and 100 barrels of oil at $30$ per barrel at $t = 3$. Accordingly, you make payments of $X$ at $1, 2, 3$ respectively.

Next, let’s calculate the level payment $X$.  

55
Your payments BEFORE the 3 forward contracts are combined:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>$2,000</td>
<td>$2,500</td>
<td>$3,000</td>
<td></td>
</tr>
</tbody>
</table>

Your payments AFTER the 3 forward contracts are combined:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

Assume that the interest rate is 10% per year. The present value of your total payments should be the same whether you pay 3 forward contracts separately or you pay level payments in a combined contract.

\[2000v + 2500v^2 + 3000v^3 = X(v + v^2 + v^3)\]

\[v = 1.1^{-1}\]

\[\Rightarrow X = 2468.28\]

What if the interest rate is not level? Let’s say that the interest rate is 10% per year during \(t \in [0, 1]\), 12% per year during \(t \in [0, 2]\), and 14% per year during \(t \in [0, 3]\). No problem. The present value of your total payments should be the same whether you pay 3 forward contracts separately or you pay level payments in a combined contract.

\[
\frac{2000}{1.1} + \frac{2500}{1.12^2} + \frac{3000}{1.14^3} = X \left( \frac{1}{1.1} + \frac{1}{1.12^2} + \frac{1}{1.14^3} \right)
\]

\[\Rightarrow X = 24650.84\]

The combined contract is called a swap. A swap is a series of exchanges between two parties that takes place in multiple dates in the future. Typically, one party of the swap pays level cash flows and receives variable cash flows (or receives something of variable values); the other party receives level cash flows and pays variable cash flows (or delivers something of variable values). In our example, you (owner of an airline) pay the supplier \(X\) at \(t = 1, 2, 3\) respectively; the supplier gives you 100 barrels of oil at \(t = 1, 2, 3\) respectively.

### 8.1.1 Physical versus financial settlement

Next, let’s talk about how a swap is settled. Use our example and assume the interest rate is 10% for all years. The swap can be settled if you pay the supplier $2,468.28 at \(t = 1, 2, 3\) respectively and the supplier has you come to its warehouse to pick up 100 barrels of oil at \(t = 1, 2, 3\) respectively. This is physical settlement.

However, the swap can be settled in a simpler way called financial settlement. You still write a check of $2,468.28 at \(t = 1, 2, 3\) respectively. However, instead of giving you 100 barrels of oil at \(t = 1, 2, 3\), the supplier simply gives you the market price of 100 barrels at


\( t = 1, 2, 3 \). You just go to the market and buy 100 barrels using the money the supplier gives you. For example, if the market price is $20 per barrel at \( t = 1 \), $25 per barrel at \( t = 2 \), and $30 per barrel at \( t = 3 \), the supplier gives you a check of $2,000 at \( t = 1 \), another check of $2,500 at \( t = 2 \), and the third check of $3,000 at \( t = 3 \). You can go to the market place and purchase 100 barrels of oil at \( t = 1, 2, 3 \) yourself.

The settlement can be simplified further. Instead of you writing the supplier a check of $2,468.28 at \( t = 1, 2, 3 \) respectively and the supplier writing you a check of $2,000 at \( t = 1 \), $2,500 at \( t = 2 \), and $3,000 at \( t = 3 \), you and the supplier can focus on the net payment. The swap can be settled this way:

- You write the supplier a check of \( 2,468.28 - 2,000 = 468.28 \) at \( t = 1 \)
- The supplier writes you a check of \( 2,500 - 2,468.28 = 31.72 \) at \( t = 2 \)
- The supplier writes you a check of \( 3,000 - 2,468.28 = 531.72 \) at \( t = 3 \)

Typically in a swap the two parties don’t write checks to each other back and forth. At each settlement date, the net payment is made from the party who owes more obligation to the other party who owes less.

### 8.1.2 Pricing swaps

Now let’s think about how we calculated the price of the level payment from the fixed-payer to the floating-payer. The equation used for solving the fixed level payments is:

\[
\text{PV of fixed payments} = \text{PV of floating payments}
\]

This equation says that when the swap contract is signed at \( t = 0 \), two parties promise to exchange equal value of cash flows; nobody wins and nobody loses. The market value of the swap is zero. The swap is a fair game.

Remember this equation. Whenever you need to calculate the fixed payment, use this equation.

Though the market value of a swap is zero at \( t = 0 \), as time passes, the market value of the swap is generally not zero. Reasons include:

- **The supply and demand may change.** For example, one day after the swap contract is signed, a war breaks out in a major oil-producing country. Suddenly, there’s a shortage of oil worldwide. The oil price goes up. If this happens, the supplier’s cost of delivering 100 barrels of oil to you at \( t = 1, 2, 3 \) will go up. However, your payment to the supplier at \( t = 1, 2, 3 \) is fixed at \( X \), which was based the old market condition. You get a good deal from the swap. If, on the other hand, one day after the swap contract is signed, there’s an over-abundance of oil supply in the market and the oil price plummets. No matter how low the oil price turns out to be at \( t = 1, 2, 3 \), you still have to pay the supplier the pre-set price \( X \) at \( t = 1, 2, 3 \). You will suffer a loss.
• **The interest rate may change.** The fixed payment \( X \) is calculated according to the expected yield curve at \( t = 0 \). If, after the swap contract is signed, the actual yield curve is different from the yield curve expected at \( t = 0 \), the present value of the fixed payments will be different from the present value of the floating payments.

• **Even if the supply/demand and interest rate won’t change, the value of the swap is zero only before the first swap payment \( X \) is made.** Once the first payment \( X \) is made, the present value of the fix-payer’s cash flows for the remaining duration of the swap is no longer equal to the present value of the float-payer’s cash flows for the remaining duration of the swap.

For example, if the interest rate is 10%, you need to pay the level amount of $2,468.28 at \( t = 1, 2, 3 \). If you pay as you go (instead of paying level payments $2,468.28 at \( t = 1, 2, 3 \)), you will pay $2,000 at \( t = 1 \), $2,500 at \( t = 2 \), and $3,000 at \( t = 3 \). Compared with the level payment $2,468.28 at \( t = 1, 2, 3 \), you overpay $468.28 at \( t = 1 \), underpay \( 2,500 - 2,468.28 = 31.72 \) at \( t = 2 \), and underpay \( 3,000 - 2,468.28 = 531.72 \) at \( t = 3 \). The present value of your overpayment and under-payments should be zero:

\[
468.28v - 31.72v^2 - 531.72v^3 = 0
\]

where \( v = 1.1^{-1} \)

If the two parties want to back out from the swap after \( t = 1 \), the floating-payer needs to refund the fixed-payer $468.28. If the two parties want to back out from the swap after \( t = 2 \), the fixed-payer needs to pay the floating-payer $531.72v = 483.38$.

### 8.2 Interest rate swap

The most common interest rate swap is “fixed-for-floating” swap, commonly referred to as a “plain vanilla swap.”

#### 8.2.1 Key features of an interest rate swap

- The notional principal is fixed at \( t = 0 \).

- The notional principal is never exchanged. It’s a scaling factor to calculate the interest rate payments.

- One party agrees to pay a fixed interest rate applied to the notional principal regularly during the life of the swap. The life of a swap is called the swap term or tenor.

- The other party agrees to pay a floating (ie. variable) interest rate applied to the notional principal regularly during the life of a swap. The floating rate is typically based on a benchmark rate such as LIBOR.

- The floating rate is “set in advance, paid in arrears.” The floating rate is determined at the beginning of a settlement period but is paid at the end of the settlement period.
8.2.2 Example of a plain vanilla interest rate swap

Consider a hypothetical 5-year swap initiated on 3/1/2006 between Microsoft and Intel. The details of the swap are as follows:

- Swap initiation date: March 1, 2006
- Swap tenor: 3 years
- Settlement dates: September 1 and March 1
- Notional principal: $100 million
- Fixed rate payer: Microsoft pays 6.04% per year
- Floating rate payer: Intel pays 6-month LIBOR

Cash flows received by Microsoft (in one million dollars):

<table>
<thead>
<tr>
<th>Time</th>
<th>Dates</th>
<th>6-month LIBOR</th>
<th>Intel to Microsoft</th>
<th>Microsoft to Intel</th>
<th>Net $ to Microsoft</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3/1/2006</td>
<td>LIBOR(_0) = 5.80%</td>
<td>$2.9</td>
<td>$3.02</td>
<td>($0.12)</td>
</tr>
<tr>
<td>0.5</td>
<td>9/1/2006</td>
<td>LIBOR(_{0.5}) =?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3/1/2007</td>
<td>LIBOR(_1) =?</td>
<td></td>
<td>$3.02</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>9/1/2007</td>
<td>LIBOR(_{1.5}) =?</td>
<td></td>
<td>$3.02</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3/1/2008</td>
<td>LIBOR(_2) =?</td>
<td></td>
<td>$3.02</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>9/1/2008</td>
<td>LIBOR(_{2.5}) =?</td>
<td></td>
<td>$3.02</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3/1/2009</td>
<td>LIBOR(_3) =?</td>
<td></td>
<td>$3.02</td>
<td></td>
</tr>
</tbody>
</table>

Explanations:

- 2.9 = 5.8\% \times \frac{180}{360} \times 100. This is the 6-month interest on $100 million principal with LIBOR rate. The 6-month LIBOR rates are simple annual rates. Here we assume 1 year=360 days and 6 months =180 days. In reality, the two parties in the swap can determine the day counting method for calculating the interest payment. For example, they can use $\frac{ActualDays}{365}$ to calculate the interest payment.

- 3.02 = 6.04\% \times \frac{180}{365} \times 100. This is the 6-month interest on $100 million principal with 6.04\% rate. The 6% rate is a simple annual interest rate.

- When the swap contract is signed on 3/1/2006, only the LIBOR for the next 6 months is known. The LIBOR rates beyond 9/1/2006 are unknown.

- LIBOR is set in advance and paid in arrears. For example, on 3/1/2006, the LIBOR rate 5.8\% is used for calculating Intel’s payment to Microsoft on 9/1/2006. On 9/1/2006, the next 6-month LIBOR rate is used for calculating the interest payment on 3/1/2007.

- LIBOR rates beyond 3/1/2009 are irrelevant to the swap. The floating interest payment on the final settlement date 3/1/2009 is based on LIBOR from 9/1/2008 to 3/1/2009.
Suppose the LIBOR rates are known. Then the cash flow diagrams are as follows:

<table>
<thead>
<tr>
<th>Time</th>
<th>Dates</th>
<th>6-month</th>
<th>Intel to Microsoft</th>
<th>Microsoft to Intel</th>
<th>Net $ to Microsoft</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3/1/2006</td>
<td>5.80%</td>
<td>$2.90</td>
<td>($3.02)</td>
<td>($0.12)</td>
</tr>
<tr>
<td>0.5</td>
<td>9/1/2006</td>
<td>5.90%</td>
<td>$2.95</td>
<td>($3.02)</td>
<td>($0.07)</td>
</tr>
<tr>
<td>1</td>
<td>3/1/2007</td>
<td>6.00%</td>
<td>$3.00</td>
<td>($3.02)</td>
<td>($0.02)</td>
</tr>
<tr>
<td>1.5</td>
<td>9/1/2007</td>
<td>6.10%</td>
<td>$3.05</td>
<td>($3.02)</td>
<td>$0.03</td>
</tr>
<tr>
<td>2</td>
<td>3/1/2008</td>
<td>6.20%</td>
<td>$3.10</td>
<td>($3.02)</td>
<td>$0.08</td>
</tr>
<tr>
<td>2.5</td>
<td>9/1/2008</td>
<td>6.30%</td>
<td>$3.15</td>
<td>($3.02)</td>
<td>$0.13</td>
</tr>
<tr>
<td>3</td>
<td>3/1/2009</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8.2.3 Motivations for an interest swap

Example 1 Interest rate swap changes floating liabilities to fixed liabilities or vice versa.

Consider a bank that receives deposits from small investors like you and me and lends out money in mortgage loans. Here the bank has a mismatch of interest rates. On the one hand, it receives fixed interest rate on mortgages; most new homeowners avoid variable rate loans and want to lock in a fixed interest on their mortgages for the next 15 or 30 years. On the other hand, the bank pays floating interest rate to individual depositors. We want our banks to give us interest rate that reflects the market interest rate. If the market interest is high but we get a low interest rate from a bank, we can deposit our money elsewhere and earn a higher interest rate.

Suppose the bank pays deposits at a floating rate of LIBOR - 1% but receives mortgage payments at fixed 6%. If LIBOR=7%, the bank will break even. If LIBOR > 7%, the bank will lose money. To hedge the risk that LIBOR may go up, the bank can enter a swap and “pay fixed, and get float.” In the swap, the bank pays fixed 6% and receives LIBOR. The counterparty receives fixed 6% and pays LIBOR.

Before the swap, the bank faces the risk that LIBOR may go up and exceed 7%.

After the swap, the bank receives 6% mortgage payments from new homeowners and passes the payments to the counterparty. The counterparty receives 6% fixed and sends LIBOR to the bank. The bank pays its depositors LIBOR - 1%, earning 1% profit no matter how high LIBOR may be.
Similarly, a firm can use interest rate swaps to change its floating assets to fixed assets or vice versa. Consider an firm which holds a floating rate note (FRN). An FRN is a bond except that the coupon rate is a variable rate (such as LIBOR + 1%) instead of a fixed rate (such as 8%). The firm originally bought an FRN instead of a fixed coupon bond because it forecast that the interest rate would rise. Suppose two years later after purchasing the FRN, the firm forecast that the interest rate would fall. The firm can use an interest rate swap to change its floating interest rate payment received from the FRN issuer to a set of fixed cash flows.

Example 2 Interest rate swap reduces a firm’s borrowing cost (comparative advantage) First, a few words on comparative advantage. The idea behind comparative advantage is “Do what you the very best, no even your second best.” A classic example is that the best attorney in town happens to be the best typist in down. Since the attorney earns more income by doing litigation rather than typing, he should work exclusively on litigation and give up typing. He can hire a secretary to type for him.

Suppose there are two companies. One is a well known company, which can borrow money at lower rate. The other is a new startup and doesn’t have a track record. It has to borrow money at a higher rate.

Their borrowing rates are as follows:
### Company | Fixed borrowing rate | Floating borrowing rate
--- | --- | ---
Mr. Established | 10% | LIBOR+0.3%
Mr. StartUp | 11.20% | LIBOR+1%
Mr. StartUp minus Mr. Established | △fixed = 1.2% | △float = 0.7%
Comparative advantage | △fixed - △float = 0.5%

Mr. Established has absolute advantage in borrowing fixed and floating rate debt. Mr. StartUp, on the other hand, has comparative advantage in borrowing floating rate: it pays only 0.7% more than Mr. Established in borrowing floating rate debt yet pays 1.2% more than Mr. Established in borrowing at a fixed rate.

Suppose Mr. Established wants to borrow some money at LIBOR and Mr. StartUp wants to borrow the same amount of money at a fixed rate.

**Total borrowing cost BEFORE using an interest rate swap:**

- Mr. Established wants to borrow at a floating rate. So it goes out, finds a lender, and borrows $1 million at a floating rate of LIBOR+0.3% per year.
- Mr. StartUp wants to borrow at a fixed rate. So it goes out, finds a lender, and borrows at 11.2%.

Total borrowing cost:

\[
\text{LIBOR} + 0.3\% + 11.2\% = \text{LIBOR} + 11.5\%
\]

**Total borrowing cost AFTER using an interest rate swap:**

- Mr. Established goes out, finds a lender, and borrow $1 million at a fixed rate of 10%. Mr. StartUp goes out, finds a lender, and borrows $1 million at a floating rate of LIBOR+1%. Then Mr. Established and Mr. StartUp enter an interest rate swap where Mr. Established pays fixed and gets float and Mr. StartUp pays float and gets fixed.

Total borrowing cost:

\[
10\% + \text{LIBOR} + 1\% = \text{LIBOR} + 11\%
\]

By entering a swap, Mr. Established and Mr. StartUp achieve 0.5% saving. Let’s assume that Mr. Established and Mr. StartUp share the 0.5% saving equally (i.e. each gains 0.25% saving).

Mr. Established’s cash flows are as follows:

- Go to a lender and borrow $1M at 10%. Pay 10% interest rate to this lender. So the annual payment to the lender is \(1 \times 10\% = 0.1M\)
- Pay Mr. StartUp LIBOR on $1M principal. For example, if LIBOR happens to be 9% in Year 1, Mr. Established will pay Mr. StartUp 9% \(\times 1M = 0.09M\) at the end of Year 1.
- Receive 9.95% interest payment on $1M principal (i.e. receive $0.0995M) annual payment from Mr. StartUp.

Mr. StartUp’s cash flows are as follows:

- Go to a lender and borrow $1 M at LIBOR+1%.
- Pay Mr. Established 9.95% annual interest on $1 M principal (i.e. Pay $0.0995M)
- Receive LIBOR interest payment on $1M principal from Mr. Established.
The net result of direct borrowing:

- Mr. Established gets $1M loan and pays LIBOR+0.3%.

- Mr. StartUp gets $1M loan and pays 11.2%.

- Total borrowing cost: LIBOR+0.3% + 11.2% = LIBOR+11.5%

**INTEREST SWAP WITHOUT A SWAP DEALER** Assume Mr. Established and Mr. StartUp directly negotiate a swap without a swap dealer.
Result after the swap:

- Mr. Established gets $1M loan. Its annual interest rate for loan repayment on $1M principal is $10\% - 9.95\% + LIBOR = LIBOR + 0.05\%$. Mr. Established borrows $1M and pays $LIBOR + 0.05\%$. This is 0.25% less than the direct borrowing rate of $LIBOR + 0.3\%$ without a swap.

- Mr. StartUp gets $1M loan. Its annual interest rate for loan repayment on $1M principal is $LIBOR + 1\% + 9.95\% - LIBOR = 10.95\%$. This is 0.25% less than the direct borrowing rate of 11.2%.

- Total borrowing cost: $LIBOR + 0.05\% + 10.95\% = LIBOR + 11\%$

**INTEREST SWAP WITH A DEALER** Suppose that the dealer receives 9.97% from Mr. StartUp but passes on only 9.93% to Mr. Established (swap bid/ask spread). The dealer takes away 9.97% - 9.93% = 0.04% as a compensation for setting up the swap.
The net result after the swap using a swap dealer:

- Mr. Established gets $1M loan. Its annual interest rate for loan repayment on $1M principal is $10% - 9.93% + \text{LIBOR} = \text{LIBOR} + 0.07\%$. Mr. Established borrows $1M and pays LIBOR+0.07\%. This is 0.23\% less than the direct borrowing rate of LIBOR +0.3\% without a swap.

- Mr. StartUp gets $1M loan. Its annual interest rate for loan repayment on $1M principal is \text{LIBOR}+1\% + 9.97\% - \text{LIBOR} = 10.97\%. This is 0.23\% less than the direct borrowing rate of 11.2\%.

- Total borrowing cost: LIBOR+0.07\% + 10.97\% = LIBOR+11.04\%

### 8.2.4 How to price an interest rate swap

Let’s walk through the example on *Derivatives Markets* page 255 “Pricing and the swap counterparty.” This example relies on Table 7.1. Make sure you understand Table 7.1.

Excerpt of Table 7.1 from *Derivatives Markets*

<table>
<thead>
<tr>
<th>Yrs to maturity</th>
<th>Zero-coupon bond yield</th>
<th>Zero-coupon bond price</th>
<th>1-Yr implied forward rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.00%</td>
<td>$P_0(0, 1) = 0.943396$</td>
<td>$r(0, 1) = 0.06$</td>
</tr>
<tr>
<td>2</td>
<td>6.50%</td>
<td>$P_0(0, 2) = 0.881659$</td>
<td>$r(1, 2) = 0.0700236$</td>
</tr>
<tr>
<td>3</td>
<td>7.00%</td>
<td>$P_0(0, 3) = 0.816298$</td>
<td>$r(2, 3) = 0.0800705$</td>
</tr>
</tbody>
</table>
Here the zero-coupon bond prices with maturity of 1 year, 2 years, and 3 years are bond selling prices in the market. They are the known values. The remaining values are based on bond price.

<table>
<thead>
<tr>
<th>Yrs to maturity</th>
<th>Zero-coupon bond yield</th>
<th>Zero-coupon bond price</th>
<th>1-Yr implied forward rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>?</td>
<td>$P_0(0, 1) = 0.943396$</td>
<td>$r(0, 1) =$?</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
<td>$P_0(0, 2) = 0.881659$</td>
<td>$r(1, 2) =$?</td>
</tr>
<tr>
<td>3</td>
<td>?</td>
<td>$P_0(0, 3) = 0.816298$</td>
<td>$r(2, 3) =$?</td>
</tr>
</tbody>
</table>

Zero-coupon bond yields are calculated as follows. Consider a 1-year zero coupon bond. For each dollar to be received at $t = 1$, the buyer has to pay 0.943396 at $t = 0$. So the bond yield is calculated as follows:

$$0.943396(1 + i) = 1$$

$$i \approx 6\%$$

Consider a 2-year zero coupon bond. For each dollar to be received at $t = 2$, the buyer has to pay 0.881659 at $t = 0$. So the bond yield is calculated as follows:

$$0.881659(1 + i)^2 = 1$$

$$i \approx 6.5\%$$

Consider a 3-year zero coupon bond. For each dollar to be received at $t = 3$, the buyer has to pay 0.816298 at $t = 0$. So the bond yield is calculated as follows:

$$0.816298(1 + i)^3 = 1$$

$$i \approx 7\%$$

The 1-year implied forward rates are calculated as follows.

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r(0,1)$</td>
<td>$r(1,2)$</td>
<td>$r(2,3)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$6%$</td>
<td></td>
<td>$6.5%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$7%$</td>
</tr>
</tbody>
</table>
At $t = 0$, the 1-year implied forward rate during $t \in [0, 1]$ is obviously 6%. So $r(0, 1) = 6\%$

At $t = 1$, the 1-year implied forward rate during $t \in [1, 2]$ is $r(1, 2)$.

$$(1 + 6\%)[1 + r(1, 2)] = (1 + 6.5\%)^2$$

$$r(1, 2) = 7.00236\%$$

At $t = 2$, the 1-year implied forward rate during $t \in [2, 3]$ is $r(2, 3)$.

$$(1 + 6.5\%)^2[1 + r(2, 3)] = (1 + 7\%)^3$$

$$r(2, 3) = 8.00705\%$$

Next, let’s determine the price of the swap (ie. finding the constant rate $R$ by which the swap dealer gets paid by the counterparty). Remember the swap dealer gets fixed and pays float. And the counterparty (the investor) gets float and pays fixed.

The annual payment by the swap dealer to the counterparty is: $R \times$ Notional Principal

The annual payment by counterparty to the swap dealer is: LIBOR $\times$ Notional Principal

As explained before, when two parties enter an interest rate swap, only the LIBOR rate during the 1st settlement period is known. The LIBOR rates beyond the first settlement is not known. Then how can we determine the floating payments if we know only the LIBOR rate for the first settlement period?

When pricing a swap, we can create various LIBOR interest rates using forward interest rate agreements. For the non-arbitrage principal to hold, the LIBOR rates must be close to the 1-year forward interest rate extracted from the forward interest rate agreements.
Set up the equation: PV of fixed payments = PV of floating payments. Set the notional principal to $1. We have:

\[
R \left( \frac{1}{1.06} + \frac{1}{1.065^2} + \frac{1}{1.07^3} \right) = \frac{6\%}{1.06} + \frac{7.0024\%}{1.065^2} + \frac{8.0071\%}{1.07^3}
\]

\[
R = \frac{0.06 + 0.070024 + 0.080071}{1.06 + 1.065^2 + 1.07^3}
\]

\[
R = \frac{0.943396 \times 6\% + 0.881659 \times 7.0024\% + 0.816298 \times 8.0071\%}{0.943396 + 0.881659 + 0.816298}
\]

\[
R = \frac{P_0(0,1)r(0,1) + P_0(0,2)r(1,2) + P_0(0,3)r(2,3)}{P_0(0,1) + P_0(0,2) + P_0(0,3)} = \frac{\sum_{i=1}^{n} P_0(0,t_i) r(t_{i-1},t_i)}{\sum_{i=1}^{n} P_0(0,t_i)} = 6.9543\%
\]

The general formula is:

\[
R = \frac{\sum_{i=1}^{n} P_0(0,t_i) r(t_{i-1},t_i)}{\sum_{i=1}^{n} P_0(0,t_i)}
\]

Here $n$ is the total number of settlements (i.e. the number of times when cash flows change hand between two parties in a swap); $t_i$ represents the time the $i$-th settlement takes place.

$P_0(0,t_i)$ is the present value of $1$ at $t_i$ discounted to $t = 0$. It is also the price of a zero-coupon bond maturing on the date $t_i$. $r(t_{i-1},t_i)$ is the implied forward rate during $t \in [t_{i-1},t_i]$.

Next, let’s use a shortcut to calculate the present value of the floating payments. The floating payments in an interest rate swap are very similar to the cash flows in a floating rate
bond. In a floating rate bond, the coupon rates are floating (instead of being fixed like in a
typical bond). The floating coupon rates are set in advance and paid in arrears.

Consider a floating rate bond. Its annual coupon rates for Year 1, Year 2, and Year 3 are
6%, 7.0024%, and 8.0071% respectively. Suppose the face amount of this floating rate bond
is $1. Then the bond holder will get $0.06 at \( t = 1 \), $0.070024 at \( t = 2 \), and $1.080071 at
\( t = 3 \).

\[
\begin{array}{c|c|c|c}
\text{Time } t & 0 & 1 & 2 & 3 \\
\hline
\leftarrow r(0,1) & \rightarrow & \leftarrow r(1,2) & \rightarrow & \leftarrow r(2,3) \\
\hline
\leftarrow 6\% & \rightarrow & \leftarrow 7.0024\% & \rightarrow & \leftarrow 8.0071\% \\
\leftarrow 6.5\% \text{ per Yr} & \rightarrow & \leftarrow 7\% \text{ per Yr} & \rightarrow \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow \\
\end{array}
\]

Cash flow received by the bondholder:

\[
\begin{array}{c|c|c|c}
\hline
\text{Time } t & 0 & 1 & 2 & 3 \\
\hline
\leftarrow r(0,1) & \rightarrow & \leftarrow r(1,2) & \rightarrow & \leftarrow r(2,3) \\
\hline
\leftarrow 6\% & \rightarrow & \leftarrow 7.0024\% & \rightarrow & \leftarrow 8.0071\% \\
\leftarrow 6.5\% \text{ per Yr} & \rightarrow & \leftarrow 7\% \text{ per Yr} & \rightarrow \\
\leftarrow & \rightarrow & \leftarrow & \rightarrow \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\hline
\text{Cash flow received} & \$6\% & \$7.0024\% & \$8.0071\% + \$1 \\
\hline
\text{by the bondholder} & =\$0.06 & =\$0.070024 & =\$1.080071 \\
\end{array}
\]

Assuming the face amount of the floating rate note is $1.

What’s the present value of this floating rate bond? Surprisingly, the present value is the
face amount $1. To see why, notice that if you discount $1.080071 from \( t = 3 \) to \( t = 2 \), you’ll
get $1:

\[
\frac{1.080071}{1 + 8.0071\%} = 1
\]

This $1 is combined with the second floating payment $0.070024, becoming $1.070024. If
we discount this cash flow from \( t = 2 \) to \( t = 1 \), once again we get $1:

\[
\frac{1.070024}{1 + 7.0024\%} = 1
\]

This $1 is combined with the 1st floating payment $0.06, becoming $1.06. If we discount
this cash flow from \( t = 1 \) to \( t = 0 \), once again we get $1:

\[
\frac{1.06}{1 + 6\%} = 1
\]

Here is another way to see why the present value of the floating note is its face amount.
The present value of our floating note is:

\[
\frac{6\%}{1 + 6\%} + \frac{7.0024\%}{(1 + 6.5\%)^2} + \frac{1 + 8.0071\%}{(1 + 7\%)^3}
\]

Notice that

\[
(1 + 7\%)^3 = (1 + 6\%) \times (1 + 7.0024\%) \times (1 + 8.0071\%)
\]
Then you can verify for yourself that
\[
(1 + 6.5\%)^2 = (1 + 6\%) \times (1 + 7.0024\%)
\]

Key point to remember: The present value of a floating rate bond is its face amount.

Next, we’re going to quickly calculate the present value of the floating payments using the present value formula for a floating rate bond. At the final settlement time \(t = 3\), suppose the floating-payer (the swap dealer in this case) gives the fixed-payer $1 and immediately gets $1 back from the fixed payer. The cash flow diagram is:

\[
\begin{array}{cccc}
\text{Time } t & 0 & 1 & 2 & 3 \\
\text{\quad Payment to the swap dealer} & \text{R} & \text{R} & \text{R} \\
\text{\quad Pay by the dealer} & \text{\$6\%} & \text{\$7.0024\%} & \text{\$1 + 8.0071\% - \$1} \\
\end{array}
\]

The equation to solve for \(R\) is: \(\text{PV fixed payments} = \text{PV floating payments}\).

\[
R \left( \frac{1}{1.06} + \frac{1}{1.065^2} + \frac{1}{1.073^3} \right) = \frac{6\%}{1.06} + \frac{7.0024\%}{1.065^2} + \frac{1 + 8.0071\%}{1.073^3} - \frac{1}{1.073^3}
\]

The present value of a floating rate bond is its face amount:

\[
\frac{6\%}{1.06} + \frac{7.0024\%}{1.065^2} + \frac{1 + 8.0071\%}{1.073^3} = 1
\]

This gives us:

\[
R = \frac{1 - \frac{1}{1.073^3}}{\frac{1}{1.06} + \frac{1}{1.065^2} + \frac{1}{1.073^3}} = \frac{1 - P_0(0, 3)}{P_0(0, 1) + P_0(0, 2) + P_0(0, 3) = \sum_{t=1}^{n=3} P_0(0, t)}
\]

The general formula is:
$$R = \frac{1 - P_0(0, t_n)}{\sum_{i=1}^{n} P_0(0, t_i)} = \frac{\sum_{i=1}^{n} P_0(0, t_i) r(t_{i-1}, t_i)}{\sum_{i=1}^{n} P_0(0, t_i)}$$

Make sure you understand the symbols in this formula. If you find it hard to memorize this formula, don’t worry. You can just use the following procedure to calculate the fixed rate $R$ in an interest rate swap:

1. Identify the LIBOR yield curve.
2. Identify fixed and floating payments.
3. Set up the equation that PV of fixed payments = PV of floating payments

### 8.2.5 The swap curve

To understand this section, you’ll need to read Section 5.7. The most important part of Section 5.7 is Equation 5.19. Though Section 5.7 is excluded from the syllabus, I recommend that you read it any way to help you understand the swap curve.

In addition, you’ll need to know what’s Eurodollars. Here is some background on Eurodollars. Eurodollars is not the joint European currency, the Euro. Eurodollars are simply US dollars deposited in commercial banks outside the United States, and thus are not under the jurisdiction of the Federal Reserve. As a result, such deposits are subject to much less regulation than similar deposits within the United States, allowing for higher margins.

Historically, such deposits were held mostly by European banks and financial institutions, and hence were called “Eurodollars”. Such deposits are now available in many countries worldwide. However, they continue to be called as “Eurodollars” regardless of the location.

Eurodollar interest rate is the interest rate earned on Eurodollars deposited by one bank with another bank. Three-month Eurodollar futures contracts are futures contracts on three-month Eurodollar interest rate.

Make sure you know how to read the quote of the Eurodollar futures contract. If $F$ is a Eurodollars futures quote, then $(100 - F)\%$ is the Eurodollars futures interest rate for a 3-month period. This formula is explained in Derivatives Markets Section 5.7, which says that the Eurodollars futures price at expiration of the contract is:

$$100 - \text{Annualized 3-month LIBOR}$$

Now you see that if the annualized 3-month LIBOR rate goes up, the quote $F$ goes down; if the annualized 3-month LIBOR rate goes down, the quote $F$ goes up.

For example, a Eurodollars futures price 95.53 corresponds to $(100 - 95.53)\% = 4.47\%$ annualized three-month interest rate or $\frac{4.47\%}{4} = 1.1175\%$ interest rate for a 3-month period on $1,000,000$ loan. The interest due on $1,000,000$ loan during the 3-month borrowing period is $1.1175\% \times 1,000,000 = 11,175$.

Now you shouldn’t have trouble understanding Equation 5.19:

$$r_{91} = (100 - F) \times \frac{1}{100} \times \frac{1}{4} \times \frac{91}{90}$$
This formula can be rewritten as:

\[ r_{91} = \frac{(100 - F)}{4} \times \frac{91}{90} \]

Here \( \frac{(100 - F)}{4} \) is the un-annualized or the actual interest rate per 3-month period. The factor \( \frac{91}{90} \) scales the 90-day actual interest rate into a 91-day actual interest rate.

Now you should be able to follow the textbook and understand Table 8.4. For example, the 0.0037% interest rate is calculated as follows:

\[ r_{91} = \frac{(100 - F)}{4} \times \frac{91}{90} = \frac{(100 - 98.555)}{4} \times \frac{91}{90} = 0.37\% \]

One not obvious point to know is that the 3-month LIBOR rate starts when the Eurodollars futures contract expires. In other words, you enter into a Eurodollars futures contract to lock in the next 3-month LIBOR rate. This is why in Table 8.4 the maturity date is \( t_i \) (Column 1 in Table 5.4) and the implied quarterly rate is \( r(t_i, r_{i+1}) \).

The swap spread represents the credit risk in the swap relative to the corresponding risk-free Treasury yield. It is the price tag on the risk that one of the parties to the swap will fail to make a payment.

### 8.2.6 Swap’s implicit loan balance

At its inception, an interest rate swap has zero value to both parties. However, as time passes, the market value of the swap may no longer be zero.

### 8.2.7 Deferred swaps

The equation for solving the fixed rate \( R \) in a deferred swap is still

\[ \text{PV of fixed payments} = \text{PV of floating payments}. \]

The formula is:

\[ R = \frac{P_0(0, t_{k-1}) - P_0(0, t_n)}{\sum_{i=k}^{n} P_0(0, t_i)} = \frac{\sum_{i=k}^{T} P_0(0, t_i)r(t_{i-1}, t_i)}{\sum_{i=1}^{n} P_0(0, t_i)} \]

### 8.2.8 Why swap interest rates?

The main idea of this section is that by using an interest rate swap a firm can lower its borrowing cost.

Firms like to borrow short-term loans; borrowing a loan for a short period of time is generally less risky and cheaper (i.e paying lower interest rate) than borrowing the same amount of money for a longer period. However, it’s hard for firms to borrow lot of money on the short-term basis. Lenders still worry that the borrower won’t pay back the loan, even if the loan is a short term loan.

In an interest rate swap, the notional principal doesn’t change hands and there’s no risk that any party to the swap may default the principal. As a result, an interest rate swap has lower credit risk than a short term loan. By entering an interest rate swap, a firm can borrow at the short term interest rate.
8.2.9 Amortizing and accrediting swaps

If the notional principal in an interest rate swap decreases over time, the swap is called an amortizing swap. If the notional principal increases over time, the interest rate swap is called an accrediting swap.

When the notional principal is not a constant, then the equation for solving the fixed rate $R$ becomes:

$$R = \frac{\sum_{i=1}^{n} Q_{t_i} P_0(0, t_i) r(t_{i-1}, t_i)}{\sum_{i=1}^{n} Q_{t_i} P_0(0, t_i)}$$

where $Q_{t_i}$ is notional principal at time $t_i$. 

$\Diamond$
Solution to *Derivatives Markets*: for Exam FM

Yufeng Guo

June 5, 2008
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Preface

This is Guo’s solution to Derivatives Markets (2nd edition ISBN 0-321-28030-X) for Exam FM. Unlike the official solution manual published by Addison-Wesley, this solution manual provides solutions to both the even-numbered and odd-numbered problems for the chapters that are on the Exam FM syllabus. Problems that are out of the scope of the FM syllabus are excluded.

Please report any errors to yufeng_guo@msn.com.

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Introduction

Recommendations on using this solution manual:

1. Obviously, you’ll need to buy Derivatives Markets (2nd edition) to see the problems.

2. Make sure you download the textbook errata from http://www.kellogg.northwestern.edu/faculty/mcdonald/htm/typos2e_01.html
Chapter 1

Introduction to derivatives

Problem 1.1.

Derivatives on weather are not as farfetched as it might appear. Visit http://www.cme.com/trading/ and you'll find more than a dozen weather derivatives currently traded at CME such as "CME U.S. Monthly Weather Heating Degree Day Futures" and "CME U.S. Monthly Weather Cooling Degree Day Futures."

a. Soft drink sales greatly depend on weather. Generally, warm weather boosts soft drink sales and cold weather reduces sales. A soft drink producer can use weather futures contracts to reduce the revenue swing caused by weather and smooth its earnings. Shareholders of a company generally want the earnings to be steady. They don't want the management to use weather as an excuse for poor earnings or wild fluctuations of earnings.

b. The recreational skiing industry greatly depends on weather. A ski resort can lose money due to warm temperatures, bitterly cold temperatures, no snow, too little snow, or too much snow. A resort can use weather derivatives to reduce its revenue risk.

c. Extremely hot or cold weather will result in greater demand for electricity. An electric utility company faces the risk that it may have to buy electricity at a higher spot price.

d. Fewer people will visit an amusement park under extreme weather conditions. An amusement park can use weather derivatives to manage its revenue risk.

How can we buy or sell weather? No one can accurately predict weather. No one can deliver weather. For people to trade on weather derivatives, weather indexes need to be invented and agreed upon. Once we have weather indexes, we can link the payoff of a weather derivative to a weather index. For more information on weather derivatives, visit:

- http://hometown.aol.com/gml1000/wrms.htm
- http://www.investopedia.com
Problem 1.2.

- Anyone (such as speculators and investors) who wants to earn a profit can enter weather futures. If you can better predict a weather index than does the market maker, you can enter weather futures and make a profit. Of course, it’s hard to predict a weather index and hence loss may occur.

- Two companies with opposite risks may enter weather futures as counter parties. For example, a soft drink company and a ski-resort operator have opposite hedging needs and can enter a futures contract. The soft drink company can have a positive payoff if the weather is too cold and a negative payoff if warm. This way, when the weather is too cold, the soft drink company can use the gain from the weather futures to offset its loss in sales. Since the soft drink company makes good money when the weather is warm, it doesn’t mind a negative payoff when the weather is cold. On the other hand, the ski resort can have a negative payoff if the weather is too cold and a positive payoff if too warm. The ski resort can use the gain from the futures to offset its loss in sales.

Problem 1.3.

a. \(100 \times 41.05 + 20 = 4125\)

b. \(100 \times 40.95 - 20 = 4075\)

c. For each stock, you buy at $41.05 and sell it an instant later for $40.95. The total loss due to the ask-bid spread: \(100 (41.05 - 40.95) = 10\). In addition, you pay $20 twice. Your total transaction cost is \(100 (41.05 - 40.95) + 2 (20) = 50\)

Problem 1.4.

a. \(100 \times 41.05 + 100 \times 41.05 \times 0.003 = 4117.315\)

b. \(100 \times 40.95 - 100 \times 40.95 \times 0.003 = 4082.715\)

c. For each stock, you buy at $41.05 and sell it an instant later for $40.95. The total loss due to the ask-bid spread: \(100 (41.05 - 40.95) = 10\). In addition, your pay commission \(100 \times 41.05 \times 0.003 + 100 \times 40.95 \times 0.003 = 24.6\). Your total transaction cost is \(10 + 24.6 = 34.6\)

Problem 1.5.

The market maker buys a security for $100 and sells it for $100.12. If the market maker buys 100 securities and immediately sells them, his profit is \(100 (100.12 - 100) = 12\)

Problem 1.6.
CHAPTER 1. INTRODUCTION TO DERIVATIVES

Your sales proceeds: 300 (30.19) – 300 (30.19) (0.005) = 9011.715
Your cost of buying 300 shares from the market to close your short position is:
300 (29.87) + 300 (29.87) (0.005) = 9005.805
Your profit: 9011.715 – 9005.805 = 5.91

Problem 1.7.

a. Consider the bid-ask spread but ignore commission and interest.
Your sales proceeds: 400 (25.12) = 10048
Your cost of buying back: 400 (23.06) = 9224
Your profit: 10048 – 9224 = 824

b. If the bid-ask spread and 0.3% commission
Your sales proceeds: 400 (25.12) – 400 (25.12) (0.003) = 10017.856
Your cost of buying back: 400 (23.06) + 400 (23.06) (0.003) = 9251.672
Your profit: 10017.856 – 9251.672 = 766.184
Profit drops by: 824 – 766.184 = 57.816

c. Your sales proceeds stay in your margin account, serving as a collateral.
Since you earn zero interest on the collateral, your lost interest is
If ignore commission: 10048 (0.03) = 301.44
If consider commission: 10017.856 (0.03) = 300.54

Problem 1.8.

By signing the agreement, you allow your broker to act as a bank, who lends your stocks to someone else and possibly earns interest on the lent stocks. Short sellers typically leave the short sale proceeds on deposit with the broker, along with additional capital called a haircut. The short sale proceeds and the haircut serve as a collateral. The short seller earns interest on this collateral. This interest is called the short rebate in the stock market.

The rebate rate is often equal to the prevailing market interest rate. However, if a stock is scarce, the broker will pay far less than the prevailing interest rate, in which case the broker earns the difference between the short rate and the prevailing interest rate.

This arrangement makes short selling easy. Also short selling can be used to hedge financial risks, which is good for the economy. By the way, you are not hurt in any way by allowing your broker to lend your shares to short sellers.

Problem 1.9.

According to http://www.investorwords.com, the ex-dividend date was created to allow all pending transactions to be completed before the record date. If an investor does not own the stock before the ex-dividend date, he or she will
be ineligible for the dividend payout. Further, for all pending transactions that have not been completed by the ex-dividend date, the exchanges automatically reduce the price of the stock by the amount of the dividend. This is done because a dividend payout automatically reduces the value of the company (it comes from the company’s cash reserves), and the investor would have to absorb that reduction in value (because neither the buyer nor the seller are eligible for the dividend).

If you borrow stock to make a short sale, you’ll need to pay the lender the dividend distributed while you maintain your short position. According to the IRS, you can deduct these payments on your tax return only if you hold the short sale open for a minimum period (such as 46 days) and you itemize your deductions.

In a perfect market, if a stock pays $5 dividend, after the ex-dividend date, the stock price will be reduced by $5. Then you could buy back stocks from the market at a reduced price to close your short position. So you don’t need to worry whether the dividend is $3 or $5.

However, in the real world, a big increase in the dividend is a sign that a company is doing better than expected. If a company pays a $5 dividend instead of the expected $3 dividend, the company’s stock price may go up after the announcement that more dividend will be paid. If the stock price goes up, you have to buy back stocks at a higher price to close your short position. So an unexpected increase of the dividend may hurt you.

In addition, if a higher dividend is distributed, you need to pay the lender the dividend while you maintain your short position. This requires you to have more capital on hand.

In the real world, as a short seller, you need to watch out for unexpected increases of dividend payout.

Problem 1.10.

http://www.investopedia.com/articles/01/082201.asp offers a good explanation of short interest:

*Short Interest*

> Short interest is the total number of shares of a particular stock that have been sold short by investors but have not yet been covered or closed out. This can be expressed as a number or as a percentage.

> When expressed as a percentage, short interest is the number of shorted shares divided by the number of shares outstanding. For example, a stock with 1.5 million shares sold short and 10 million shares outstanding has a short interest of 15% (1.5 million/10 million = 15%).

Most stock exchanges track the short interest in each stock and issue reports at month’s end. These reports are great because by showing what short sellers are doing, they allow investors to gauge overall market sentiment surrounding a particular stock. Or alternatively, most exchanges provide an online tool to calculate short interest for a particular security.
CHAPTER 1. INTRODUCTION TO DERIVATIVES

Reading Short Interest

A large increase or decrease in a stock's short interest from the previous month can be a very telling indicator of investor sentiment. Let's say that Microsoft's (MSFT) short interest increased by 10% in one month. This means that there was a 10% increase in the amount of people who believe the stock will decrease. Such a significant shift provides good cause for us to find out more. We would need to check the current research and any recent news reports to see what is happening with the company and why more investors are selling its stock.

A high short-interest stock should be approached for buying with extreme caution but not necessarily avoided at all costs. Short sellers (like all investors) aren't perfect and have been known to be wrong from time to time.

In fact, many contrarian investors use short interest as a tool to determine the direction of the market. The rationale is that if everyone is selling, then the stock is already at its low and can only move up. Thus, contrarians feel that a high short-interest ratio (which we will discuss below) is bullish - because eventually there will be significant upward pressure on the stock's price as short sellers cover their short positions (i.e. buy back the stocks they borrowed to return to the lender).

The more likely that investors can speculate on the stock, the higher the demand for the stock and the higher the short interest.

A broker can short sell more than his existing inventory. For example, if a broker has 500 shares of IBM stocks, he can short sell 600 shares of IBM stocks as long as he knows where to find the additional 100 shares of IBM stocks. If all the brokers simultaneously lend out more than what they have in their stock inventories, then the number of stocks sold short might exceed the total number of the stocks outstanding.

NASDAQ short interest is available by issue for a rolling twelve months and is based on a mid-month settlement date. For more information, visit http://www.nasdaqtrader.com/asp/short_interest.asp.

Problem 1.11.

You go to a bank. The bank uses its customers' deposits and lends you an asset worth $100. Then 90 days later you buy back the asset at $102 from the open market (i.e. you come up with $102 from whatever sources) and return $102 to the bank. Now your short position is closed.

Problem 1.12.

We need to borrow an asset called money from a bank (the asset owner) to pay for a new house. The asset owner faces credit risk (the risk that we may not be able to repay the loan). To protect itself, the bank needs collateral.

The house is collateral. If we don't pay back our loan, the bank can foreclose the house.
CHAPTER 1. INTRODUCTION TO DERIVATIVES

To protect against the credit risk, the bank requires a haircut (i.e. requires that the collateral is greater than the loan). Typically, a bank lends only 80% of the purchase price of the house, requiring the borrower to pay a 20% down payment.
Chapter 2

Introduction to forwards and options

Problem 2.1.

Long a stock=Own a stock (or buy a stock).
If you own a stock, your payoff at any time is the stock’s market price because you can sell it any time at the market price. Let $S$ represent the stock price at $T = 1$.
Your payoff at $T = 1$ is $S$.
Your profit at $T = 1$ is:
Payoff - FV(initial investment) = $S - 50 (1.1) = S - 55$. 
You can see that the profit is zero when the stock price $S = 55$. Alternatively, set $S - 55 = 0 \rightarrow S = 55$. 
Payoff = S  Profit = S - 55

Payoff and profit: Long one stock

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Problem 2.2.

Short a stock = Short sell a stock. If you short sell a stock, your payoff at any time after the short sale is the negative of the stock’s market price. This is because to close your short position you’ll need to buy the stock back at the market price and return it to the broker. Your payoff at $T = 1$ is $-S$. Your profit at $T = 1$ is: Payoff - FV(initial investment) = $-S + 50 (1.1) = 55 - S$

You can see that the profit is zero when the stock price $S = 55$. Alternatively, set $55 - S = 0 \rightarrow S = 55$.
Problem 2.3.

The opposite of a purchased call is written call (or sold call). The opposite of a purchased put is written put (or sold put).

The main idea of this problem is:
- The opposite of a purchased call ≠ a purchased put
- The opposite of a purchased put ≠ a purchased call

Problem 2.4.

a. Long forward means being a buyer in a forward contract. Payoff of a buyer in a forward at $T$ is
\[
Payoff = S_T - F = S_T - 50
\]

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Payoff = $S_T - 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>-10</td>
</tr>
<tr>
<td>45</td>
<td>-5</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>55</td>
<td>5</td>
</tr>
<tr>
<td>60</td>
<td>10</td>
</tr>
</tbody>
</table>

b. Payoff of a long call (i.e. owning a call) at expiration $T$ is:
\[
Payoff = \max(0, S_T - K) = \max(0, S_T - 50)
\]

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Payoff = $\max(0, S_T - 50)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>45</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
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</tr>
<tr>
<td>55</td>
<td>5</td>
</tr>
<tr>
<td>60</td>
<td>10</td>
</tr>
</tbody>
</table>

c. A call option is a privilege. You exercise a call and buy the stock only if your payoff is positive.
In contrast, a forward is an obligation. You need to buy the stock even if your payoff is negative.
A privilege is better than an obligation.
Consequently, a long call is more expensive than a long forward on the same underlying stock with the same time to expiration.
CHAPTER 2.  INTRODUCTION TO FORWARDS AND OPTIONS

Problem 2.5.

a. Short forward = Enter into a forward as a seller

Payoff of a seller in a forward at $T$ is

$$\text{Payoff} = F - S_T = 50 - S_T$$

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Payoff $= 50 - S_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>10</td>
</tr>
<tr>
<td>45</td>
<td>5</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>55</td>
<td>-5</td>
</tr>
<tr>
<td>60</td>
<td>-10</td>
</tr>
</tbody>
</table>

b. Payoff of a long put (i.e. owning a put) at expiration $T$ is:

$$\text{Payoff} = \max(0, K - S_T) = \max(0, 50 - S_T)$$

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Payoff $= \max(0, 50 - S_T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>10</td>
</tr>
<tr>
<td>45</td>
<td>5</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>55</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>0</td>
</tr>
</tbody>
</table>

c. A put option is a privilege. You exercise a put and sell the stock only if your payoff is positive.

In contrast, a forward is an obligation. You need to sell the stock even if your payoff is negative.

A privilege is better than an obligation.

Consequently, a long put is more expensive than a short forward on the same underlying stock with the same time to expiration.
CHAPTER 2. INTRODUCTION TO FORWARDS AND OPTIONS

Problem 2.6.

\[ 91 (1 + r) = 100 \rightarrow r = 0.0989 \]

The effective annual interest rate is 9.89%.

If you buy the bond at \( t = 0 \), your payoff at \( t = 1 \) is 100.

Your profit at \( t = 1 \) is \( 100 - 91 (1 + 0.0989) = 0 \) regardless of the stock price at \( t = 1 \).

If you buy a bond, you just earn the risk-free interest rate. Beyond this, your profit is zero.

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**Payoff: Long a bond**

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Profit of longing a bond is zero.
If you sell the bond at $t = 0$, your payoff at $t = 1$ is $-100$ (you need to pay the bond holder 100).

Your profit at $t = 1$ is $91 \times (1 + 0.0989) - 100 = 0$ regardless of the stock price at $t = 1$

If you sell a bond, you just earn the risk-free interest rate. Beyond this, your profit is zero.

Payoff: Shorting a bond
Profit of shorting a bond is zero.
Problem 2.7.

a. It costs nothing for one to enter a forward contract. Hence the payoff of a forward is equal to the profit.

Suppose we long a forward (i.e. we are the buyer in the forward). Our payoff and profit at expiration is:

\[ S_T - F = S_T - 55 \]

Payoff (and profit) of a long forward
Suppose we short a forward (i.e. we are the seller in the forward), our payoff and profit at expiration is:
\[ F - S_T = 55 - S_T \]
b. If the stock doesn’t pay dividend, buying a stock outright at $t = 0$ and getting a stock at $T = 1$ through a forward are identical. There’s no benefit to owning a stock early.

c. Suppose the stock pays dividend before the forward expiration date $T = 1$. Please note that if you own a stock prior to the dividend date, you will receive the dividend. In contrast, if you are a buyer in a forward contract, at $T = 1$, you’ll get a stock but you won’t receive any dividend.

- If the stock is expected to pay dividend, then the stock price is expected to drop after the dividend is paid. The forward price agreed upon at $t = 0$ already considers that a dividend is paid during $(0, T)$; the dividend will reduce the forward rate. There’s no advantage to buying a stock outright over buying a stock through a forward. Otherwise, there will be arbitrage opportunities.

- If the stock is not expected to pay dividend but actually pays dividend (a surprise dividend), then the forward price $F$ agreed upon at $t = 0$ was set without knowing the surprise dividend. So $F$ is the forward price on a non-dividend paying stock. Since dividend reduces the value of a stock, $F$ is higher than the forward price on an otherwise identical but dividend-paying stock. If you own a stock at $t = 0$, you’ll receive the windfall dividend. If you buy a stock through a forward, you’ll pay $F$, which is higher than the forward price on an otherwise identical but dividend-paying stock. Hence owning a stock outright is more beneficial than buying a stock through a forward.

Problem 2.8.

$r =$ risk free interest rate
Under the no-arbitrage principle, you get the same profit whether you buy a stock outright or through a forward.

Profit at $T = 1$ if you buy a stock at $t = 0$ is: $S_T - 50 (1 + r)$
Profit at $T = 1$ if you buy a stock through a forward: $S_T - 53$

$\rightarrow S_T - 50 (1 + r) = S_T - 53 \\
50 (1 + r) = 53 \\
r = 0.06$
Chapter 2. Introduction to Forwards and Options

Problem 2.9.

a. Price of an index forward contract expiring in one year is:

\[ F^{\text{Index}} = 1000 \times (1.1) = 1100 \]

To see why: If the seller borrows 1000 at \( t = 0 \), buys an index, and holds it for one year, then he’ll have one stock to deliver at \( T = 1 \). The seller’s cost is \( 1000 \times (1.1) = 1100 \). To avoid arbitrage, the forward price must be 1100.

Profit at \( T = 1 \) of owning a forward on an index:

\[ S_T - F^{\text{Index}} = S_T - 1100 \]

If you buy an index at \( t = 0 \), your profit at \( T = 1 \) is \( S_T - 1000 \times (1.1) = S_T - 1100 \)

So you get the same profit whether you buy the index outright or buy the index through a forward. This should make sense. If owning a stock outright and buying it through a forward have different profits, arbitrage opportunities exist.

b. The forward price 1200 is greater than the fair forward price 1100. No rational person will want to enter such an unfair forward contract. Thus the seller needs to pay the buyer an up-front premium to incite the buyer. The buyer in the forward needs to receive \( \frac{1200 - 1100}{1.1} = 90.91 \) at \( t = 0 \) to make the forward contract fair. Of course, the buyer needs to pay the forward price 1200 at \( T = 1 \).

c. Now the forward price 1000 is lower than the fair forward price 1100. You can imagine thousands of bargain hunters are waiting in line to enter this forward contract. If you want to enter the forward contract, you have to pay the seller a premium in the amount of \( \frac{1100 - 1000}{1.1} = 90.91 \) at \( t = 0 \). In addition, you’ll need to pay the forward price 1000 at \( T = 1 \).

Problem 2.10.

a. Profit=\( \max (0, S_T - 1000) - 95.68 \)

Set profit to zero:

\[ \max (0, S_T - 1000) - 95.68 = 0 \]
\[ \rightarrow S_T - 1000 - 95.68 = 0 \quad S_T = 1000 + 95.68 = 1095.68 \]

b. The profit of a long forward (i.e. being a buyer in a forward): \( S_T - 1020 \)

\( S_T - 1020 = \max (0, S_T - 1000) - 95.68 \)

If \( S_T > 1000 \), there’s no solution

If \( S_T \leq 1000 \):

\( S_T - 1020 = 0 - 95.68 \quad \rightarrow S_T = 1020 - 95.68 = 924.32 \)
Problem 2.11.

a. Profit of a long (i.e. owning) put is \( \max(0, 1000 - S_T) - 75.68 \)

\[
\max(0, 1000 - S_T) - 75.68 = 0
\]

\[
1000 - S_T - 75.68 = 0 \quad S_T = 924.32
\]

b. Profit of a short forward (i.e. being a seller in a forward) is \( 1020 - S_T \)

\[
\max(0, 1000 - S_T) - 75.68 = 1020 - S_T
\]

If \( 1000 \geq S_T \)

\[
1000 - S - 75.68 = 1020 - S \quad \text{no solution}
\]

If \( 1000 \leq S_T \)

\[
-75.68 = 1020 - S_T \quad S_T = 1020 + 75.68 = 1095.68
\]
Problem 2.12.

Table 2.4 is:

<table>
<thead>
<tr>
<th>Position</th>
<th>Maximum Loss</th>
<th>Maximum Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long forward (buyer in forward)</td>
<td>-Forward price</td>
<td>Unlimited</td>
</tr>
<tr>
<td>Short forward (seller in forward)</td>
<td>Unlimited</td>
<td>Forward Price</td>
</tr>
<tr>
<td>Long call (own a call)</td>
<td>-FV (premium)</td>
<td>Unlimited</td>
</tr>
<tr>
<td>Short call (sell a call)</td>
<td>Unlimited</td>
<td>FV(premium)</td>
</tr>
<tr>
<td>Long put (own a put)</td>
<td>-FV(premium)</td>
<td>Strike Price - FV(premium)</td>
</tr>
<tr>
<td>Short put (sell a put)</td>
<td>PV(premium)-Strike Price</td>
<td>FV(premium)</td>
</tr>
</tbody>
</table>

- If you are a buyer in a forward, the worst that can happen to you is $S_T = 0$ (i.e. stock price at $T$ is zero). If this happens, you still have to pay the forward price $F$ at $T$ to buy the stock which is worth zero. You’ll lose $F$. Your best case is $S_T = \infty$, where you have an unlimited gain.

- If you are a seller in a forward, the worst case is that $S_T = \infty$; you’ll incur unlimited loss. Your best case is that $S_T = 0$, in which case you sell a worthless asset for the forward price $F$.

- If you buy a call, your worst case is $S_T < K$, where $K$ is the strike price. If this happens, you just let the call expire worthless. You’ll lose the future value of your premium (if you didn’t buy the call and deposit your money in a bank account, you could earn the future value of your deposit). Your best case is that $S_T = \infty$, where you’ll have an unlimited gain.

- If you sell a call, your worst case is $S_T = \infty$, in which case you’ll incur an unlimited loss. Your best case is $S_T < K$, in which case the call expires worthless; the call holder wastes his premium and your profit is the future value of the premium you received from the buyer.

- If you buy a put, your worst case is that $S_T \geq K$, in which case you’ll let your put expire worthless and you’ll lose the future value of the put premium. Your best case is $S_T = 0$, in which case you sell a worthless stock for $K$. Your profit is $K - FV\,(premium)$.

- If you sell a put, your worst case is $S_T = 0$, in which case the put holder sells you a worthless stock for $K$; your profit is $FV\,(premium) - K$. Your best case is $S_T \geq K$, where the written put expires worthless and your profit is $FV\,(premium)$. 
Problem 2.13.

Let $S$ represent the stock price at the option expiration date. I’ll draw a separate diagram for the payoff and a separate diagram for the profit.

a. Suppose you long a call (i.e. buy a call).

(i) Payoff at expiration is

$$\max(0, S - 35) = \begin{cases} 
0 & \text{if } S < 35 \\
S - 35 & \text{if } S \ge 35
\end{cases}$$

Your profit at expiration = Payoff - FV (premium)

$$= \max(0, S - 35) - 9.12 \times 1.08 = \max(0, S - 35) - 9.8496$$

$$= \begin{cases} 
0 & \text{if } S < 35 \\
S - 35 - 9.8496 & \text{if } S \ge 35
\end{cases} = \begin{cases} 
-9.8496 & \text{if } S < 35 \\
S - 44.8496 & \text{if } 35 \le S
\end{cases}$$

![Payoff and Profit: Long a 35 strike call diagram](image-url)
(ii) Payoff at expiration is \( \max(0, S - 40) = \) \[
\begin{cases} 
0 & \text{if } S < 40 \\
S - 40 & \text{if } S \geq 40 
\end{cases}
\]

Your profit at expiration = Payoff - FV (premium)

\[
= \max(0, S - 40) - 6.22(1.08) = \max(0, S - 40) - 6.7176 \\
= \begin{cases} 
0 & \text{if } S < 40 \\
S - 40 & \text{if } S \geq 40 
\end{cases} - 6.7176 = \begin{cases} 
-6.7176 & \text{if } S < 40 \\
S - 46.7176 & \text{if } S \geq 40 
\end{cases}
\]
(iii) Payoff at expiration is $\max(0, S - 45) = \begin{cases} 0 & \text{if } S < 45 \\ S - 45 & \text{if } S \geq 45 \end{cases}$

Your profit at expiration $=$ Payoff - FV (premium)
$= \max(0, S - 45) - 4.08(1.08) = \max(0, S - 45) - 4.4064$

$= \begin{cases} 0 & \text{if } S < 45 \\ S - 45 - 4.4064 & \text{if } S \geq 45 \end{cases}$

b. The payoff of a long call is $\max(0, S - K)$. As $K$ increases, the payoff gets worse and the option becomes less valuable. Everything else equal, the higher the strike price, the lower the call premium.
Problem 2.14.

Suppose we own a put (i.e. long put).

a. Payoff at expiration is $\max (0, 35 - S) = \begin{cases} 35 - S & \text{if } S \leq 35 \\ 0 & \text{if } S > 35 \end{cases}$

Your profit at expiration = Payoff - FV (premium)

$= \max (0, 35 - S) - 1.53(1.08) = \max (0, 35 - S) - 1.6524$

$= \begin{cases} 35 - S & \text{if } S \leq 35 \\ 0 & \text{if } S > 35 \end{cases} - 1.6524 = \begin{cases} 33.3476 - S & \text{if } S \leq 35 \\ -1.6524 & \text{if } S > 35 \end{cases}$

The blue line is the payoff. The black line is the profit.
b. Payoff at expiration is \( \max (0, 40 - S) = \begin{cases} 
40 - S & \text{if } S \leq 40 \\
0 & \text{if } S > 40 
\end{cases} \)

Your profit at expiration = Payoff - FV (premium)
= \( \max (0, 40 - S) - 3.26 \times 1.08 \) = \( \max (0, 40 - S) - 3.5208 \)
= \( \begin{cases} 
40 - S & \text{if } S \leq 40 \\
0 & \text{if } S > 40 
\end{cases} - 3.5208 \) = \( \begin{cases} 
36.4792 - S & \text{if } S \leq 40 \\
-3.5208 & \text{if } S > 40 
\end{cases} \)

Payoff and profit: Long a 40 strike put

The blue line is the payoff. The black line is the profit.
c. Payoff at expiration is $\max(0, 45 - S) = \begin{cases} 45 - S & \text{if } S \leq 45 \\ 0 & \text{if } S > 45 \end{cases}$

Your profit at expiration = Payoff - FV (premium)
= $\max(0, 45 - S) - 5.75 (1.08) = \max(0, 45 - S) - 6.21$
= $\begin{cases} 45 - S & \text{if } S \leq 45 \\ 0 & \text{if } S > 45 \end{cases}$ - 6.21
= $\begin{cases} 38.79 - S & \text{if } S \leq 45 \\ -6.21 & \text{if } S > 45 \end{cases}$

The blue line is the payoff. The black line is the profit.

As the strike price increases, the payoff of a put goes up and the more valuable a put is. Everything else equal, the higher the strike price, the more expensive a put is.
Problem 2.15.

If you borrow money from a bank to buy a $1000 S&R index, your borrowing cost is known at the time of borrowing. Suppose the annual effective risk free interest rate is $r$. If you borrow $1000 at $t = 0$, then at $T$ you just pay the bank $1000(1 + r)^T$. You can sleep well knowing that your borrowing cost is fixed in advance.

In contrast, if you short-sell $n$ number of IBM stocks and use the short sale proceeds to buy a $1000 S&R$ index, you own the brokerage firm $n$ number of IBM stocks. If you want to close your short position at time $T$, you need to buy $n$ stocks at $T$. The cost of $n$ stocks at $T$ is $nS_T$, where $S_T$ is the price of IBM stocks per share at $T$. Since $S_T$ is not known in advance, if you use short selling to finance your purchase of a $1000 S&R$ index, your borrowing cost $nS_T$ cannot be known in advance. This brings additional risk to your position. As such, you can’t determine your profit.

Problem 2.16.

Skip this problem. SOA is unlikely to ask you to design a spreadsheet on the exam.
Chapter 3

Insurance, collars, and other strategies

Problem 3.1.

The put premium is $74.201$. At $t = 0$, you

• spend $1000$ to buy an S&R index
• spend $74.201$ to buy a 1000-strike put
• borrow $980.39$
• take out $(1000 + 74.201) - 980.39 = 93.811$ out of your own pocket.

So your total borrowing is $980.39 + 93.811 = 1074.20$.
The future value is $1074.20 (1.02) = 1095.68$

<table>
<thead>
<tr>
<th>S&amp;R index</th>
<th>S&amp;R Put</th>
<th>Payoff</th>
<th>-(Cost+Interest)</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>900</td>
<td>100</td>
<td>1000</td>
<td>$-1095.68$</td>
<td>$1000 - 1095.68 = -95.68$</td>
</tr>
<tr>
<td>950</td>
<td>50</td>
<td>1000</td>
<td>$-1095.68$</td>
<td>$1000 - 1095.68 = -95.68$</td>
</tr>
<tr>
<td>1000</td>
<td>0</td>
<td>1000</td>
<td>$-1095.68$</td>
<td>$1000 - 1095.68 = -95.68$</td>
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<tr>
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<td>0</td>
<td>1050</td>
<td>$-1095.68$</td>
<td>$1050 - 1095.68 = -45.68$</td>
</tr>
<tr>
<td>1100</td>
<td>0</td>
<td>1100</td>
<td>$-1095.68$</td>
<td>$1100 - 1095.68 = 4.32$</td>
</tr>
<tr>
<td>1150</td>
<td>0</td>
<td>1150</td>
<td>$-1095.68$</td>
<td>$1150 - 1095.68 = 54.32$</td>
</tr>
<tr>
<td>1200</td>
<td>0</td>
<td>1200</td>
<td>$-1095.68$</td>
<td>$1200 - 1095.68 = 104.32$</td>
</tr>
</tbody>
</table>
CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES

Payoff. The payoff of owning an index is $S$, where $S$ is the price of the index at the put expiration.

The payoff of owning a put is \( \max(0, 1000 - S) \) at expiration.

Total payoff:

\[
S + \max(0, 1000 - S) = S + \begin{cases} 
1000 - S & \text{if } S \leq 1000 \\
0 & \text{if } S > 1000
\end{cases} = \begin{cases} 
1000 & \text{if } S \leq 1000 \\
S & \text{if } S > 1000
\end{cases}
\]

Profit is:

\[
\begin{cases} 
1000 & \text{if } S \leq 1000 \\
S & \text{if } S > 1000
\end{cases} - 1095.68 = \begin{cases} 
-95.68 & \text{if } S \leq 1000 \\
S - 1095.68 & \text{if } S > 1000
\end{cases}
\]

Payoff and Profit: index + put
CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES

Problem 3.2.

At $t = 0$ you

- short sell one S&R index, receiving $1000
- sell a 1000-strike put, receiving $74.201
- deposit $1000 + 74.201 = 1074.201 in a savings account. This grows into $1074.201(1.02) = 1095.68$ at $T = 1$

The payoff of the index sold short is $-S$

The payoff of a sold put: $-\max (0, 1000 - S)$

The total payoff at expiration is:

$$-S - \max (0, 1000 - S) = -S - \left\{ \begin{array}{ll}
1000 - S & \text{if } S \leq 1000 \\
0 & \text{if } S > 1000
\end{array} \right. = \left\{ \begin{array}{ll}
-1000 & \text{if } S \leq 1000 \\
-S & \text{if } S > 1000
\end{array} \right.$$  

The profit at expiration is:

$$\left\{ \begin{array}{ll}
-1000 & \text{if } S \leq 1000 \\
-S & \text{if } S > 1000
\end{array} \right. + 1095.68 = \left\{ \begin{array}{ll}
95.68 & \text{if } S \leq 1000 \\
1095.68 - S & \text{if } S > 1000
\end{array} \right.$$  

<table>
<thead>
<tr>
<th>S&amp;R index</th>
<th>S&amp;R Put</th>
<th>Payoff</th>
<th>-(Cost+Interest)</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>-900</td>
<td>-100</td>
<td>-1000</td>
<td>1095.68</td>
<td>-1000 + 1095.68 = 95.68</td>
</tr>
<tr>
<td>-950</td>
<td>-50</td>
<td>-1000</td>
<td>1095.68</td>
<td>-1000 + 1095.68 = 95.68</td>
</tr>
<tr>
<td>-1000</td>
<td>0</td>
<td>-1000</td>
<td>1095.68</td>
<td>-1000 + 1095.68 = 95.68</td>
</tr>
<tr>
<td>-1050</td>
<td>0</td>
<td>-1050</td>
<td>1095.68</td>
<td>-1050 + 1095.68 = 45.68</td>
</tr>
<tr>
<td>-1100</td>
<td>0</td>
<td>-1100</td>
<td>1095.68</td>
<td>-1100 + 1095.68 = -4.32</td>
</tr>
<tr>
<td>-1150</td>
<td>0</td>
<td>-1150</td>
<td>1095.68</td>
<td>-1150 + 1095.68 = -54.32</td>
</tr>
<tr>
<td>-1200</td>
<td>0</td>
<td>-1200</td>
<td>1095.68</td>
<td>-1200 + 1095.68 = -104.32</td>
</tr>
</tbody>
</table>

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CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES

Payoff = \[
\begin{cases} 
-1000 & \text{if } S \leq 1000 \\
-S & \text{if } S > 1000 
\end{cases}
\]

Profit = \[
\begin{cases} 
-1000 & \text{if } S \leq 1000 \\
-S & \text{if } S > 1000 + 1095.68 
\end{cases}
\]

You can verify that the profit diagram above matches the textbook Figure 3.5 (d).
CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES

Problem 3.3.

Option 1: Buy S&R index for 1000 and buy a 950-strike put
Option 2: Invest 931.37 in a zero-coupon bond and buy a 950-strike call.
Verify that Option 1 and 2 have the same payoff and the same profit.

Option 1:
If you own an index, your payoff at any time is the spot price of the index $S$. The payoff of owning a 950-strike put is $\max(0, 950 - S)$. Your total payoff at the put expiration is

$$S + \max(0, 950 - S) = \begin{cases} 950 - S & \text{if } S \leq 950 \\ 0 & \text{if } S > 950 \end{cases} = \begin{cases} 950 & \text{if } S \leq 950 \\ S & \text{if } S > 950 \end{cases}$$

To calculate the profit, we need to know the initial investment. At $t = 0$, we spend 1000 to buy an index and 51.777 to buy the 950-strike put. The total investment is $1000 + 51.777 = 1051.777$. The future value of the investment is $1051.777 \times (1.02) = 1072.81$

So the profit is:

$$\begin{cases} 950 & \text{if } S \leq 950 \\ S & \text{if } S > 950 \end{cases} - 1072.81 = \begin{cases} -122.81 & \text{if } S \leq 950 \\ S - 1072.81 & \text{if } S > 950 \end{cases}$$
Payoff = \[
\begin{cases} 
950 & \text{if } S \leq 950 \\ 
S & \text{if } S > 950
\end{cases}
\]

Profit = \[
\begin{cases} 
950 & \text{if } S \leq 950 \\ 
S & \text{if } S > 950
\end{cases} - 1072.81
\]
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Option 2:
Payoff of the zero-coupon bond at $T = 0.5$ year is: $931.37 \times (1.02) = 950$
Payoff of owning a 950-strike call: $\max(0, S - 950)$

Total payoff:
$950 + \max(0, S - 950) = 950 + \begin{cases} 
0 & \text{if } S \leq 950 \\
S - 950 & \text{if } S > 950
\end{cases} = \begin{cases} 
950 & \text{if } S \leq 950 \\
S & \text{if } S > 950
\end{cases}$

To calculate the profit, we need to know the initial investment. We spend $931.37$ to buy a bond and $120.405$ to buy a 950-strike call. The future value of the investment is $(931.37 + 120.405) \times 1.02 = 1072.81$. The profit is:

$\begin{cases} 
950 & \text{if } S \leq 950 \\
S & \text{if } S > 950
\end{cases} - 1072.81 = \begin{cases} 
-122.81 & \text{if } S \leq 950 \\
S - 1072.81 & \text{if } S > 950
\end{cases}$

Option 1 and 2 have the same payoff and the same profit. But why? It’s because the put-call parity:

$C(K, T) + PV(K) = P(K, T) + S_0$

Option 1 consists of buying S&R index and a 950-strike put
Option 2 consists of investing $PV(K) = 950 \times (1.02^{-1}) = 931.37$ and buying a 950-strike call. Due to the put-call parity, Option 1 and 2 have the same payoff and the same profit.
Problem 3.4.

Option 1: Short sell S&R index for 1000 and buy a 950-strike call
Option 2: Borrow 931.37 and buy a 950-strike put
Verify that Option 1 and 2 have the same payoff and the same profit.

Option 1: At \( t = 0 \), your payoff from the short sale of an index is \(-S\), where \( S \) is the index price at \( T = 0.5 \). At \( T = 0.5 \), your payoff from owning a call is \( \max(0, S - 950) = \begin{cases} 
0 & \text{if } S < 950 \\
S - 950 & \text{if } S \geq 950 
\end{cases} \).

Your total payoff is
\[
-S + \begin{cases} 
0 & \text{if } S < 950 \\
S - 950 & \text{if } S \geq 950 
\end{cases} = \begin{cases} 
-S & \text{if } S < 950 \\
-950 & \text{if } S \geq 950 
\end{cases}
\]

Please note that when calculating the payoff, we’ll ignore the sales price of the index $1,000 and the call purchase price $120.405. These two numbers affect your profit, but they don’t affect your payoff. Your payoff is the same no matter whether you sold your index for $1 or $1000, and no matter whether you buy the 950-strike call for $10 or $120.41.

Next, let’s find the profit at \( T = 0.5 \). At \( t = 0 \), you sell an index for 1000. Of 1000 you get, you spend 120.405 \approx 120.41 to buy a 950-strike call. You have 1000 – 120.41 = 879.59 left. This will grow into 879.59 \times 1.02 = 897.18 at \( T = 0.5 \). At \( T = 0.5 \), your profit is 897.18 plus the payoff:

\[
\text{Profit} = 897.18 + \begin{cases} 
-S & \text{if } S < 950 \\
-950 & \text{if } S \geq 950 
\end{cases} = \begin{cases} 
897.18 - S & \text{if } S < 950 \\
-52.82 & \text{if } S \geq 950 
\end{cases}
\]
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Payoff = \begin{cases} 
-S & \text{if } S < 950 \\
-950 & \text{if } S \geq 950 
\end{cases}

Profit = 897.18 + \begin{cases} 
-S & \text{if } S < 950 \\
-950 & \text{if } S \geq 950 
\end{cases}

Payoff and Profit: Short index + Long call
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Option 2 payoff. At $t = 0.5$, you need to pay the lender $931.37 \times 1.02 = 950$, so, a payoff of $-950$ (you’ll write the lender a check of 950). At $T = 0.5$, the payoff from buying a 950-strike put is $\max(0, 950 - S) = \begin{cases} 950 - S & \text{if } S < 950 \\ 0 & \text{if } S \geq 950 \end{cases}$.

Your total payoff at $T = 0.5$ is:

$-950 + \begin{cases} 950 - S & \text{if } S < 950 \\ 0 & \text{if } S \geq 950 \end{cases} = \begin{cases} -S & \text{if } S < 950 \\ -950 & \text{if } S \geq 950 \end{cases}$.

This is the same as the payoff in Option 1.

Option 2 profit. There are two ways to calculate the profit.
Method 1. The total profit is the sum of profit earned from borrowing 931.37 and the profit earned by buying a 950-strike put. The profit from borrowing 931.37 is zero; you borrow 931.37 at $t = 0$. This grows into $931.37 \times 1.02 = 950$ at $T = 0.5$ in your savings account. Then at $T = 0.5$, you take out 950 from your savings account and pay the lender. Now your savings account is zero. So the profit earned from borrowing 931.37 is zero.

Next, let’s calculate the profit from buying the put. The put premium is $51.78$. So your profit earned from buying the put option is

$-51.78 \times 1.02 + \max(0, 950 - S) = -52.82 + \max(0, 950 - S)$

$= -52.82 + \begin{cases} 950 - S & \text{if } S < 950 \\ 0 & \text{if } S \geq 950 \end{cases} = \begin{cases} 897.18 - S & \text{if } S < 950 \\ -52.82 & \text{if } S \geq 950 \end{cases}$

Method 2. We already know the payoff is $\begin{cases} -S & \text{if } S < 950 \\ -950 & \text{if } S \geq 950 \end{cases}$. We just need to deduct the future value of the initial investment. At $t = 0$, you receive 931.37 from the lender and pay 51.78 to buy the put. So your total cash is $931.37 - 51.78 = 879.59$, which grows into $879.59 \times 1.02 = 897.18$ at $t = 0.5$. Hence, your profit is:

$\begin{cases} -S & \text{if } S < 950 \\ -950 & \text{if } S \geq 950 \end{cases} + 897.18 = \begin{cases} 897.18 - S & \text{if } S < 950 \\ -52.82 & \text{if } S \geq 950 \end{cases}$

No matter whether you use Method 1 or Method 2, the Option 2 profit is the same as the Option 1 profit.

You might wonder why Option 1 and Option 2 have the same payoff and the same profit. The parity formula is

$Call(K,T) - Put(K,T) = PV(F_{0,T} - K) = PV(F_{0,T}) - PV(K)$

Rearranging this equation, we get:

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\[ \text{Call}(K,T) + PV(K) = \text{Put}(K,T) + PV(F_{0,T}) \]

Since \( PV(F_{0,T}) = S_0 \), now we have:

\[ \text{Call}(K,T) + PV(K) = \text{Put}(K,T) + S_0 \]

The above equation can also be read as:

\[ \text{Call}(K,T) + PV(K) = \text{Put}(K,T) + S_0 \]

Rearranging the above formula, we get:

\[ \text{Call}(K,T) + S_0 = \text{Put}(K,T) + PV(K) \]

The above equation can also be read as:

\[ \text{Call}(K,T) + S_0 = \text{Put}(K,T) + PV(K) \]

According to the parity equation, Option 1 and Option 2 are identical portfolios and should have the same payoff and the same profit. In this problem, Option 1 consists of shorting an S&R index and buying a 950-strike call. Option 2 consists of borrowing \( PV(K) = 950 \cdot (1.02^{-1}) = 931.37 \) and buying a 950-strike put. As a result, Option 1 and 2 have the same payoff and the same profit.
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Problem 3.5.

Option 1: Short sell index for 1000 and buy 1050-strike call
Option 2: Borrow 1029.41 and buy a 1050-strike put.
Verify that Option 1 and 2 have the same payoff and the same profit.
Option 1:
Payoff:
\[-S + \max(0, S - 1050) = -S + \begin{cases} 0 & \text{if } S < 1050 \\ S - 1050 & \text{if } S \geq 1050 \end{cases} = \begin{cases} -S & \text{if } S < 1050 \\ -1050 & \text{if } S \geq 1050 \end{cases} \]

Profit:
Your receive 1000 from the short sale and spend 71.802 to buy the 1050-strike call.
The future value is: \((1000 - 71.802) \cdot 1.02 = 946.76\)
So the profit is:
\[\begin{cases} -S & \text{if } S < 1050 \\ -1050 & \text{if } S \geq 1050 \end{cases} + 946.76 = \begin{cases} 946.76 - S & \text{if } S < 1050 \\ -103.24 & \text{if } S \geq 1050 \end{cases} \]
CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES

\[
\text{Payoff} = \begin{cases} 
-S & \text{if } S < 1050 \\
-1050 & \text{if } S \geq 1050
\end{cases}
\]

\[
\text{Profit} = \begin{cases} 
-S & \text{if } S < 1050 \\
-1050 & \text{if } S \geq 1050 + 946.76
\end{cases}
\]
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Option 2:
Payoff:
If you borrow $1029.41, you’ll need to pay $1029.41 (1.02) = 1050 at $T = 0.5$
So the payoff of borrowing 1029.41 is 1050.

Payoff of the purchased put is $\max (0, 1050 - S)$

Total payoff is:
$$= -1050 + \begin{cases} 
1050 - S & \text{if } S < 1050 \\
0 & \text{if } S \geq 1050 
\end{cases} = \begin{cases} 
-S & \text{if } S < 1050 \\
-1050 & \text{if } S \geq 1050 
\end{cases}$$

Initially, you receive $1029.41 from a bank and spend $101,214 to buy a 950-strike put. So your net receipt at $t = 0$ is $1029.41 - 101,214 = 928.196$. Its future value is $928.196 \times (1.02) = 946.76$. Your profit is:

$$\begin{cases} 
-S & \text{if } S < 1050 \\
-1050 & \text{if } S \geq 1050 
\end{cases} + 946.76 = \begin{cases} 
946.76 - S & \text{if } S < 1050 \\
-103.24 & \text{if } S \geq 1050 
\end{cases}$$

Option 1 and 2 have the same payoff and the same profit. This is because the put-call parity:

$$\underbrace{\text{Call}(K, T)} + \underbrace{-S_0} = \underbrace{\text{Put}(K, T)} + \underbrace{-PV(K)}$$

buy a call sell one index buy a put borrow PV of strike price
Problem 3.6.

(a) buy an index for 1000
(b) buy a 950-strike call, sell a 950-strike put, and lend 931.37
Verify that (a) and (b) have the same payoff and the same profit.

(a)'s payoff is $S$. Profit is $S - 1000 (1.02) = S - 1020$

(b)'s payoff:

<table>
<thead>
<tr>
<th>Payoff</th>
<th>Initial receipt</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy a 950-strike call</td>
<td>max $(0, S - 950)$</td>
</tr>
<tr>
<td>sell a 950-strike put</td>
<td>$- \max (0, 950 - S)$</td>
</tr>
<tr>
<td>lend 931.37</td>
<td>$931.37 (1.02) = 950$</td>
</tr>
<tr>
<td>Total</td>
<td>$-120.405 + 51.777 - 931.37 = -1000$</td>
</tr>
</tbody>
</table>

Total payoff:

$$
\max (0, S - 950) - \max (0, 950 - S) + 950 \\
= \begin{cases} 
0 & \text{if } S < 950 \\
S - 950 & \text{if } S \geq 950
\end{cases} - \begin{cases} 
950 - S & \text{if } S < 950 \\
0 & \text{if } S \geq 950
\end{cases} + 950 \\
= \begin{cases} 
-(950 - S) + 950 & \text{if } S < 950 \\
S - 950 + 950 & \text{if } S \geq 950
\end{cases} = S
$$

Total profit: $S - 1000 (1.02) = S - 1020$

(a) and (b) have the same payoff and the same profit. Why?

$$
\text{Call} (K, T) + \left( -S_0 \right) = \text{Put} (K, T) + \left( -PV (K) \right)
$$

$$
\rightarrow S_0 = \text{Call} (K, T) + \left( -\text{Put} (K, T) + \left( PV (K) \right) \right)
$$

\begin{align*}
\text{buy a call} & \quad \text{sell one index} & \quad \text{buy a put} & \quad \text{borrow PV of strike price} \\
\rightarrow & \quad S_0 & \quad \text{Call} (K, T) & \quad -\text{Put} (K, T) & \quad PV (K) \\
\text{buy one index} & \quad \text{buy a call} & \quad \text{sell a put} & \quad \text{lend PV of strike price}
\end{align*}
(a) and (b) have the following common payoff and profit.

Payoff = $S$
Profit = $S - 1020$
Problem 3.7.

(a) short index for 1000
(b) sell 1050-strike call, buy a 1050-strike put, and borrow 1029.41
Verify that (a) and (b) have the same payoff and the same profit.

(a) Payoff is $-S$. Profit is $-S + 1000 \times (1.02) = 1020 - S$
(b) Payoff:
max (0, 1050 - S) - max (0, S - 1050) - 1029.41 \times (1.02)

\[
= \begin{cases} 
1050 - S & \text{if } S < 1050 \\
0 & \text{if } S \geq 1050 
\end{cases} 
- \begin{cases} 
0 & \text{if } S < 1050 \\
S - 1050 & \text{if } S \geq 1050 
\end{cases} 
1029.41 &= \begin{cases} 
1050 - S & \text{if } S < 1050 \\
0 & \text{if } S \geq 1050 
\end{cases} 
- \begin{cases} 
(S - 1050) - 1050 & \text{if } S < 1050 \\
-S & \text{if } S \geq 1050 
\end{cases} = -S
\]

We need to calculate the initial investment of (b).
At $t = 0$, we
- Receive 71.802 from selling a 1050-strike call
- Pay 101.214 to buy a 1050-strike put
- Receive 1029.41 from a lender

Our net receipt is $71.802 - 101.214 + 1029.41 = 1000$.
The future value is $1000 \times (1.02) = 1020$

So the profit at $T = 0.5$ is $-S + 1020 = 1020 - S$.

You see that (a) and (b) have the same payoff and the same profit. Why?
From the put-call parity, we have:

\[
\underbrace{\text{Call} (K, T)}_{\text{buy a call}} + \underbrace{S_0}_{\text{sell one index}} = \underbrace{\text{Put} (K, T)}_{\text{buy a put}} + \underbrace{-PV (K)}_{\text{borrow PV of strike price}}
\]

\[
\rightarrow \underbrace{-S_0}_{\text{sell one index}} = \underbrace{-\text{Call} (K, T)}_{\text{sell a call}} + \underbrace{\text{Put} (K, T)}_{\text{buy a put}} + \underbrace{-PV (K)}_{\text{borrow PV of strike price}}
\]
Payoff $= -S$. Profit $= 1020 - S$
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Problem 3.8.

Put-call parity: $\text{Call}(K, T) + PV(K) = \text{Put}(K, T) + S_0$

$109.2 + PV(K) = 60.18 + 1000$
$PV(K) = 60.18 + 1000 - 109.2 = 950.98$

$PV(K) = \frac{K}{1.02}$
$K = 950.98 (1.02) = 970$

Problem 3.9.

Buy a call (put) a lower strike + Sell an otherwise identical call (put) with a higher strike = Bull call (put) spread

Option 1: buy 950-strike call and sell 1000-strike call
Option 2: buy 950-strike put and sell 1000-strike put.
Verify that option 1 and 2 have the same profit.

Option 1:
Payoff $= \max(0, S - 950) - \max(0, S - 1000) = \begin{cases} 0 & \text{if } S < 950 \\ S - 950 & \text{if } S \geq 950 \end{cases} - \begin{cases} 0 & \text{if } S < 1000 \\ S - 1000 & \text{if } S \geq 1000 \end{cases}$

$= \begin{cases} 0 & \text{if } S < 950 \\ S - 950 & \text{if } 1000 > S \geq 950 \end{cases} - \begin{cases} 0 & \text{if } S < 1000 \\ S - 1000 & \text{if } S \geq 1000 \end{cases}$

$= \begin{cases} 0 & \text{if } S < 950 \\ S - 950 & \text{if } 1000 > S \geq 950 \end{cases} - \begin{cases} 0 & \text{if } S < 1000 \\ S - 1000 & \text{if } S \geq 1000 \end{cases}$

$= 0$ if $S < 950$
$= S - 950$ if $1000 > S \geq 950$
$= S - 950 - (S - 1000)$ if $S \geq 1000$

Initial cost:
- Spend 120.405 to buy 950-strike call
- Sell 1000-strike call receiving 93.809

Total initial investment: 120.405 - 93.809 = 26.596
The future value is 26.596 (1.02) = 27.12792

Profit is:
$= \begin{cases} 0 & \text{if } S < 950 \\ S - 950 & \text{if } 1000 > S \geq 950 \end{cases} - 27.13 = \begin{cases} -27.13 & \text{if } S < 950 \\ S - 950 - 27.13 & \text{if } 1000 > S \geq 950 \end{cases}$

$= \begin{cases} -27.13 & \text{if } S < 950 \\ S - 977.13 & \text{if } 1000 > S \geq 950 \end{cases}$
$= 22.87$ if $S \geq 1000$
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Payoff = \[
\begin{cases} 
0 & \text{if } S < 950 \\
S - 950 & \text{if } 1000 > S \geq 950 \\
50 & \text{if } S \geq 1000 
\end{cases}
\]

Profit = \[
\begin{cases} 
0 & \text{if } S < 950 \\
S - 950 & \text{if } 1000 > S \geq 950 - 27.13 \\
50 & \text{if } S \geq 1000 
\end{cases}
\]
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Option 2: buy 950-strike put and sell 1000-strike put.
The payoff:
\[
\max (0, 950 - S) - \max (0, 1000 - S)
\]
\[
= \begin{cases} 
 950 - S & \text{if } S < 950 \\
 0 & \text{if } S \geq 950 
\end{cases} - \begin{cases} 
 1000 - S & \text{if } S < 1000 \\
 0 & \text{if } S \geq 1000 
\end{cases}
\]
\[
= \begin{cases} 
 950 - S & \text{if } S < 950 \\
 0 & \text{if } 1000 > S \geq 950 \\
 0 & \text{if } S \geq 1000 
\end{cases} - \begin{cases} 
 1000 - S & \text{if } S < 950 \\
 1000 - S & \text{if } 1000 > S \geq 950 \\
 0 & \text{if } S \geq 1000 
\end{cases}
\]
\[
= \begin{cases} 
 (950 - S) - (1000 - S) & \text{if } S < 950 \\
 - (1000 - S) & \text{if } 1000 > S \geq 950 \\
 0 & \text{if } S \geq 1000 
\end{cases}
\]

Initial cost:
- Buy 950-strike put. Pay 51.777
- Sell 1000-strike put. Receive 74.201

Net receipt: 74.201 - 51.777 = 22.424
Future value: 22.424 (1.02) = 22.87248
The profit is:
\[
\begin{cases} 
 -50 & \text{if } S < 950 \\
 S - 1000 & \text{if } 1000 > S \geq 950 + 22.87 \\
 0 & \text{if } S \geq 1000 
\end{cases}
\]
\[
= \begin{cases} 
 -50 + 22.87 & \text{if } S < 950 \\
 S - 1000 + 22.87 & \text{if } 1000 > S \geq 950 \\
 22.87 & \text{if } S \geq 1000 
\end{cases}
\]
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\[
\begin{align*}
\text{Payoff} &= \begin{cases} 
-50 & \text{if } S < 950 \\
S - 1000 & \text{if } 1000 > S \geq 950 \\
0 & \text{if } S \geq 1000 
\end{cases} \\
\text{Profit} &= \begin{cases} 
-50 & \text{if } S < 950 \\
S - 1000 & \text{if } 1000 > S \geq 950 \\
0 & \text{if } S \geq 1000 
\end{cases} + 22.87
\end{align*}
\]
The payoff of the first option is $50 greater than the payoff of the second option. However, at \( t = 0 \), we pay 26.596 to set up option 1; we pay \(-22.424\) (i.e. we receive 22.424) to set up option 2. It costs us \(26.596 - (-22.424) = 49.02\) more initially to set up option 1 than option 2. The future value of this initial set up cost is \(49.02(1.02) = 50\). As a result, option 1 and 2 have the same profit at \( T = 0.5 \).

This should make sense in a world of no arbitrage. Consider two portfolios A and B. If for any stock price \(\text{Payoff}(A) = \text{Payoff}(B) + c\), then \(\text{InitialCost}(A) = \text{InitialCost}(B) + PV(c)\) to avoid arbitrage.

\[
\begin{align*}
\text{Profit}(A) &= \text{Payoff}(A) - FV[\text{InitialCost}(A)] \\
&= \text{Payoff}(B) + c - FV[\text{InitialCost}(B)] - c \\
&= \text{Payoff}(B) - FV[\text{InitialCost}(B)] = \text{Profit}(B)
\end{align*}
\]

Similarly, if \(\text{InitialCost}(A) = \text{InitialCost}(B) + PV(c)\)

\[
\rightarrow \text{Payoff}(A) = \text{Payoff}(B) + c \rightarrow \text{Profit}(A) = \text{Profit}(B)
\]

Finally, let’s see why option 1 is always $50 higher than option 2 in terms of the payoff and the initial set up cost. The put-call parity is:

\[
\begin{align*}
\text{Call}(K; T) &+ \text{PV}(K) = \text{Put}(K; T) + S_0 \\
\text{Call}(K_1; T) &+ \text{PV}(K_1) = \text{Put}(K_1; T) + S_0 \\
\text{Call}(K_2; T) &+ \text{PV}(K_2) = \text{Put}(K_2; T) + S_0
\end{align*}
\]

The timing of the put-call parity is at \( t = 0 \). The above equation means

\[
\begin{align*}
\text{Cost of buying a call at } t=0 &+ \text{Cost of investing PV of strike price at } t=0 \\
= \text{Cost of buying a put at } t=0 &+ \text{Cost of buying an index at } t=0
\end{align*}
\]

If we are interested in the payoff at expiration date \( T \), then the put-call parity is:

\[
\begin{align*}
\text{Payoff a call at } T &+ \text{strike price at } T = \text{Payoff of a put at } T &+ \text{Index price at } T
\end{align*}
\]

Now we set up the initial cost parity for two strike prices \(K_1 < K_2\)

\[
\begin{align*}
\text{cost of buying a call } &+ \text{PV of } K_1 = \text{cost of buying a put } &+ \text{cost of buying one index at } t=0 \\
\text{cost of buying a call } &+ \text{PV of } K_2 = \text{cost of buying a put } &+ \text{cost of buying one index at } t=0
\end{align*}
\]

\[
\rightarrow \left[ \begin{array}{c}
\text{Call}(K_1; T) \\
\text{Call}(K_2; T)
\end{array} \right] + [\text{PV}(K_1) - \text{PV}(K_2)]
\]
CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES

\[
= \left[ \frac{\text{Put}(K_1, T)}{\text{cost of buying a put}} - \frac{\text{Put}(K_2, T)}{\text{cost of buying a put}} \right]
\]

\[
\rightarrow \left[ \frac{\text{Call}(K_1, T)}{\text{cost of buying a call}} - \frac{\text{Call}(K_2, T)}{\text{cost of buying a call}} \right]
\]

Call spread

\[
= \left[ \frac{\text{Put}(K_1, T)}{\text{cost of buying a put}} - \frac{\text{Put}(K_2, T)}{\text{cost of buying a put}} \right] + [PV(K_2) - PV(K_1)]
\]

Put spread

So the initial cost of setting up a call bull spread always exceeds the initial set up cost of a bull put spread by a uniform amount \( PV(K_2) - PV(K_1) \).

In this problem, \( PV(K_2) - PV(K_1) = (1000 - 950) \cdot 1.02^{-1} = 49.02 \)

Set up the payoff parity at \( T \):

\[
\text{Call}(K_1, T) + K_1 = \frac{\text{Put}(K_1, T)}{\text{Payoff of a put at } T} + S_{\text{Index price at } T}
\]

\[
\text{Call}(K_2, T) + K_1 = \frac{\text{Put}(K_2, T)}{\text{Payoff of a put at } T} + S_{\text{Index price at } T}
\]

\[
\rightarrow \left[ \frac{\text{Call}(K_1, T)}{\text{Payoff a long call at } T} - \frac{\text{Call}(K_2, T)}{\text{Payoff a long call at } T} \right]
\]

Call bull payoff at \( T \)
\[ \text{Put spread payoff at } T = \left( \text{Payoff of a long put at } T \right) - \left( \text{Payoff of a long put at } T \right) + K_2 - K_1 \]

\[ \rightarrow \left( \text{Payoff of a long call at } T \right) + \left( \text{Payoff short call at } T \right) \]

\[ = \left( \text{Payoff of a long put at } T \right) - \left( \text{Payoff of a short put at } T \right) + K_2 - K_1 \]

In this problem, \( K_2 - K_1 = 1000 - 950 = 50 \)
So the payoff of a call bull spread at \( T = 0.5 \) always exceeds the payoff of a put bull spread by a uniform amount 50.
Problem 3.10.

Buy a call (put) a higher strike + Sell an otherwise identical call (put) with lower strike = Bear call (put) spread.

In this problem, \( K_1 = 1050 \), \( K_2 = 950 \) (a bear spread)

\[
PV(K_2) - PV(K_1) = (950 - 1050) 1.02^{-1} = (-100) 1.02^{-1} = -98.04
\]

\[
\begin{align*}
\text{Call spread} & = \left[ \frac{\text{Call}(K_1, T)}{\text{cost of buying a call}} + \frac{-\text{Call}(K_2, T)}{\text{cost of selling a call}} \right] \\
& = \left[ \frac{\text{Put}(K_1, T)}{\text{cost of buying a put}} + \frac{-\text{Put}(K_2, T)}{\text{cost of selling a put}} \right] + [PV(K_2) - PV(K_1)] \\
& = \left[ \frac{\text{Put}(K_1, T)}{\text{cost of buying a put}} + \frac{-\text{Put}(K_2, T)}{\text{cost of selling a put}} \right] - 98.04
\end{align*}
\]

\[
\begin{align*}
\text{Put spread} & = \left[ \frac{\text{Call}(K_1, T)}{\text{Payoff a long call at } T} + \frac{-\text{Call}(K_2, T)}{\text{Payoff short call at } T} \right] \\
& = \left[ \frac{\text{Put}(K_2, T)}{\text{Payoff of a long put at } T} + \frac{-\text{Put}(K_1, T)}{\text{Payoff of a short put at } T} \right] + K_2 - K_1 \\
& = \left[ \frac{\text{Put}(K_2, T)}{\text{Payoff of a long put at } T} + \frac{-\text{Put}(K_1, T)}{\text{Payoff of a short put at } T} \right] - 100
\end{align*}
\]

For any index price at expiration, the payoff of the call bear spread is always 100 less than the payoff of the put bear spread. Consequently, as we have seen, to avoid arbitrage, the initial set-up cost of the call bear spread is less than the initial set-up cost of the put bear spread by the amount of present value of the 100. The call bear spread and the put bear spread have the same profit at expiration.
Next, let’s draw the payoff and profit diagram for each spread.

Payoff of the call bear spread:
Payoff = \max (0, S - 1050) - \max (0, S - 950)

<table>
<thead>
<tr>
<th></th>
<th>( S &lt; 950 )</th>
<th>( 950 \leq S &lt; 1050 )</th>
<th>( 1050 \leq S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1050-strike call</td>
<td>0</td>
<td>0</td>
<td>( S - 1050 )</td>
</tr>
<tr>
<td>Sell 950-strike call</td>
<td>0</td>
<td>( 950 - S )</td>
<td>( 950 - S )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>( 950 - S )</td>
<td>( -100 )</td>
</tr>
</tbody>
</table>

\[
= \begin{cases} 
0 & \text{if } S < 950 \\
950 - S & \text{if } 950 \leq S < 1050 \\
-100 & \text{if } S \geq 1050 
\end{cases}
\]

The initial set-up cost of the call bear spread:

- Buy 1050-strike call. Pay 71.802
- Sell 950-strike call. Receive 120.405

Net receipt: \( 120.405 - 71.802 = 48.603 \)
Future value: \( 48.603 (1.02) = 49.575 \) 06

So the profit of the call bear spread at expiration is

\[
= \begin{cases} 
0 & \text{if } S < 950 \\
950 - S & \text{if } 950 \leq S < 1050 \\
-100 & \text{if } S \geq 1050 
\end{cases} + 49.58 = \begin{cases} 
49.58 & \text{if } S < 950 \\
999.58 - S & \text{if } 950 \leq S < 1050 \\
-50.42 & \text{if } S \geq 1050 
\end{cases}
\]
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\[
\text{Payoff} = \begin{cases} 
0 & \text{if } S < 950 \\
950 - S & \text{if } 950 \leq S < 1050 \\
-100 & \text{if } S \geq 1050
\end{cases}
\]

\[
\text{Profit} = \begin{cases} 
0 & \text{if } S < 950 \\
950 - S & \text{if } 950 \leq S < 1050 + 49.58 \\
-100 & \text{if } S \geq 1050
\end{cases}
\]

Payoff and Profit: Call bear spread
Payoff of the put bear spread:
Payoff = max \((0, 1050 - S)\) − max \((0, 950 - S)\)

\[
\begin{array}{c|ccc}
\text{Buy 1050-strike put} & S < 950 & 950 \leq S < 1050 & 1050 \leq S \\
1050 - S & 1050 - S & 0 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{Sell 950-strike put} & S < 950 & 950 \leq S < 1050 & 1050 \leq S \\
S - 950 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{Total} & S < 950 & 950 \leq S < 1050 & 1050 \leq S \\
100 & 1050 - S & 0 \\
\end{array}
\]

\[
= \begin{cases} 
100 & \text{if } S < 950 \\
1050 - S & \text{if } 950 \leq S < 1050 \\
0 & \text{if } S \geq 1050 
\end{cases}
\]

Payoff of the put bear spread
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The initial set-up cost:

- Buy 1050-strike put. Pay 101.214
- Sell 950-strike put. Receive 51.777

Net cost: 101.214 – 51.777 = 49.437
Future value: 49.437 (1.02) = 50.42

The profit at expiration is:

\[
= \begin{cases} 
100 & \text{if } S < 950 \\
1050 - S & \text{if } 950 \leq S < 1050 \\
0 & \text{if } S \geq 1050 
\end{cases} = \begin{cases} 
49.58 & \text{if } S < 950 \\
999.58 - S & \text{if } 950 \leq S < 1050 \\
-50.42 & \text{if } S \geq 1050 
\end{cases}
\]

We see that the call bear spread and the put bear spread have the same profit.
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Problem 3.11.

<table>
<thead>
<tr>
<th></th>
<th>Initial Cost</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy S&amp;R index</td>
<td>1000</td>
<td>$S$</td>
</tr>
<tr>
<td>Buy 950-strike put</td>
<td>51.777</td>
<td>$\max (0, 950 - S)$</td>
</tr>
<tr>
<td>Sell 1050-strike call</td>
<td>$-71.802$</td>
<td>$-\max (0, S - 1050)$</td>
</tr>
<tr>
<td>Total</td>
<td>$1000 + 51.777 - 71.802 = 979.975$</td>
<td>$999.5745$</td>
</tr>
<tr>
<td>FV (initial cost)</td>
<td>$979.975 (1.02) = 999.5745$</td>
<td></td>
</tr>
</tbody>
</table>

The net option premium is: $51.777 - 71.802 = -20.025$. So we receive $-20.025$ if we enter this collar.

Payoff

<table>
<thead>
<tr>
<th></th>
<th>$S &lt; 950$</th>
<th>$950 \leq S &lt; 1050$</th>
<th>$1050 \leq S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy S&amp;R index</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
</tr>
<tr>
<td>Buy 950-strike put</td>
<td>$950 - S$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Sell 1050-strike call</td>
<td>$0$</td>
<td>$0$</td>
<td>$1050 - S$</td>
</tr>
<tr>
<td>Total</td>
<td>$950$</td>
<td>$S$</td>
<td>$1050$</td>
</tr>
</tbody>
</table>

The payoff at expiration is:

$$\begin{cases} 
50 & \text{if } S < 950 \\
S & \text{if } 950 \leq S < 1050 \\
1050 & \text{if } S \geq 1050 
\end{cases}$$

The profit at expiration is:

$$\begin{cases} 
950 - 999.57 & \text{if } S < 950 \\
S - 999.57 & \text{if } 950 \leq S < 1050 \\
1050 - 999.57 & \text{if } S \geq 1050 
\end{cases}$$ 

$$ = \begin{cases} 
-49.57 & \text{if } S < 950 \\
S - 999.57 & \text{if } 950 \leq S < 1050 \\
50.43 & \text{if } S \geq 1050 
\end{cases}$$
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\[
\text{Profit} = \begin{cases} 
-49.57 & \text{if } S < 950 \\
S - 999.57 & \text{if } 950 \leq S < 1050 \\
50.43 & \text{if } S \geq 1050
\end{cases}
\]

The net option premium is $-20.025$. So we receive $20.025$ if we enter this collar. To construct a zero-cost collar and keep $950$-strike put, we need to increase the strike price of the call such that the call premium is equal to the put premium of $51.777$. 

![Diagram of profit vs. index price](image-url)
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Problem 3.12.

<table>
<thead>
<tr>
<th></th>
<th>Initial Cost</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy S&amp;R index</td>
<td>1000</td>
<td>$S$</td>
</tr>
<tr>
<td>Buy 950-strike put</td>
<td>51.777</td>
<td>$\max(0, 950 - S)$</td>
</tr>
<tr>
<td>Sell 1107-strike call</td>
<td>-51.873</td>
<td>$-\max(0, S - 1050)$</td>
</tr>
<tr>
<td>Total</td>
<td>1000 + 51.777 - 51.873 = 999.904</td>
<td>$999.904 (1.02) = 1019.90208$</td>
</tr>
</tbody>
</table>

The net option premium is: $51.777 - 51.873 = -0.096$. So we receive 0.096 if we enter this collar. This is very close to a zero-cost collar, where the net premium is zero.

Payoff

<table>
<thead>
<tr>
<th></th>
<th>$S &lt; 950$</th>
<th>$950 \leq S &lt; 1107$</th>
<th>$1107 \leq S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy S&amp;R index</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
</tr>
<tr>
<td>Buy 950-strike put</td>
<td>$950 - S$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sell 1050-strike call</td>
<td>0</td>
<td>0</td>
<td>$1107 - S$</td>
</tr>
<tr>
<td>Total</td>
<td>$950$</td>
<td>$S$</td>
<td>$1107$</td>
</tr>
</tbody>
</table>

The profit is:

\[
\begin{align*}
950 & \quad \text{if} \quad S < 950 \\
S & \quad \text{if} \quad 950 \leq S < 1107 \quad -1019.90 = \begin{cases} 
-69.9 & \quad \text{if} \quad S < 950 \\
S - 1019.90 & \quad \text{if} \quad 950 \leq S < 1107 \\
87.1 & \quad \text{if} \quad S \geq 1107
\end{cases}
\end{align*}
\]
\[
\text{Profit} = \begin{cases} 
-69.9 & \text{if } S < 950 \\
S - 1019.90 & \text{if } 950 \leq S < 1107 \\
87.1 & \text{if } S \geq 1107
\end{cases}
\]
Problem 3.13.

a. 1050-strike S&R straddle

Straddle = buy a call and put with the same strike price and time to expiration.

<table>
<thead>
<tr>
<th></th>
<th>Initial Cost</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1050-strike call</td>
<td>71.802</td>
<td>max (0, S - 1050)</td>
</tr>
<tr>
<td>Buy 1050-strike put</td>
<td>101.214</td>
<td>max (0, 1050 - S)</td>
</tr>
<tr>
<td>Total</td>
<td>71.802 + 101.214 = 173.016</td>
<td></td>
</tr>
<tr>
<td>FV (initial cost)</td>
<td>173.016 (1.02) = 176.47632</td>
<td></td>
</tr>
</tbody>
</table>

Payoff

<table>
<thead>
<tr>
<th></th>
<th>S &lt; 1050</th>
<th>S ≥ 1050</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1050-strike call</td>
<td>0</td>
<td>S - 1050</td>
</tr>
<tr>
<td>Buy 1050-strike put</td>
<td>1050 - S</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>1050 - S</td>
<td>S - 1050</td>
</tr>
</tbody>
</table>

The profit is:

\[
\begin{cases} 
1050 - S & \text{if } S < 1050 \\
S - 1050 & \text{if } S \geq 1050 
\end{cases}
\]

\[
873.52 - S & \text{if } S < 1050 \\
S - 1226.48 & \text{if } S \geq 1050
\]

Profit: long 1050-strike call and long 1050-strike put
b. written 950-strike S&R straddle

<table>
<thead>
<tr>
<th></th>
<th>Initial revenue</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>short 950-strike call</td>
<td>120.405</td>
<td>(-\max (0, S - 950))</td>
</tr>
<tr>
<td>short 950-strike put</td>
<td>51.777</td>
<td>(-\max (0, 950 - S))</td>
</tr>
<tr>
<td>Total</td>
<td>120.405 + 51.777 = 172.182</td>
<td></td>
</tr>
<tr>
<td>FV (initial cost)</td>
<td>172.182 (1.02) = 175.62564\</td>
<td></td>
</tr>
</tbody>
</table>

Payoff

<table>
<thead>
<tr>
<th></th>
<th>(S &lt; 950)</th>
<th>(S \geq 950)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell 950-strike call</td>
<td>0</td>
<td>(950 - S)</td>
</tr>
<tr>
<td>sell 950-strike put</td>
<td>(S - 950)</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>(S - 950)</td>
<td>(950 - S)</td>
</tr>
</tbody>
</table>

The profit is:

\[
\begin{cases} 
S - 950 & \text{if } S < 950 \\
950 - S & \text{if } S \geq 950 
\end{cases}
+ 175.66 = \begin{cases} 
S - 774.34 & \text{if } S < 950 \\
1125.66 - S & \text{if } S \geq 950 
\end{cases}
\]

Profit: short 950-strike call and short 950-strike put
c. simultaneous purchase of 1050-straddle and sale of 950-straddle

\[
\text{Profit} = \text{Profit of purchase of 1050-straddle} + \text{Profit of sale of 950-straddle}
\]

\[
\begin{align*}
\text{Profit} &= \begin{cases} 
873.52 - S & \text{if } S < 950 \\
S - 1226.48 & \text{if } S \geq 1050 
\end{cases} + \begin{cases} 
S - 774.34 & \text{if } S < 950 \\
1125.66 - S & \text{if } S \geq 1050 
\end{cases} \\
&= \begin{cases} 
873.52 - S + (S - 774.34) & \text{if } S < 950 \\
873.52 - S + (1125.66 - S) & \text{if } 950 \leq S < 1050 \\
S - 1226.48 + (1125.66 - S) & \text{if } S \geq 1050 
\end{cases} \\
&= \begin{cases} 
99.18 & \text{if } S < 950 \\
1999.18 - 2S & \text{if } 950 \leq S < 1050 \\
-100.82 & \text{if } S \geq 1050 
\end{cases}
\]

Profit: long 1050-strike straddle and short 950 straddle

The put-call parity is:
\[
Call(K, T) + \frac{PV(K)}{S_0} = Put(K, T) + S_0
\]

\[
\rightarrow Call(K_1, T) + \frac{PV(K_1)}{S_0} = Put(K_1, T) + S_0
\]

\[
\rightarrow Call(K_2, T) + \frac{PV(K_2)}{S_0} = Put(K_2, T) + S_0
\]

\[
\rightarrow \left[ Call(K_1, T) - Call(K_2, T) \right] + PV(K_1 - K_2) = \left[ Put(K_1, T) - Put(K_2, T) \right]
\]

\[
\rightarrow \left[ Call(K_1, T) + \frac{PV(K_1)}{S_0} - Call(K_2, T) \right] = \left[ Put(K_1, T) + \frac{PV(K_2)}{S_0} - Put(K_2, T) \right]
\]

\[
\rightarrow \left[ Call(K_1, T) + \frac{PV(K_1)}{S_0} - Call(K_2, T) \right] - \left[ Put(K_1, T) + \frac{PV(K_2)}{S_0} - Put(K_2, T) \right] = PV(K_2 - K_1)
\]

\[
\rightarrow \left[ Call(K_1, T) + \frac{PV(K_1)}{S_0} - Call(K_2, T) \right] + \left[ \frac{PV(K_1)}{S_0} - Put(K_1, T) + Put(K_2, T) \right] = PV(K_2 - K_1)
\]

The initial cost is \( PV(K_2 - K_1) \) at \( t = 0 \). The payoff at expiration \( T = 0.5 \) is \( K_2 - K_1 \). The transaction is equivalent to investing \( PV(K_2 - K_1) \) in a savings account at \( t = 0 \) and receiving \( K_2 - K_1 \) at \( T \), regardless of the S&R price at expiration. So the transaction doesn’t have any S&R price risk. We just earn the risk free interest rate over the 6-month period.

In this problem, \( K_1 = 950 \) and \( K_2 = 1000 \)

\[
\left[ Call(K_1, T) + \frac{PV(K_1)}{S_0} - Call(K_2, T) \right] + \left[ \frac{PV(K_1)}{S_0} - Put(K_1, T) + Put(K_2, T) \right] = PV(1000 - 950) = PV(50) = 50 \left( 1.02^{-1} \right) = 49.02
\]

So the total initial cost is 49.02. The payoff is 50.02 (1.02) = 50. The profit is 0. We earn a 2% risk-free interest rate over the 6-month period.

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Problem 3.15.

a. Buy a 950-strike call and sell two 1050-strike calls

<table>
<thead>
<tr>
<th></th>
<th>Initial cost</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy a 950-strike call</td>
<td>120.405</td>
<td>$\max(0, S - 950)$</td>
</tr>
<tr>
<td>sell two 1050-strike calls</td>
<td>-2 (71.802)</td>
<td>$-2 \max(0, S - 1050)$</td>
</tr>
<tr>
<td>Total</td>
<td>120.405 - 143.604 = -23.199</td>
<td></td>
</tr>
<tr>
<td>FV (initial cost)</td>
<td>-23.199 (1.02) = -23.662 98</td>
<td></td>
</tr>
</tbody>
</table>

Payoff

<table>
<thead>
<tr>
<th></th>
<th>$S &lt; 950$</th>
<th>$950 \leq S &lt; 1050$</th>
<th>$S \geq 1050$</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy a 950-strike call</td>
<td>0</td>
<td>$S - 950$</td>
<td>$S - 950$</td>
</tr>
<tr>
<td>sell two 1050-strike calls</td>
<td>0</td>
<td>0</td>
<td>$-2 \left(S - 1050\right)$</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>$S - 950$</td>
<td>$S - 950 - 2\left(S - 1050\right) = 1150 - S$</td>
</tr>
</tbody>
</table>

The profit is:

$$
\begin{align*}
0 & \quad \text{if} \quad S < 950 \\
S - 950 & \quad \text{if} \quad 950 \leq S < 1050 \\
1150 - S & \quad \text{if} \quad S \geq 1050
\end{align*}
$$

$$
+23.66 = \begin{cases} 
23.66 & \text{if} \quad S < 950 \\
S - 950 & \text{if} \quad 950 \leq S < 1050 \\
1150 - S + 23.66 & \text{if} \quad S \geq 1050
\end{cases}
$$

$$
= \begin{cases} 
23.66 & \text{if} \quad S < 950 \\
S - 926.34 & \text{if} \quad 950 \leq S < 1050 \\
1173.66 - S & \text{if} \quad S \geq 1050
\end{cases}
$$

long 950-strike call and short two 1050-strike calls
CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES

b. Buy two 950-strike calls and sell three 1050-strike calls

<table>
<thead>
<tr>
<th></th>
<th>Initial cost</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy two 950-strike</td>
<td>2(120.405) =</td>
<td>2 max (0, S - 950)</td>
</tr>
<tr>
<td>calls</td>
<td>240.81</td>
<td></td>
</tr>
<tr>
<td>sell three 1050-strike</td>
<td>-3(71.802) =</td>
<td>-3 max (0, S - 1050)</td>
</tr>
<tr>
<td>calls</td>
<td>-215.406</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>240.81 - 215.406 = 25.404</strong></td>
<td></td>
</tr>
<tr>
<td><strong>FV (initial cost)</strong></td>
<td><strong>25.404 (1.02) = 25.91208</strong></td>
<td></td>
</tr>
</tbody>
</table>

Payoff

<table>
<thead>
<tr>
<th></th>
<th>S &lt; 950</th>
<th>950 ≤ S &lt; 1050</th>
<th>S ≥ 1050</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy two 950-strike</td>
<td>0</td>
<td>2(S - 950)</td>
<td>2(S - 950)</td>
</tr>
<tr>
<td>calls</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sell three 1050-strike</td>
<td>0</td>
<td>0</td>
<td>-3(S - 1050)</td>
</tr>
<tr>
<td>calls</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>2(S - 950)</td>
<td>2(S - 950) - 3(S - 1050) = 1250 - S</td>
</tr>
</tbody>
</table>

Profit:

\[
\begin{align*}
0 & \quad \text{if} \quad S < 950 \\
2(S - 950) & \quad \text{if} \quad 950 \leq S < 1050 \\
1250 - S & \quad \text{if} \quad S \geq 1050
\end{align*}
\]

\[
= \begin{cases} 
-25.91 & \quad \text{if} \quad S < 950 \\
2S - 1925.91 & \quad \text{if} \quad 950 \leq S < 1050 \\
1224.09 - S & \quad \text{if} \quad S \geq 1050
\end{cases}
\]

long two 950-strike calls and short three 1050-strike calls

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c. Buy \( n \) 950-strike calls and short \( m \) 1050-strike calls such that the initial premium is zero.

\[
120.405n = 71.802m \quad \Rightarrow \quad \frac{n}{m} = \frac{71.802}{120.405} = 0.5963
\]

Problem 3.16.

A spread consists of buying one option at one strike price and selling an otherwise identical option with a different strike price.

A bull spread consists of buying one option at one strike and selling an otherwise identical option but at a higher strike price.

A bear spread consists of buying one option at one strike and selling an otherwise identical option but at a lower strike price.

A bull spread and a bear spread will never have zero premium because the two options don’t have the same premium.

A butterfly spread might have a zero net premium.

Problem 3.17.

According to \( \text{http://www.daytradeteam.com/dtt/butterfly-options-trading.asp} \), a butterfly spread combines a bull and a bear spread. It uses three strike prices. The lower two strike prices are used in the bull spread, and the higher strike price in the bear spread. Both puts and calls can be used. A very large profit is made if the stock is at or very near the middle strike price on expiration day.

When you enter a butterfly spread, you are entering 3 options orders at once. If the stock remains or moves into a defined range, you profit, and if the stock moves out of the desired range, you lose. The closer the stock is to the middle strike price on expiration day, the larger your profit.

For the strike price \( K_1 < K_2 < K_3 \), \( K_2 = \lambda K_1 + (1 - \lambda) K_3 \)

So for each \( K_2 \)-strike option sold, there needs to be \( \lambda \) units of \( K_1 \)-strike options bought and \( (1 - \lambda) \) units of \( K_3 \)-strike options bought. In this problem, \( K_1 = 950, \ K_2 = 1020, \ K_3 = 1050 \)

\( 1020 = 950\lambda + (1 - \lambda) 1050 \quad \Rightarrow \lambda = 0.3 \)

For every ten 1020-strike calls written, there needs to be three 950-strike calls purchased and seven 1050-strike calls purchased (so we buy three 950 – 1020 bull spreads and seven 1020 – 1050 bear spreads).

<table>
<thead>
<tr>
<th></th>
<th>Initial cost</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell ten 1020-strike calls</td>
<td>(-844.7)</td>
<td>(-10 \max (0, S - 1020))</td>
</tr>
<tr>
<td>three 950-strike calls</td>
<td>(3 \times 120.405)</td>
<td>(3 \max (0, S - 950))</td>
</tr>
<tr>
<td>seven 1050-strike calls</td>
<td>(7 \times 71.802)</td>
<td>(7 \max (0, S - 1050))</td>
</tr>
<tr>
<td>Total</td>
<td>(-844.7 + 361.215 + 502.614 = 19.129)</td>
<td></td>
</tr>
<tr>
<td>( \text{FV (initial cost)} )</td>
<td>(19.129 \times 1.02 = 19.51158)</td>
<td></td>
</tr>
</tbody>
</table>


**CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES**

<table>
<thead>
<tr>
<th>Payoff</th>
<th>$S &lt; 950$</th>
<th>$950 \leq S &lt; 1020$</th>
<th>$1020 \leq S &lt; 1050$</th>
<th>$1050 \leq S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>950-strike calls</td>
<td>0</td>
<td>$3(S - 950)$</td>
<td>$3(S - 950)$</td>
<td>$3(S - 950)$</td>
</tr>
<tr>
<td>1020-strike calls</td>
<td>0</td>
<td>0</td>
<td>$-10(S - 1020)$</td>
<td>$-10(S - 1020)$</td>
</tr>
<tr>
<td>1050-strike calls</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$7(S - 1050)$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>$3(S - 950)$</td>
<td>7350 - 7S</td>
<td>0</td>
</tr>
</tbody>
</table>

$3(S - 950) - 10(S - 1020) = 7350 - 7S$

$3(S - 950) - 10(S - 1020) + 7(S - 1050) = 0$

The payoff = \[
\begin{cases}
0 & \text{if } S < 950 \\
3(S - 950) & \text{if } 950 \leq S < 1020 \\
7350 - 7S & \text{if } 1020 \leq S < 1050 \\
0 & \text{if } 1050 \leq S
\end{cases}
\]

A key point to remember is that for a butterfly spread $K_1 < K_2 < K_3$, the payoff is zero if $S \leq K_1$ or $S \geq K_3$.

The profit is:

\[
\begin{cases}
0 & \text{if } S < 950 \\
3(S - 950) & \text{if } 950 \leq S < 1020 \\
7350 - 7S & \text{if } 1020 \leq S < 1050 \\
0 & \text{if } 1050 \leq S
\end{cases}
\]

$$
= \begin{cases}
-19.51 & \text{if } S < 950 \\
3S - 2869.51 & \text{if } 950 \leq S < 1020 \\
7330.49 - 7S & \text{if } 1020 \leq S < 1050 \\
-19.51 & \text{if } 1050 \leq S
\end{cases}
$$

---

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\[
\text{Payoff} = \begin{cases} 
0 & \text{if } S < 950 \\
3(S - 950) & \text{if } 950 \leq S < 1020 \\
7350 - 7S & \text{if } 1020 \leq S < 1050 \\
0 & \text{if } 1050 \leq S 
\end{cases}
\]

\[
\text{Profit} = \begin{cases} 
0 & \text{if } S < 950 \\
3(S - 950) & \text{if } 950 \leq S < 1020 \\
7350 - 7S & \text{if } 1020 \leq S < 1050 - 19.51 \\
0 & \text{if } 1050 \leq S 
\end{cases}
\]

The black line is the profit line. The blue line is the payoff line.

Butterfly spread \(K_1 = 950, K_2 = 1020, K_3 = 1050\)

The black line is the profit line. The blue line is the payoff line.
Problem 3.18.

The option price table is:

<table>
<thead>
<tr>
<th>Strike</th>
<th>Call premium</th>
<th>Put premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>6.13</td>
<td>0.44</td>
</tr>
<tr>
<td>40</td>
<td>2.78</td>
<td>1.99</td>
</tr>
<tr>
<td>45</td>
<td>0.97</td>
<td>5.08</td>
</tr>
</tbody>
</table>

Time to expiration is \( T = \frac{91}{365} \approx 0.25 \)
The annual effective rate is 8.33%
The quarterly effective rate is \( \sqrt[4]{1.0833} - 1 = 2.02\% \)

a. Buy 35–strike call, sell two 40-strike calls, and buy 45-strike call. Let’s reproduce the textbook Figure 3.14.

<table>
<thead>
<tr>
<th>Initial cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy a 35-strike call</td>
</tr>
<tr>
<td>sell two 40-strike calls</td>
</tr>
<tr>
<td>buy a 45-strike call</td>
</tr>
<tr>
<td>Total</td>
</tr>
<tr>
<td>FV (initial cost)</td>
</tr>
</tbody>
</table>

Payoff

<table>
<thead>
<tr>
<th>( S \leq 35 )</th>
<th>(35 \leq S &lt; 40)</th>
<th>(40 \leq S &lt; 45)</th>
<th>(45 \leq S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>35-strike call</td>
<td>0</td>
<td>(S - 35)</td>
<td>(S - 35)</td>
</tr>
<tr>
<td>40-strike calls</td>
<td>0</td>
<td>0</td>
<td>(-2(S - 40))</td>
</tr>
<tr>
<td>45-strike call</td>
<td>0</td>
<td>0</td>
<td>(S - 45)</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>(S - 35)</td>
<td>(45 - S)</td>
</tr>
</tbody>
</table>

\( S - 35 - 2(S - 40) = 45 - S \)
\( S - 35 - 2(S - 40) + S - 45 = 0 \)

The payoff is:

\[
\begin{align*}
0 & \quad \text{if} \quad S < 35 \\
S - 35 & \quad \text{if} \quad 35 \leq S < 40 \\
45 - S & \quad \text{if} \quad 40 \leq S < 45 \\
0 & \quad \text{if} \quad 45 \leq S 
\end{align*}
\]

The profit is:

\[
\begin{align*}
0 & \quad \text{if} \quad S < 35 \\
S - 35 & \quad \text{if} \quad 35 \leq S < 40 \\
45 - S & \quad \text{if} \quad 40 \leq S < 45 \\
0 & \quad \text{if} \quad 45 \leq S 
\end{align*}
\]
CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES

\[
\text{Payoff} = \begin{cases} 
0 & \text{if } S < 35 \\
S - 35 & \text{if } 35 \leq S < 40 \\
45 - S & \text{if } 40 \leq S < 45 \\
0 & \text{if } 45 \leq S 
\end{cases}
\]

\[
\text{Profit} = \begin{cases} 
0 & \text{if } S < 35 \\
S - 35 & \text{if } 35 \leq S < 40 \\
45 - S & \text{if } 40 \leq S < 45 - 1.57 \\
0 & \text{if } 45 \leq S 
\end{cases}
\]

Butterfly spread $K_1 = 35$, $K_2 = 40$, $K_3 = 45$
b. Buy a 35–strike put, sell two 40-strike puts, and buy a 45-strike put.
Let’s reproduce the textbook Figure 3.14.

<table>
<thead>
<tr>
<th>Initial cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy a 35-strike put</td>
</tr>
<tr>
<td>sell two 40-strike puts</td>
</tr>
<tr>
<td>buy a 45-strike put</td>
</tr>
<tr>
<td>Total</td>
</tr>
<tr>
<td>FV (initial cost)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Payoff</th>
<th>S &lt; 35</th>
<th>35 ≤ S &lt; 40</th>
<th>40 ≤ S &lt; 45</th>
<th>45 ≤ S</th>
</tr>
</thead>
<tbody>
<tr>
<td>35-strike put</td>
<td>35 − S</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>40-strike puts</td>
<td>−2 (40 − S)</td>
<td>−2 (40 − S)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>45-strike put</td>
<td>45 − S</td>
<td>45 − S</td>
<td>45 − S</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>S − 35</td>
<td>45 − S</td>
<td>0</td>
</tr>
</tbody>
</table>

35 − S − 2 (40 − S) + 45 − S = 0
−2 (40 − S) + 45 − S = S − 35

The payoff =
\[
\begin{cases} 
0 & \text{if } S < 35 \\
S - 35 & \text{if } 35 \leq S < 40 \\
45 - S & \text{if } 40 \leq S < 45 \\
0 & \text{if } 45 \leq S 
\end{cases}
\]

The profit =
\[
\begin{cases} 
0 & \text{if } S < 35 \\
S - 35 & \text{if } 35 \leq S < 40 \\
45 - S & \text{if } 40 \leq S < 45 \\
-1.57 & \text{if } 45 \leq S 
\end{cases}
\]
\textbf{CHAPTER 3. INSURANCE, COLLARS, AND OTHER STRATEGIES}

\[
\text{Payoff} = \begin{cases} 
0 & \text{if } S < 35 \\
S - 35 & \text{if } 35 \leq S < 40 \\
45 - S & \text{if } 40 \leq S < 45 \\
0 & \text{if } 45 \leq S 
\end{cases}
\]

\[
\text{Profit} = \begin{cases} 
0 & \text{if } S < 35 \\
S - 35 & \text{if } 35 \leq S < 40 \\
45 - S & \text{if } 40 \leq S < 45 \\
0 & \text{if } 45 \leq S 
\end{cases} - 1.57
\]

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    xtick={25,30,35,40,45,50,55,60},
    xticklabels={25,30,35,40,45,50,55,60},
    ytick={0,1,2,3,4,5},
    yticklabels={-1,0,1,2,3,4,5},
    xlabel={Stock Price},
    ylabel={Butterfly spread $K_1 = 35, K_2 = 40, K_3 = 45$}
]
\addplot[black,thick] coordinates {
(25,0) (30,0) (35,0) (40,5) (45,0) (50,0) (55,0) (60,0)
};
\addplot[blue,thick] coordinates {
(25,0) (30,0) (35,0) (40,5) (45,0) (50,0) (55,0) (60,0)
};
\end{axis}
\end{tikzpicture}
\end{center}

The black line is the payoff; the blue line is the profit.
c. Buy one stock, buy a 35 put, sell two 40 calls, and buy a 45 call.

The put-call parity is:

\[
\text{Call} (K, T) + PV (K) = \text{Put} (K, T) + S_0
\]

\[
\text{buy a call} + \text{invest PV of strike price} = \text{buy a put} + \text{buy one stock}
\]

\[
\rightarrow \text{Buy stock + buy 35 put} = \text{buy 35 call + PV(35)}
\]

Buy one stock, buy a 35 put, sell two 40 calls, and buy a 45 call is the same as:

- buy a 35 call, sell two 40 calls, and buy a 45 call, and deposit PV(35) in a savings account.

We already know from Part a. that "buy a 35 call, sell two 40 calls, and buy a 45 call" reproduces the textbook profit diagram Figure 3.14.

Depositing PV(35) = 35 \times (1.0202^{-1}) = 34.31 won’t change the profit because any deposit in a savings account has zero profit.

Hence the profit diagram of "Buy one stock, buy a 35 put, sell two 40 calls, and buy a 45 call" is Figure 3.14.
Problem 3.19.

a. The parity is:
\[ \text{Call}(K,T) - \text{Put}(K,T) = PV(F_{0,T} - K) \]
We are told that \[ \text{Call}(K,T) - \text{Put}(K,T) = 0 \]
\[ \rightarrow PV(F_{0,T} - K) = 0 \]
\[ \Rightarrow PV(F_{0,T}) = PV(K) \]
Since \[ PV(F_{0,T}) = S_0 \]
\[ \rightarrow PV(K) = S_0 \]

b. Buying a call and selling an otherwise identical put creates a synthetic long forward.

c. We buy the call at the ask price and sell the put at the bid price. So we have to pay the dealer a little more than the fair price of the call when we buy the call from the dealer; we’ll get less than the fair price of put when we sell a put to the dealer. To ensure that the call premium equals the put premium given there’s a bid-ask spread, we need to make the call less valuable and the put more valuable. To make the call less valuable and the put more valuable, we can increase the strike price. In other words, if there’s no bid-ask spread, then \( K = F_{0,T} \). If there’s bid-ask spread, \( K > F_{0,T} \).

d. A synthetic short stock position means "buy put and sell call." To have zero net premium after the bid-ask spread, we need to make the call more valuable and the put less valuable. To achieve this, we can decrease the strike price. In other words, if there’s no bid-ask spread, then \( K = F_{0,T} \). If there’s bid-ask spread, \( K < F_{0,T} \).

e. Transaction fees is not really a wash because there’s a bid-ask spread. We pay more if we buy an option and we get less if we sell an option.

Problem 3.20.

This problem is about building a spreadsheet. You won’t be asked to build a spreadsheet in the exam. Skip this problem.
Chapter 4

Introduction to risk management

Problem 4.1.

Let

- \( S \) = the price of copper per pound at \( T = 1 \)
- \( P^{BH} \) = Profit per pound of copper at \( T = 1 \) before hedging
- \( P^{AH} \) = Profit per pound of copper at \( T = 1 \) after hedging

For each pound of copper produced, XYZ incurs $0.5 fixed cost and $0.4 variable cost.

\[
P^{BH} = S - (0.5 + 0.4) = S - 0.9
\]

XYZ sells a forward. The profit of the forward at \( T = 1 \) is:

\[
F_{0,T} - S = 1 - S
\]

\[
\rightarrow P^{AH} = (S - 0.9) + (F_{0,T} - S) = F_{0,T} - 0.9
\]

We are told that \( F_{0,T} = 1 \)

\[
\rightarrow P^{AH} = 1 - 0.9 = 0.1
\]
$P^{BH} = S - 0.9 \quad P^{AH} = 0.1$

Profit: before hedging and after hedging

Before hedging

After hedging

Copper price

Profit
Problem 4.2.

\[ P^{AH} = F_{0,T} - 0.9 \]

If \( F_{0,T} = 0.8 \)
\[ \rightarrow P^{AH} = 0.8 - 0.9 = -0.1 \]

If XYZ shuts down its production, its profit at \( T = 1 \) is \(-0.5\) (it still has to pay the fixed cost)
If XYZ continues its production, its after hedging profit at \( T = 1 \) is \(-0.1\)
\(-0.1 > -0.5\)
\[ \rightarrow \text{XYZ should continue its production} \]

If \( F_{0,T} = 0.45 \)
\[ \rightarrow P^{AH} = 0.45 - 0.9 = -0.45 \]

If XYZ shuts down its production, its profit at \( T = 1 \) is \(-0.5\) (it still has to pay the fixed cost)
If XYZ continues its production, its after hedging profit at \( T = 1 \) is \(-0.45\)
\(-0.45 > -0.5\)
\[ \rightarrow \text{XYZ should continue its production} \]

Problem 4.3.

The profit of a long \( K \)-strike put at \( T = 1 \):

\[
\max (0, K - S) - FV(Premium) = \begin{cases} 
K - S & \text{if } S < K \\
0 & \text{if } S \geq K
\end{cases} - FV(Premium)
\]

\[ \rightarrow P^{AH} = P^{BH} + \begin{cases} 
K - S & \text{if } S < K \\
0 & \text{if } S \geq K
\end{cases} - FV(Premium) \]

\[ = S - 0.9 + \begin{cases} 
K - S & \text{if } S < K \\
0 & \text{if } S \geq K
\end{cases} - FV(Premium) \]

\[ = \begin{cases} 
K & \text{if } S < K \\
S & \text{if } S \geq K
\end{cases} - 0.9 - FV(Premium) \]
$K = 0.95 \quad FV\, (Premium) = 0.0178 \times (1.06) = 0.02$

$$P^{AH} = \begin{cases} 0.95 & \text{if } S < 0.95 \\ S & \text{if } S \geq 0.95 \end{cases} - 0.92$$

Long 0.95-strike put
$K = 1 \quad FV (\text{Premium}) = 0.0376 (1.06) = 0.04$

$\rightarrow P^{AH} = \begin{cases} 1 & \text{if } S < 1 \\ S & \text{if } S \geq 1 \end{cases} - 0.9 - 0.04 = \begin{cases} 0.06 & \text{if } S < 1 \\ S - 0.94 & \text{if } S \geq 1 \end{cases}$

Long 1-strike put
$K = 1.05 \quad FV (Premium) = 0.0665 \times (1.06) = 0.07$

$\rightarrow P^{AH} = \begin{cases} 1.05 & \text{if } S < 1.05 \\ S & \text{if } S \geq 1.05 \end{cases}$

$-0.9 - 0.07 = \begin{cases} 0.08 & \text{if } S < 1.05 \\ S - 0.97 & \text{if } S \geq 1.05 \end{cases}$
Problem 4.4.

The profit of a short $K$-strike call at $T = 1$:

$$- \max (0, S - K) + FV(\text{Premium}) = - \begin{cases} 0 & \text{if } S < K \\ S - K & \text{if } S \geq K \end{cases} + FV(\text{Premium})$$

$$\rightarrow P_{AH} = P_{BH} = - \begin{cases} 0 & \text{if } S < K \\ S - K & \text{if } S \geq K \end{cases} + FV(\text{Premium})$$

$$= S - 0.9 - \begin{cases} 0 & \text{if } S < K \\ S - K & \text{if } S \geq K \end{cases} + FV(\text{Premium})$$

$$= \begin{cases} S & \text{if } S < K \\ K & \text{if } S \geq K \end{cases} - 0.9 + FV(\text{Premium})$$
\[ K = 0.95 \quad FV (\text{Premium}) = 0.0649 \times (1.06) = 0.07 \]

\[ \to P_{\text{AH}} = \begin{cases} 
S & \text{if } S < 0.95 \\
0.95 & \text{if } S \geq 0.95
\end{cases} 
\]

\[ -0.9 + 0.07 = \begin{cases} 
S - 0.83 & \text{if } S < 0.95 \\
0.12 & \text{if } 0.95 \leq S
\end{cases} \]
$K = 1 \quad FV (Premium) = 0.0376 (1.06) = 0.04$

$\rightarrow P^{AH} = \begin{cases} S & \text{if } S < 1 \\ 1 & \text{if } S \geq 1 \end{cases} - 0.9 + 0.04 = \begin{cases} S - 0.86 & \text{if } S < 1 \\ 0.14 & \text{if } 1 \leq S \end{cases}$
$K = 1.05 \quad FV (Premium) = 0.0194 \times (1.06) = 0.02$

$\rightarrow P^{AH} = \begin{cases} 
S & \text{if } S < 1.05 \\
1.05 & \text{if } S \geq 1.05
\end{cases} \quad \begin{cases} 
S - 0.88 & \text{if } S < 1.05 \\
0.17 & \text{if } 1.05 \leq S
\end{cases}$

Short 1.05-strike call

![Copper Price vs Profit Graph]
Problem 4.5.

\[ P^{AH} = P^{BH} + P^{Collar} \quad P^{BH} = S - 0.9 \]

Suppose XYZ buy a \( K_1 \)-strike put and sells a \( K_2 \)-strike call.

The profit of the collar is:

\[ P^{Collar} = \text{Payoff} - FV(\text{Net Initial Premium}) \]
\[ = [\max(0, K_1 - S) - \max(0, S - K_2)] - FV(\text{Put Premium}) + FV(\text{Call Premium}) \]

a. Buy 0.95-strike put and sell 1-strike call

The payoff is \( \max(0, 0.95 - S) - \max(0, S - 1) \)

<table>
<thead>
<tr>
<th>( S )</th>
<th>( S &lt; 0.95 )</th>
<th>( 0.95 \leq S &lt; 1 )</th>
<th>( S \geq 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>long 0.95-strike put</td>
<td>( 0.95 - S )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>short 1-strike call</td>
<td>0</td>
<td>0</td>
<td>1 - ( S )</td>
</tr>
<tr>
<td>Total</td>
<td>( 0.95 - S )</td>
<td>0</td>
<td>( 1 - S )</td>
</tr>
</tbody>
</table>

\[-FV(\text{Put Premium}) + FV(\text{Call Premium}) \]
\[ = (-0.0178 + 0.0376) \times 1.06 = 0.02 \]

\[ P^{Collar} = \begin{cases} 
0.95 - S & \text{if } S < 0.95 \\
0 & \text{if } 0.95 \leq S < 1 + 0.02 \\
1 - S & \text{if } S \geq 1 
\end{cases} \]

\[ P^{AH} = P^{BH} + P^{Collar} \]
\[ = S - 0.9 + \begin{cases} 
0.95 - S & \text{if } S < 0.95 \\
0 & \text{if } 0.95 \leq S < 1 + 0.02 \\
1 - S & \text{if } S \geq 1 
\end{cases} \]
\[ = \begin{cases} 
0.07 & \text{if } S < 0.95 \\
S - 0.88 & \text{if } 0.95 \leq S < 1 \\
0.12 & \text{if } 1 \leq S 
\end{cases} \]
$P_{AH} = \begin{cases} 
0.07 & \text{if } S < 0.95 \\
S - 0.88 & \text{if } 0.95 \leq S < 1 \\
0.12 & \text{if } 1 \leq S 
\end{cases}$

Long 0.95-strike put and short 1-strike call
b. Buy 0.975-strike put and sell 1.025-strike call

The payoff is \( \max(0, 0.975 - S) - \max(0, S - 1.025) \)

<table>
<thead>
<tr>
<th></th>
<th>( S &lt; 0.975 )</th>
<th>( 0.975 \leq S &lt; 1.025 )</th>
<th>( S \geq 1.025 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>long 0.975-strike put</td>
<td>( 0.975 - S )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>short 1-strike call</td>
<td>0</td>
<td>0</td>
<td>( 1.025 - S )</td>
</tr>
<tr>
<td>Total</td>
<td>( 0.975 - S )</td>
<td>0</td>
<td>( 1.025 - S )</td>
</tr>
</tbody>
</table>

\[-FV(\text{Put Premium}) + FV(\text{Call Premium})\]
\[= (-0.0265 + 0.0274) \times 1.06 = 0.000954 = 0.001\]

\[p^{\text{Collar}} = \begin{cases} 
0.975 - S & \text{if } S < 0.975 \\
0 & \text{if } 0.975 \leq S < 1.025 \\
1.025 - S & \text{if } S \geq 1.025
\end{cases} + 0.001\]

\[p^{\text{Collar}} \]

\[= \begin{cases} 
0.975 - S & \text{if } S < 0.975 \\
0 & \text{if } 0.975 \leq S < 1.025 \\
1.025 - S & \text{if } S \geq 1.025
\end{cases} + 0.001\]

\[= \begin{cases} 
0.076 & \text{if } S < 0.975 \\
S - 0.899 & \text{if } 0.975 \leq S < 1.025 \\
0.126 & \text{if } 1.025 \leq S
\end{cases}\]
\[ P^{AH} = \begin{cases} 
0.076 & \text{if } S < 0.975 \\
S - 0.899 & \text{if } 0.975 \leq S < 1.025 \\
0.126 & \text{if } 1.025 \leq S 
\end{cases} \]

Long 0.975-strike put and short 1.025-strike call
c. Buy 1.05-strike put and sell 1.05-strike call

The payoff is \( \max (0, 1.05 - S) - \max (0, S - 1.05) \)

<table>
<thead>
<tr>
<th>( S &lt; 1.05 )</th>
<th>( S \geq 1.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>long 1.05-strike put</td>
<td>( 1.05 - S )</td>
</tr>
<tr>
<td>short 1.05-strike call</td>
<td>( 0 )</td>
</tr>
<tr>
<td>Total</td>
<td>( 1.05 - S )</td>
</tr>
</tbody>
</table>

\[-FV(Put \ Premium) + FV(Call \ Premium)\]
\[= (-0.0665 + 0.0194) 1.06 = -0.05\]

\[P_{\text{Collar}} = (1.05 - S) - 0.05 = 1 - S\]
\[P_{\text{AH}} = P_{\text{BH}} + P_{\text{Collar}} = S - 0.9 + 1 - S = 0.1\]
$P^{AH} = 0.1$

Long 1.05-strike put and short 1.05-strike call
Problem 4.6.

a. Sell one 1.025-strike put and buy two 0.975-strike puts

The payoff is \[ 2 \max(0, 0.975 - S) - \max(0, 1.025 - S) \]

<table>
<thead>
<tr>
<th></th>
<th>( S &lt; 0.975 )</th>
<th>( 0.975 \leq S &lt; 1.025 )</th>
<th>( S \geq 1.025 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>long two 0.975-strike puts</td>
<td>( 2 \times (0.975 - S) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>short 1.025-strike put</td>
<td>( S - 1.025 )</td>
<td>( S - 1.025 )</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>( 0.925 - S )</td>
<td>( S - 1.025 )</td>
<td>0</td>
</tr>
</tbody>
</table>

Initial net premium paid = \( 2 \times (0.0265) - 0.0509 = 0.0021 \)
Future value: \( 0.0021 \times (1.06) = 0.0022 \)

\[
P_{\text{Paylater}} = \begin{cases} 
0.925 - S & \text{if } S < 0.975 \\
S - 1.025 & \text{if } 0.975 \leq S < 1.025 - 0.0022 \\
0 & \text{if } S \geq 1.025 
\end{cases}
\]

\[
P^{\text{AH}} = P^{\text{BH}} + P_{\text{Paylater}}
\]

\[
= S - 0.9 + \begin{cases} 
0.925 - S & \text{if } S < 0.975 \\
S - 1.025 & \text{if } 0.975 \leq S < 1.025 - 0.0022 \\
0 & \text{if } S \geq 1.025 
\end{cases}
\]

\[
= \begin{cases} 
0.0228 & \text{if } S < 0.975 \\
2S - 1.9272 & \text{if } 0.975 \leq S < 1.025 \\
S - 0.9022 & \text{if } 1.025 \leq S 
\end{cases}
\]
\[ P^{AH} = \begin{cases} 
0.0228 & \text{if } S < 0.975 \\
2S - 1.9272 & \text{if } 0.975 \leq S < 1.025 \\
S - 0.9022 & \text{if } 1.025 \leq S 
\end{cases} \]
b. Sell two 1.034-strike puts and buy three 1-strike puts

The payoff is \(3 \max (0, 1 - S) - 2 \max (0, 1.034 - S)\)

<table>
<thead>
<tr>
<th></th>
<th>(S &lt; 1)</th>
<th>(1 \leq S &lt; 1.034)</th>
<th>(S \geq 1.034)</th>
</tr>
</thead>
<tbody>
<tr>
<td>long three 1-strike puts</td>
<td>(3 (1 - S))</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>short two 1.034-strike puts</td>
<td>(2 (S - 1.034))</td>
<td>(2 (S - 1.034))</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>(3 (1 - S) + 2 (S - 1.034) = 0.932 - S)</td>
<td>(2 (S - 1.034))</td>
<td>0</td>
</tr>
</tbody>
</table>

Initial premium paid: \(3 (0.0376) - 2 (0.0563) = 0.0002\)
Future value: \(0.0002 (1.06) = 0.000212\)

\[
\text{Profit: } \begin{cases} 
0.932 - S & \text{if } S < 1 \\
2 (S - 1.034) & \text{if } 1 \leq S < 1.034 \\
0 & \text{if } S \geq 1.034 
\end{cases} - 0.0002
\]

\[
p_{AH} = p_{BH} + p_{T\text{playlater}}
\]

\[
= S - 0.9 + \begin{cases} 
0.932 - S & \text{if } S < 1 \\
2 (S - 1.034) & \text{if } 1 \leq S < 1.034 \\
0 & \text{if } S \geq 1.034 
\end{cases} - 0.0002
\]

\[
= \begin{cases} 
0.0318 & \text{if } S < 1 \\
3.9 - 2.968 & \text{if } 1 \leq S < 1.034 \\
S - 0.9002 & \text{if } S \geq 1.034 
\end{cases}
\]
\[ P^{AH} = \begin{cases} 
0.0318 & \text{if } S < 1 \\
3S - 2.9682 & \text{if } 1 \leq S < 1.034 \\
S - 0.9002 & \text{if } 1.034 \leq S 
\end{cases} \]
CHAPTER 4. INTRODUCTION TO RISK MANAGEMENT

Problem 4.7.

Telco buys copper wires from Wirco. Telco’s purchase price per unit of wire is $5 plus the price of copper. Telco collects $6.2 per unit of wire used.

Telco’s unhedged profit is:

\[ P^{BH} = 6.2 - (5 + S) = 1.2 - S \]

If Telco buys a copper forward, the profit from the forward at \( T = 1 \) is

\[ S - F_{0,T} \]

The profit after hedging is

\[ P^{AH} = P^{BH} + S - F_{0,T} = 1.2 - S + S - F_{0,T} = 1.2 - F_{0,T} \]

\[ F_{0,T} = 1 \quad \rightarrow \quad P^{AH} = 1.2 - 1 = 0.2 \]
$P^{AH} = 1.2 - F_{0,T}$ \quad $P^{AH} = 1.2 - 1 = 0.2$

![Graph showing profit, hedged profit, and unhedged profit against copper price](image-url)
Problem 4.8.

\[ P^{BH} = 1.2 - S \]

Telco buys a \( K \)-strike call. The profit from the long call is:

\[ \max (0, S - K) - FV (Premium) \]

\[ \rightarrow P^{AH} = 1.2 - S + \max (0, S - K) - FV (Premium) \]

\[ = 1.2 - S + \begin{cases} 
0 & \text{if } S < K \\
S - K & \text{if } S \geq K
\end{cases} - FV (Premium) \]

\[ \rightarrow P^{AH} = 1.2 - FV (Premium) - \begin{cases} 
S & \text{if } S < K \\
K & \text{if } S \geq K
\end{cases} \]
a. $K = 0.95 \quad FV (Premium) = 0.0649 (1.06) = 0.069$

$\rightarrow P^{AH} = 1.2 - 0.069 - \begin{cases}
S & \text{if } S < 0.95 \\
0.95 & \text{if } S \geq 0.95
\end{cases} = \begin{cases}
1.131 - S & \text{if } S < 0.95 \\
0.181 & \text{if } S \geq 0.95
\end{cases}$
b. $K = 1 \quad PV\ (Premium) = 0.0376 \times (1.06) = 0.039856$

$P_{AII} = 1.2 - 0.04 - \begin{cases} S \quad \text{if} \quad S < 1 \\ 1 \quad \text{if} \quad S \geq 1 \end{cases} = \begin{cases} 1.16 - S \quad \text{if} \quad S < 1 \\ 0.16 \quad \text{if} \quad 1 \leq S \end{cases}$
c. $K = 1.05$  \hspace{1cm} FV (Premium) = 0.0194 (1.06) = 0.020564

$$
\rightarrow P^{AH} = 1.2 - 0.02 - \begin{cases} 
S & \text{if } S < 1.05 \\
1.05 & \text{if } S \geq 1.05
\end{cases} = \begin{cases} 
1.18 - S & \text{if } S < 1.05 \\
0.13 & \text{if } 1.05 \leq S
\end{cases}
$$
Problem 4.9.

\[ \text{Telco sells a put option with strike price } K. \]

The profit from the written put is:

\[ - \max (0, K - S) + FV (\text{Premium}) \]

\[ \Rightarrow P^{BH} = 1.2 - S \]

\[ \Rightarrow P^{AH} = 1.2 - S - \max (0, K - S) + FV (\text{Premium}) \]

\[ = 1.2 - S - \begin{cases} 
K - S & \text{if } S < K \\
0 & \text{if } S \geq K 
\end{cases} + FV (\text{Premium}) \]

\[ = \begin{cases} 
-K & \text{if } S < K \\
-S & \text{if } K \leq S 
\end{cases} + 1.2 + FV (\text{Premium}) \]

\[ \Rightarrow P^{AH} = \begin{cases} 
K & \text{if } S < K \\
S & \text{if } K \leq S 
\end{cases} + 1.2 + FV (\text{Premium}) \]
a. $K = 0.95$  

$FV (Premium) = 0.0178 \times (1.06) = 0.018868$

$P_{AH} = \begin{cases} 
0.95 & \text{if } S < 0.95 \\
S & \text{if } 0.95 \leq S 
\end{cases}$

+1.2 + 0.019 = \begin{cases} 
0.269 & \text{if } S < 0.95 \\
1.219 - S & \text{if } 0.95 \leq S 
\end{cases}$
b. $K = 1 \quad FV (Premium) = 0.0376 (1.06) = 0.039856$

$P_{AH} = \begin{cases} 1 & \text{if } S < 1 \\ S & \text{if } 1 \leq S \end{cases} + 1.2 + 0.04 = \begin{cases} 0.24 & \text{if } S < 1 \\ 1.24 - S & \text{if } 1 \leq S \end{cases}$
c. \( K = 1.05 \), \( FV(Premium) = 0.0665(1.06) = 0.07049 \)

\[ P^{\text{AH}} = \begin{cases} 
1.05 & \text{if } S < 1.05 \\
S & \text{if } 1.05 \leq S \end{cases} + 1.2 + 0.07 = \begin{cases} 
0.22 & \text{if } S < 1.05 \\
1.27 - S & \text{if } 1.05 \leq S 
\end{cases} \]
Problem 4.10.

Suppose Telco sells a $K_1$-strike put and buys a $K_2$-strike call. The profit of the collar is:

$$P_{\text{Collar}} = \text{Payoff} - FV(\text{Net Initial Premium})$$

$$= \left[ - \max(0, K_1 - S) + \max(0, S - K_2) \right] + FV(\text{Put Premium}) - FV(\text{Call Premium})$$

a. Sell 0.95-strike put and buy 1-strike call

The payoff is $-\max(0, 0.95 - S) + \max(0, S - 1)$

<table>
<thead>
<tr>
<th>$S$</th>
<th>Payoff</th>
<th>$0.95 \leq S &lt; 1$</th>
<th>$S \geq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>short 0.95-strike put</td>
<td>$-(0.95 - S)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>long 1-strike call</td>
<td>0</td>
<td>0</td>
<td>$-(1 - S)$</td>
</tr>
<tr>
<td>Total</td>
<td>$-(0.95 - S)$</td>
<td>0</td>
<td>$-(1 - S)$</td>
</tr>
</tbody>
</table>

$$FV(\text{Put Premium}) - FV(\text{Call Premium}) = (0.0178 - 0.0376) \times 1.06 = -0.02$$

$$P_{\text{Collar}} = \begin{cases} 
S - 0.95 & \text{if } S < 0.95 \\
0 & \text{if } 0.95 \leq S < 1 - 0.02 \\
S - 1 & \text{if } S \geq 1 
\end{cases}$$

$$P_{\text{AH}} = P_{BH} + P_{\text{Collar}}$$

$$= 1.2 - S + \begin{cases} 
S - 0.95 & \text{if } S < 0.95 \\
0 & \text{if } 0.95 \leq S < 1 - 0.02 = 0.23 \text{ if } S < 0.95 \\
S - 1 & \text{if } S \geq 1 \\
1.18 - S & \text{if } 0.95 \leq S < 1 \\
0.18 & \text{if } 1 \leq S 
\end{cases}$$
$P^{AH} = \begin{cases} 
0.23 & \text{if } S < 0.95 \\
1.18 - S & \text{if } 0.95 \leq S < 1 \\
0.18 & \text{if } 1 \leq S 
\end{cases}$

Short 0.95-strike put and long 1-strike call
b. Sell 0.975-strike put and buy 1.025-strike call

The payoff is \( \max(0, 0.975 - S) - \max(0, S - 1.025) \)

<table>
<thead>
<tr>
<th></th>
<th>( S &lt; 0.975 )</th>
<th>( 0.975 \leq S &lt; 1.025 )</th>
<th>( S \geq 1.025 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>short 0.975-strike put</td>
<td>( -(0.975 - S) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>long 1-strike call</td>
<td>0</td>
<td>0</td>
<td>(- (1.025 - S) )</td>
</tr>
<tr>
<td>Total</td>
<td>( -(0.975 - S) )</td>
<td>0</td>
<td>(- (1.025 - S) )</td>
</tr>
</tbody>
</table>

\[
FV(P\text{ut Premium}) - FV(Call\text{ Premium}) = (0.0265 - 0.0274) \times 1.06 = -0.001
\]

\[
p_{Collar} = \begin{cases} 
S - 0.975 & \text{if}\ S < 0.975 \\
0 & \text{if}\ 0.975 \leq S < 1.025 \\
S - 1.025 & \text{if}\ S \geq 1.025 
\end{cases} - 0.001
\]

\[
P_{AH} = P^{BH} + p_{Collar}
\]

\[
= 1.2 - S + \begin{cases} 
S - 0.975 & \text{if}\ S < 0.975 \\
0 & \text{if}\ 0.975 \leq S < 1.025 \\
S - 1.025 & \text{if}\ S \geq 1.025 
\end{cases} - 0.001
\]

\[
= \begin{cases} 
0.224 & \text{if}\ S < 0.975 \\
- S + 1.199 & \text{if}\ 0.975 \leq S < 1.025 \\
0.174 & \text{if}\ 1.025 \leq S 
\end{cases}
\]
\[
P^{AH} = \begin{cases} 
0.224 & \text{if } S < 0.975 \\
-S + 1.199 & \text{if } 0.975 \leq S < 1.025 \\
0.174 & \text{if } 1.025 \leq S 
\end{cases}
\]

Short 0.95-strike put and long 1-strike call
c. Sell 0.95-strike put and buy 0.95-strike call
The payoff is 
\[ -\max(0, 0.95 - S) + \max(0, S - 0.95) \]

<table>
<thead>
<tr>
<th></th>
<th>( S &lt; 0.95 )</th>
<th>( S \geq 0.95 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>short 0.95-strike put</td>
<td>(- (0.95 - S))</td>
<td>0</td>
</tr>
<tr>
<td>long 0.95-strike call</td>
<td>0</td>
<td>( S - 0.95 )</td>
</tr>
<tr>
<td>Total</td>
<td>( S - 0.95 )</td>
<td>( S - 0.95 )</td>
</tr>
</tbody>
</table>

\[
FV(\text{Put Premium}) - FV(\text{Call Premium}) = (0.0178 - 0.0649) \times 1.06 = -0.05
\]
\[
P_{\text{Collar}} = (S - 0.95) - 0.05 = S - 1
\]
\[
P_{\text{AH}} = P_{\text{BH}} + P_{\text{Collar}} = 1.2 - S + S - 1 = 0.2
\]
\( P^{AH} = 0.2 \)

Short 0.95-strike put and long 0.95-strike call
Problem 4.11.

a. sell 0.975-strike call and buy two 1.034-strike calls
The payoff is $2 \max (0, S - 1.034) - \max (0, S - 0.975)$

\[
\begin{array}{c|c|c|c}
 & S < 0.975 & 0.975 \leq S < 1.034 & S \geq 1.034 \\
\hline
\text{short 0.975-strike call} & 0 & -(S - 0.975) & -(S - 0.975) \\
\text{long 1.034-strike calls} & 0 & 0 & 2(S - 1.034) \\
\hline
\text{Total} & 0 & -(S - 0.975) & S - 1.093 \\
\end{array}
\]

\[-(S - 0.975) + 2(S - 1.034) = S - 1.093\]

The payoff $= \begin{cases} 0 & \text{if } S < 0.975 \\ 0.975 - S & \text{if } 0.975 \leq S < 1.034 \\ S - 1.093 & \text{if } S \geq 1.034 \end{cases}$

Initial net premium paid $= 2(0.0243) - 0.05 = -0.0014$
Future value: $-0.0014(1.06) = -0.001484 \approx -0.001$

\[P_{\text{Paylater}} = \begin{cases} 0 & \text{if } S < 0.975 \\ 0.975 - S & \text{if } 0.975 \leq S < 1.034 \\ S - 1.093 & \text{if } S \geq 1.034 \end{cases} + 0.001\]

\[= \begin{cases} 0.001 & \text{if } S < 0.975 \\ 0.976 - S & \text{if } 0.975 \leq S < 1.034 \\ S - 1.092 & \text{if } 1.034 \leq S \end{cases}\]

\[P_{AH} = P_{BH} + P_{\text{Paylater}} = 1.2 - S + \begin{cases} 0.001 & \text{if } S < 0.975 \\ 0.976 - S & \text{if } 0.975 \leq S < 1.034 \\ S - 1.092 & \text{if } 1.034 \leq S \end{cases}\]

\[= \begin{cases} 1.201 - S & \text{if } S < 0.975 \\ 2.176 - 2S & \text{if } 0.975 \leq S < 1.034 \\ 0.108 & \text{if } 1.034 \leq S \end{cases}\]
\[ P^{AH} = \begin{cases} 
1.201 - S & \text{if } S < 0.975 \\
2.176 - 2S & \text{if } 0.975 \leq S < 1.034 \\
0.108 & \text{if } 1.034 \leq S 
\end{cases} \]
b. sell two 1-strike calls and buy three 1.034-strike calls
The payoff is \(-2 \max(0, S - 1) + 3 \max(0, S - 1.034)\)

<table>
<thead>
<tr>
<th></th>
<th>(S &lt; 1)</th>
<th>(1 \leq S &lt; 1.034)</th>
<th>(S \geq 1.034)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell two 1-strike calls</td>
<td>0</td>
<td>(-2(S - 1))</td>
<td>(-2(S - 1))</td>
</tr>
<tr>
<td>buy three 1.034-strike calls</td>
<td>0</td>
<td>0</td>
<td>3 ((S - 1.034))</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>(-2(S - 1))</td>
<td>(S - 1.102)</td>
</tr>
</tbody>
</table>

\(-2(S - 1) + 3(S - 1.034) = S - 1.102\)

The payoff = \(
\begin{cases} 
0 & \text{if } S < 1 \\
-2(S - 1) & \text{if } 1 \leq S < 1.034 \\
S - 1.102 & \text{if } S \geq 1.034
\end{cases}
\)

Initial net premium paid = \(3(0.0243) - 2(0.0376) = -0.0023\)
Future value: \(-0.0023 (1.06) = -0.002438 \approx -0.0024\)

\(p_{\text{Paylater}} = \begin{cases} 
0 & \text{if } S < 1 \\
-2(S - 1) & \text{if } 1 \leq S < 1.034 \\
S - 1.102 & \text{if } S \geq 1.034
\end{cases} + 0.0024\)

\(p^{AH} = p^{BH} + p^{Paylater}\)

\(= 1.2 - S + \begin{cases} 
0 & \text{if } S < 1 \\
-2(S - 1) & \text{if } 1 \leq S < 1.034 \\
S - 1.102 & \text{if } S \geq 1.034
\end{cases} + 0.0024\)

\(= \begin{cases} 
1.2024 - S & \text{if } S < 1 \\
3.2024 - 3S & \text{if } 1 \leq S < 1.034 \\
0.1004 & \text{if } 1.034 \leq S
\end{cases}\)
\[ P^{AH} = \begin{cases} 
1.2024 - S & \text{if } S < 1 \\
3.2024 - 3S & \text{if } 1 \leq S < 1.034 \\
0.1004 & \text{if } 1.034 \leq S 
\end{cases} \]
Problem 4.12.
Wirco’s total profit per unit of wire produced:
• Revenue. $S + 5$, where $S$ is the price of copper
• Copper cost is $S$
• Fixed cost is 3
• Variable cost 1.5
Profit before hedging: $P^{BH} = S + 5 - (S + 3 + 1.5) = 0.5$
The profit is fixed regardless of copper price.
If Wirco buys a copper forward, this will introduce the copper price risk (i.e. Wirco’s profit will now depend on the copper price).
Wirco buys a copper forward. The profit from the forward is $S F_{0,T}$
The profit after hedging is $P^{AH} = P^{BH} + S - F_{0,T} = 0.5 + S - F_{0,T}$
If $F_{0,T} = 1$ → $P^{AH} = 0.5 + S - 1 = S - 0.5$
Now you the profit depends on $S$. If $S$ goes down, the profit goes down.

Problem 4.13.
The unhedged profit is $P^{BH} = 0.5$. This doesn’t depend on the copper price at $T = 1$. However, if Wirco uses any derivatives (call, put, forward, etc.), this will make the hedged profit as a function of the copper price at $T = 1$. Using any derivatives will make the hedged profit fluctuate with the copper price, increasing the variability of the profit.

The question "Did the firm earn $10 in profit (relative to accounting break-even) or lose $30 in profit (relative to the profit that could be obtained by hedging)?" portrays derivatives a way to make profit. However, most companies use derivatives to manage their risks, not to seek additional profit. If they have idle money, they would rather invest in their core business than buy call or put options to make money on stocks. This is all you need to know about this question.

Problem 4.15.

<table>
<thead>
<tr>
<th></th>
<th>Price= $9</th>
<th>Price= $11.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-tax operating income</td>
<td>$-1$</td>
<td>$1.2$</td>
</tr>
<tr>
<td>Tax at 40%</td>
<td>$-1(0.4) = -0.4$</td>
<td>$1.2(0.4) = 0.48$</td>
</tr>
<tr>
<td>After tax income</td>
<td>$-1 - (-0.4) = -0.6$</td>
<td>$1.2 - 0.48 = 0.72$</td>
</tr>
</tbody>
</table>

Because losses are fully tax deductible, we pay $-0.4$ tax (i.e. IRS will send us a check of 0.4)
Expected profit is: $0.5(-0.6 + 0.72) = 0.06$
CHAPTER 4. INTRODUCTION TO RISK MANAGEMENT

Problem 4.16.

a. Expected pre-tax profit:
Firm A: \( 0.5 (1000 - 600) = 200 \)
Firm B: \( 0.5 (300 + 100) = 200 \)

b. Expected after-tax profit:

<table>
<thead>
<tr>
<th>Firm A</th>
<th>Good State</th>
<th>Bad State</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-tax income</td>
<td>1000</td>
<td>600</td>
</tr>
<tr>
<td>Tax at 40%</td>
<td>1000 (0.4) = 400</td>
<td>600 (0.4) = 240</td>
</tr>
<tr>
<td>After tax income</td>
<td>1000 - 400 = 600</td>
<td>600 - (-240) = 840</td>
</tr>
</tbody>
</table>

Expected after-tax profit: \( 0.5 (600 - 360) = 120 \)

<table>
<thead>
<tr>
<th>Firm B</th>
<th>Good State</th>
<th>Bad State</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-tax income</td>
<td>300</td>
<td>100</td>
</tr>
<tr>
<td>Tax at 40%</td>
<td>300 (0.4) = 120</td>
<td>100 (0.4) = 40</td>
</tr>
<tr>
<td>After tax income</td>
<td>300 - 120 = 180</td>
<td>100 - 40 = 60</td>
</tr>
</tbody>
</table>

Expected after-tax profit: \( 0.5 (180 + 60) = 120 \)

Problem 4.17.

a. Expected pre-tax profit:
Firm A: \( 0.5 (1000 - 600) = 200 \)
Firm B: \( 0.5 (300 + 100) = 200 \)

b. Expected after-tax profit:

<table>
<thead>
<tr>
<th>Firm A</th>
<th>Good State</th>
<th>Bad State</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-tax income</td>
<td>1000</td>
<td>600</td>
</tr>
<tr>
<td>Tax</td>
<td>1000 (0.4) = 400</td>
<td>600 (0.4) = 0</td>
</tr>
<tr>
<td>After tax income</td>
<td>1000 - 400 = 600</td>
<td>600</td>
</tr>
</tbody>
</table>

Expected after-tax profit: \( 0.5 (600 - 600) = 0 \)

<table>
<thead>
<tr>
<th>Firm B</th>
<th>Good State</th>
<th>Bad State</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-tax income</td>
<td>300</td>
<td>100</td>
</tr>
<tr>
<td>Tax at 40%</td>
<td>300 (0.4) = 120</td>
<td>100 (0.4) = 40</td>
</tr>
<tr>
<td>After tax income</td>
<td>300 - 120 = 180</td>
<td>100 - 40 = 60</td>
</tr>
</tbody>
</table>

Expected after-tax profit: \( 0.5 (180 + 60) = 120 \)

c. This question is vague. I’m not sure to whom A or B might pay. This is what I guess the author wants us to answer:
CHAPTER 4. INTRODUCTION TO RISK MANAGEMENT

The expected cash flow Company A receives depends on the tax law.

\[ E(Profit^A) = \begin{cases} 
120 & \text{if loss is tax deductible} \\
0 & \text{if loss is not tax deductible} 
\end{cases} \]

Suppose A is unsure about the IRS’s tax policy (i.e. not sure whether the loss is deductible or not next year). Then the present value of the difference of the tax law is

\[ \frac{120}{1.1} = 109.09 \]. So the effect of the tax law has a present value 109.09.

Company B doesn’t have any loss. Its after tax profit doesn’t depend on whether a loss is tax deductible. So the effect of the tax law has a present value of zero.

Problem 4.18.

We are given: \( r = \delta = 4.879\% \quad T = 1 \)

\[ F_{0,T} = S_0 e^{-(r-\delta)T} = 420 \quad \rightarrow S_0 = 420 \]

Call strike price \( K_C = 440 \); put strike price \( K_P = 400 \)

First, find the call and put premiums. Install the CD contained in the textbook Derivatives Markets in your computer. Open the spreadsheet titled "optbasic2." Enter:

<table>
<thead>
<tr>
<th>Inputs</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Price</td>
<td>420</td>
</tr>
<tr>
<td>Exercise Price</td>
<td>440</td>
</tr>
<tr>
<td>Volatility</td>
<td>5.500%</td>
</tr>
<tr>
<td>Risk-free interest rate</td>
<td>4.879%</td>
</tr>
<tr>
<td>Time to Expiration (years)</td>
<td>1</td>
</tr>
<tr>
<td>Dividend Yield</td>
<td>4.879%</td>
</tr>
</tbody>
</table>

You should get the call premium: \( C = 2.4944 \)

<table>
<thead>
<tr>
<th>Inputs</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Price</td>
<td>420</td>
</tr>
<tr>
<td>Exercise Price</td>
<td>400</td>
</tr>
<tr>
<td>Volatility</td>
<td>5.500%</td>
</tr>
<tr>
<td>Risk-free interest rate</td>
<td>4.879%</td>
</tr>
<tr>
<td>Time to Expiration (years)</td>
<td>1</td>
</tr>
<tr>
<td>Dividend Yield</td>
<td>4.879%</td>
</tr>
</tbody>
</table>

You should get the put premium: \( P = 2.2072 \)

a. Buy 440-strike call and sell a 440-put

Let \( S \) represent the gold price at the option expiration date \( T = 1 \).
The payoff is:

<table>
<thead>
<tr>
<th></th>
<th>$S &lt; 400$</th>
<th>$400 \leq S &lt; 440$</th>
<th>$S \geq 440$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 440-strike call</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Sell 400-strike put</td>
<td>$- (400 - S)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>$S - 400$</td>
<td>0</td>
<td>$S - 440$</td>
</tr>
</tbody>
</table>

Payoff = \[
\begin{cases} 
S - 400 & \text{if } S < 400 \\
0 & \text{if } 400 \leq S < 440 \\
S - 440 & \text{if } S \geq 440 
\end{cases}
\]

The initial cost of the collar is:

*Premium* \( = 2.4944 - 2.2072 = 0.2872 \)

*FV* (*Premium*) \( = 0.2872 e^{0.04879(1)} = 0.30 \)

So the profit from the collar is:

\[C_{\text{Collar}} = \begin{cases} 
S - 400 & \text{if } S < 400 \\
0 & \text{if } 400 \leq S < 440 \\
S - 440 & \text{if } S \geq 440 
\end{cases} - 0.30\]

The profit before hedging:

- Auric sells each widget for $800
- It has fixed cost: $340
- Input (gold) cost: $S$

Profit before hedging: \(PBH = 800 - (340 + S) = 460 - S\)
So Auric’s profit after hedging:
\[ P^A_{H} = P^B_{H} + P^{Collar} \]

\[ = 460 - S + \begin{cases} S - 400 & \text{if } S < 400 \\ 0 & \text{if } 400 \leq S < 440 \\ S - 440 & \text{if } S \geq 440 \end{cases} \]

\[ = 59.7 - S & \text{if } S < 400 \\ 459.7 - S & \text{if } 400 \leq S < 440 \\ 19.7 & \text{if } 440 \leq S \]
b. We need to find
\[
\begin{align*}
C &= P \\
K_C - K_P &= 30
\end{align*}
\]
Using the spreadsheet "optbasic2," after trial-and-error, we find that:
\[
K_C = 435.52 \quad K_P = 405.52 \\
C = 3.4264 \quad P = 3.4234 \quad \rightarrow C \approx P
\]
Let \( K_C \) and \( K_P \) represent the strike price for the call and the put.

Collar width 30 \( \rightarrow K_C - K_P = 30 \)
Let \( C \) and \( P \) represent the call and put premium calculated by the Black-Scholes formula.

When we buy the call, we pay \( C' = C + 0.25 \)
When we sell the put, we get \( P' = P - 0.25 \)
Zero collar \( \rightarrow C' = P' \) \( \ \ \ C + 0.25 = P - 0.25 \) \( \quad C = P - 0.5 \)
So we need to find \( C \) and \( P \) such
\[
\begin{align*}
C &= P - 0.5 \\
K_C - K_P &= 30
\end{align*}
\]
This is all the concepts you need to know about this problem.

Using the spreadsheet "optbasic2," after trial-and-error, we find that:
\[
K_C = 436.53 \quad K_P = 406.53 \\
C = 3.1938 \quad P = 3.6938
\]

**Problem 4.19.**

a. Sell 440-strike call and buy two \( K \)-strike calls such the net premium is zero.

The 440-strike call premium is: \( C^{440} = 2.4944 \)
We need to find \( K \) such that \( C^{440} - 2C^K = 0 \)
\[
\rightarrow 2.4944 - 2C^K = 0 \quad C^K = 2.4944/2 = 1.2472
\]

We know that \( K > 440 \) (otherwise \( C^K \geq C^{440} \))
Using the spreadsheet "optbasic2," after trial-and-error, we find that:
\[
K = 448.93 \quad C^K = 1.2469 \approx 1.2472
\]

b. Profit before hedging: \( P^{BH} = 800 - (340 + S) = 460 - S \)
Zero cost collar \( \rightarrow \) Profit = Payoff
The payoff is:

<table>
<thead>
<tr>
<th></th>
<th>( S &lt; 440 )</th>
<th>( 440 \leq S &lt; 448.93 )</th>
<th>( S \geq 448.93 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell 440-strike call</td>
<td>0</td>
<td>(- (S - 440))</td>
<td>(- (S - 440))</td>
</tr>
<tr>
<td>buy two 448.93-strike calls</td>
<td>0</td>
<td>0</td>
<td>2 ( (S - 448.93))</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>440 - S</td>
<td>( S - 457.86 )</td>
</tr>
</tbody>
</table>
CHAPTER 4. INTRODUCTION TO RISK MANAGEMENT

\[- (S - 440) + 2(S - 448.93) = S - 457.86 \]

\[P^{\text{Collar}} = \begin{cases} 
0 & \text{if } S < 440 \\
440 - S & \text{if } 440 \leq S < 448.93 \\
S - 457.86 & \text{if } S \geq 448.93
\end{cases} \]

So Auric’s profit after hedging:

\[P^\text{AH} = P^\text{BH} + P^{\text{Collar}} \]

\[= 460 - S + \begin{cases} 
0 & \text{if } S < 440 \\
440 - S & \text{if } 440 \leq S < 448.93 \\
S - 457.86 & \text{if } S \geq 448.93
\end{cases} = \begin{cases} 
460 - S & \text{if } S < 440 \\
900 - 2S & \text{if } 440 \leq S < 448.93 \\
2.14 & \text{if } 448.93 \leq S
\end{cases} \]
\[
P_{AH} = \begin{cases} 
460 - S & \text{if } S < 440 \\
900 - 2S & \text{if } 440 \leq S < 448.93 \\
2.14 & \text{if } 448.93 \leq S 
\end{cases} \quad P_{BH} = 460 - S
\]
CHAPTER 4. INTRODUCTION TO RISK MANAGEMENT

Problem 4.20.
Ignore. Not on the FM syllabus.

Problem 4.21.
Ignore. Not on the FM syllabus.

Problem 4.22.
Ignore. Not on the FM syllabus.

Problem 4.23.
Ignore. Not on the FM syllabus.

Problem 4.24.
Ignore. Not on the FM syllabus.

Problem 4.25.
Ignore. Not on the FM syllabus.
Chapter 5

Financial forwards and futures

Problem 5.1.

<table>
<thead>
<tr>
<th>Description</th>
<th>Get paid at time</th>
<th>Deliver asset at time</th>
<th>payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell asset outright</td>
<td>0</td>
<td>0</td>
<td>$S_0$ at $t = 0$</td>
</tr>
<tr>
<td>Sell asset through loan</td>
<td>$T$</td>
<td>0</td>
<td>$S_0e^{rt}$ at $T$</td>
</tr>
<tr>
<td>Short prepaid forward</td>
<td>0</td>
<td>$T$</td>
<td>$S_0$ at $t = 0$</td>
</tr>
<tr>
<td>Short forward</td>
<td>$T$</td>
<td>$T$</td>
<td>$S_0e^{rt}$ at $T$</td>
</tr>
</tbody>
</table>

Problem 5.2.

a. $F_{0,T}^p = S_0e^{-\delta T} - PV(Div)$
   $= 50e^{-0.08(1)} - [e^{-0.06(3/12)} + e^{-0.06(6/12)} + e^{-0.06(9/12)} + e^{-0.06(12/12)}]$  
   $= 46.1467$

b. $F_{0,T} = F_{0,T}^p e^{rT} = 46.1467e^{0.06(1)} = 49.0003$

Problem 5.3.

a. $F_{0,T}^p = S_0e^{-\delta T} = 50e^{-0.08(1)} = 46.1558$

b. $F_{0,T} = F_{0,T}^p e^{rT} = 46.1558e^{0.06(1)} = 49.0099$

Problem 5.4.
a. \( F_{0,T} = S_0 e^{(r-\delta)T} = 35e^{(0.05-0)0.5} = 35.8860 \)

b. \( \frac{1}{T} \ln \frac{F_{0,T}}{S_0} = \frac{1}{0.5} \ln \frac{35.5}{35} = 0.02837 \)

c. \( F_{0,T} = S_0 e^{(r-\delta)T} \quad 35.5 = 35e^{(0.05-\delta)0.5} \quad \rightarrow \delta = 2.163\% \)
Problem 5.5.

a. \( F_{0,T} = S_0 e^{(r - \delta)T} = 1100e^{(0.05 - 0.09)/12} = 1142.0332 \)

b. As a buyer in a forward contract, we face the risk that the index may be worth zero at \( T \) (i.e. \( S_T = 0 \)), yet we still have to pay \( F_{0,T} \) to buy it. This is how to hedge our risk:

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>long a forward (i.e. be a buyer in forward)</td>
<td>0</td>
<td>( S_T - F_{0,T} )</td>
</tr>
<tr>
<td>short sell an index</td>
<td>( S_0 )</td>
<td>( -S_T )</td>
</tr>
<tr>
<td>deposit ( S_0 ) in savings account</td>
<td>( -S_0 )</td>
<td>( S_0 e^{r T} )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>( S_0 e^{r T} - F_{0,T} )</td>
</tr>
</tbody>
</table>

For this problem:

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy a forward</td>
<td>0</td>
<td>( S_T - 1142.03 )</td>
</tr>
<tr>
<td>short sell an index</td>
<td>( 1100 )</td>
<td>( -S_T )</td>
</tr>
<tr>
<td>deposit ( S_0 ) in savings account</td>
<td>( -1100 )</td>
<td>( 1100e^{(0.05/12)} = 1142.03 )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

After hedging, our profit is zero.

c. As a seller in the forward contract, we face the risk that \( S_T = \infty \). If \( S_T = \infty \) and we don’t already have an index on hand for delivery at \( T \), we have to pay \( S_T = \infty \) and buy an index from the open market. We’ll be bankrupt. This is how to hedge:

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell a forward (i.e. be a seller in forward)</td>
<td>0</td>
<td>( F_{0,T} - S_T )</td>
</tr>
<tr>
<td>buy an index</td>
<td>( -S_0 )</td>
<td>( S_T )</td>
</tr>
<tr>
<td>borrow ( S_0 )</td>
<td>( S_0 )</td>
<td>( -S_0 e^{r T} )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>( F_{0,T} - S_0 e^{r T} )</td>
</tr>
</tbody>
</table>

For this problem:

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell a forward (i.e. be a seller in forward)</td>
<td>0</td>
<td>( 1142.03 - S_T )</td>
</tr>
<tr>
<td>buy an index</td>
<td>( -1100 )</td>
<td>( S_T )</td>
</tr>
<tr>
<td>borrow ( S_0 )</td>
<td>( 1100 )</td>
<td>( -1100e^{(0.05/12)} = -1142.03 )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Problem 5.6.

a. \( F_{0,T} = S_0 e^{(r-\delta)T} = 1100 e^{(0.05-0.015)9/12} = 1129.26 \)

b. Transactions

<table>
<thead>
<tr>
<th>t = 0</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>long a forward (i.e. be a buyer in forward)</td>
<td>0</td>
</tr>
<tr>
<td>short sell ( e^{-\delta T} ) index</td>
<td>( S_0 e^{-\delta T} )</td>
</tr>
<tr>
<td>deposit ( S_0 e^{-\delta T} ) in savings account</td>
<td>( -S_0 e^{-\delta T} )</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
</tr>
</tbody>
</table>

For this problem:

<table>
<thead>
<tr>
<th>t = 0</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy a forward</td>
<td>0</td>
</tr>
<tr>
<td>short sell ( e^{-\delta T} ) index</td>
<td>( 1100 e^{(-0.015)9/12} = 1087.69 )</td>
</tr>
<tr>
<td>deposit ( S_0 e^{-\delta T} ) in savings</td>
<td>( -1087.69 )</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
</tr>
</tbody>
</table>

c. Transactions

<table>
<thead>
<tr>
<th>t = 0</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>short a forward (i.e. be a seller in forward)</td>
<td>0</td>
</tr>
<tr>
<td>buy ( e^{-\delta T} ) index</td>
<td>( -S_0 e^{-\delta T} )</td>
</tr>
<tr>
<td>borrow ( S_0 e^{-\delta T} )</td>
<td>( S_0 e^{-\delta T} )</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
</tr>
</tbody>
</table>

For this problem:

<table>
<thead>
<tr>
<th>t = 0</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>short a forward</td>
<td>0</td>
</tr>
<tr>
<td>buy ( e^{-\delta T} ) index</td>
<td>( -1100 e^{(-0.015)9/12} = -1087.69 )</td>
</tr>
<tr>
<td>borrow ( S_0 e^{-\delta T} )</td>
<td>( 1087.69 )</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
</tr>
</tbody>
</table>
CHAPTER 5. FINANCIAL FORWARDS AND FUTURES

Problem 5.7.

\[ F_{0,T} = S_0 e^{rT} = 1100e^{(0.05)0.5} = 1127.85 \]

a. The 6-month forward price in the market is 1135, which is greater than the fair forward price 1127.85. So we have two identical forwards (one in the open market and one that can be synthetically built) selling at different prices. To arbitrage, always buy low and sell high.

<table>
<thead>
<tr>
<th>Transactions</th>
<th>t = 0</th>
<th>T = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell expensive forward from market</td>
<td>0</td>
<td>1135 - ST</td>
</tr>
<tr>
<td>build cheap forward</td>
<td></td>
<td></td>
</tr>
<tr>
<td>buy an index</td>
<td>-1100</td>
<td>ST</td>
</tr>
<tr>
<td>borrow 1100</td>
<td>1100</td>
<td>-1100e^{(0.05)0.5} = -1127.85</td>
</tr>
<tr>
<td>Total profit</td>
<td>0</td>
<td>1135 - 1127.85 = 7.15</td>
</tr>
</tbody>
</table>

We didn’t pay anything at \( t = 0 \), but we have a profit 7.15 at \( T = 0.5 \).

b. The 6-month forward price in the market is 1115, which is cheaper than the fair forward price 1127.85. So we have two identical forwards (one in the open market and one that can be synthetically built) selling at different prices. To arbitrage, always buy low and sell high.

<table>
<thead>
<tr>
<th>Transactions</th>
<th>t = 0</th>
<th>T = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy cheap forward from market</td>
<td>0</td>
<td>ST - 1115</td>
</tr>
<tr>
<td>build expensive forward for sale</td>
<td></td>
<td></td>
</tr>
<tr>
<td>short sell an index</td>
<td>1100</td>
<td>-ST</td>
</tr>
<tr>
<td>lend 1100</td>
<td>-1100</td>
<td>1100e^{(0.05)0.5} = 1127.85</td>
</tr>
<tr>
<td>Total profit</td>
<td>0</td>
<td>1127.85 - 1115 = 12.85</td>
</tr>
</tbody>
</table>

We didn’t pay anything at \( t = 0 \), but we have a profit 12.85 at \( T = 0.5 \).
Problem 5.8.

\[ F_{0,T} = S_0 e^{(r-\delta)T} = 1100e^{(0.05-0.02)0.5} = 1116.62 \]

a. The 6-month forward price in the market is 1120, which is greater than the fair forward price 1116.62. So we have two identical forwards (one in the open market and one that can be synthetically built) selling at different prices. To arbitrage, always buy low and sell high.

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell expensive forward</td>
<td>0</td>
<td>( 1120 - S_T )</td>
</tr>
<tr>
<td>build cheap forward</td>
<td></td>
<td></td>
</tr>
<tr>
<td>buy ( e^{-\delta T} ) index</td>
<td>(-1100e^{(-0.02)0.5} = -1089.05)</td>
<td>( S_T )</td>
</tr>
<tr>
<td>borrow ( S_0e^{-\delta T} )</td>
<td>( 1100e^{(-0.02)0.5} = 1089.05)</td>
<td>(-1089.05e^{(0.05)0.5} = -1116.62)</td>
</tr>
<tr>
<td>Total profit</td>
<td>0</td>
<td>( 1120 - 1116.62 = 3.38 )</td>
</tr>
</tbody>
</table>

We didn’t pay anything at \( t = 0 \), but we have a profit 3.38 at \( T = 0.5 \).

b.

The 6-month forward price in the market is 1110, which is cheaper than the fair forward price 1116.62. So we have two identical forwards (one in the open market and one that can be synthetically built) selling at different prices. To arbitrage, always buy low and sell high.

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy cheap forward from market</td>
<td>0</td>
<td>( S_T - 1110 )</td>
</tr>
<tr>
<td>build expensive forward for sale</td>
<td></td>
<td></td>
</tr>
<tr>
<td>short sell ( e^{-\delta T} ) index</td>
<td>( 1100e^{(-0.02)0.5} = 1089.05 )</td>
<td>(-S_T )</td>
</tr>
<tr>
<td>lend ( S_0e^{-\delta T} )</td>
<td>(-1089.05 )</td>
<td>( 1089.05e^{(0.05)0.5} = 1116.62 )</td>
</tr>
<tr>
<td>Total profit</td>
<td>0</td>
<td>( 1116.62 - 1110 = 6.62 )</td>
</tr>
</tbody>
</table>

We didn’t pay anything at \( t = 0 \), but we have a profit 6.62 at \( T = 0.5 \).

Problem 5.9.

This is a poorly designed problem, more amusing than useful for passing the exam. Don’t waste any time on this. Skip.
CHAPTER 5. FINANCIAL FORWARDS AND FUTURES

Problem 5.10.

a. \( F_{0,T} = S_0e^{(r-\delta)T} \rightarrow 1129.257 = 1100e^{(0.05-\delta)0.75} \)

\[ e^{(0.05-\delta)0.75} = \frac{1129.257}{1100} \]

\[ 0.05 - \delta = \frac{1}{0.75} \ln \frac{1129.257}{1100} = 3.5\% \]

\[ \delta = 1.5\% \]

b. If you believe that the true dividend yield is 0.5\%, then the fair forward price is:

\[ F_{0,T} = 1100e^{(0.05-0.005)0.75} = 1137.759 \]

The market forward price is 1129.257, which is cheaper than the fair price. To arbitrage, buy low and sell high.

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy cheap forward from market</td>
<td>0</td>
<td>( S_T - 1129.257 )</td>
</tr>
<tr>
<td>build expensive forward for sale</td>
<td></td>
<td></td>
</tr>
<tr>
<td>short sell ( e^{-\delta T} ) index</td>
<td>( 1100e^{(-0.005)(0.75)} = 1095.883 )</td>
<td>( -S_T )</td>
</tr>
<tr>
<td>lend ( S_0e^{-\delta T} )</td>
<td>-1095.883</td>
<td>1095.883e^{(0.05)(0.75)} = 1137.759</td>
</tr>
<tr>
<td>Total profit</td>
<td>0</td>
<td>1137.759 - 1129.257 = 8.502</td>
</tr>
</tbody>
</table>

c. If you believe that the true dividend yield is 3\%, then the fair forward price is:

\[ F_{0,T} = 1100e^{(0.05-0.03)0.75} = 1116.624 \]

The market forward price is 1129.257, which is higher than the fair price. To arbitrage, buy low and sell high.

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sell expensive forward</td>
<td>0</td>
<td>1129.257 - ( S_T )</td>
</tr>
<tr>
<td>build cheap forward</td>
<td></td>
<td></td>
</tr>
<tr>
<td>buy ( e^{-\delta T} ) index</td>
<td>-1100e^{-0.03(0.75)} = -1075.526</td>
<td>( S_T )</td>
</tr>
<tr>
<td>borrow ( S_0e^{-\delta T} )</td>
<td>1075.526</td>
<td>-1075.526e^{(0.05)(0.75)} = -1116.624</td>
</tr>
<tr>
<td>Total profit</td>
<td>0</td>
<td>1129.257 - 1116.624 = 12.633</td>
</tr>
</tbody>
</table>
Problem 5.11.

a. One contract is worth 1200 points. Each point is worth $250. The notional value of 4 S&P futures is:
\[ 4 \times 1200 \times 250 = \$1,200,000 \]

b. The value of the initial margin: \$1,200,000 \times 0.1 = \$120,000

Problem 5.12.

a. Notional value of 10 S&P futures:
\[ 10 \times 950 \times 250 = \$2,375,000 \]
The initial margin: \$2375 000 \times 0.1 = \$237,500

b. The maintenance margin: 237,500 (0.8) = 190,000
At the end of Week 1, our initial margin grows to:
\[ 237500e^{0.06(1/52)} = 237774.20 \]

Suppose the futures price at the end of Week 1 is \( X \). The futures price at \( t = 0 \) is 950. After marking-to-market, we gain \( (X - 950) \) points per contract. The notional gain of the 4 futures after marking-to-market is:
\[ (X - 950)(10)(250) \]

After marking-to-market, our margin account balance is
\[ 237774.20 + (X - 950)(10)(250) = 2500X - 2137225.8 \]

We get a margin call if
\[ 2500X - 2137225.8 < 190000 \rightarrow X < 930.89032 \]
For example, \( X = 930.89 \) will lead to a margin call.

Problem 5.13.

a. 

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy forward</td>
<td>0</td>
<td>( S_T - F_{0,t} = S_T - S_0 e^{rT} )</td>
</tr>
<tr>
<td>lend ( S_0 )</td>
<td>(-S_0)</td>
<td>( S_0 e^{rT} )</td>
</tr>
<tr>
<td>Total</td>
<td>(-S_0)</td>
<td>( S_T )</td>
</tr>
</tbody>
</table>

b. 

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy forward</td>
<td>0</td>
<td>( S_T - F_{0,t} = S_T - S_0 e^{rT} + FV(Div) )</td>
</tr>
<tr>
<td>lend ( S_0 - PV(Div) )</td>
<td>(-S_0 + PV(Div))</td>
<td>( S_0 e^{rT} - FV(Div) )</td>
</tr>
<tr>
<td>Total</td>
<td>(-S_0 + PV(Div))</td>
<td>( S_T )</td>
</tr>
</tbody>
</table>

c. 

<table>
<thead>
<tr>
<th>Transactions</th>
<th>( t = 0 )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy forward</td>
<td>0</td>
<td>( S_T - F_{0,t} = S_T - S_0 e^{(r-s)T} )</td>
</tr>
<tr>
<td>lend ( S_0 e^{-sT} )</td>
<td>(-S_0 e^{-sT})</td>
<td>( S_0 e^{(r-s)T} )</td>
</tr>
<tr>
<td>Total</td>
<td>(-S_0 e^{-sT})</td>
<td>( S_T )</td>
</tr>
</tbody>
</table>
CHAPTER 5. FINANCIAL FORWARDS AND FUTURES

Problem 5.14.

If the forward price $F_{0,T}$ is too low, this is how to make some free money.

1. Buy low. At $t = 0$, enter a forward to buy one stock. Incur transaction cost $k$.

2. Sell high. At $t = 0$, sell one stock short and receive $S_0^b - k$.

3. The net cash flow after 1 and 2 is $S_0^b - 2k$. Lend $S_0^b - 2k$ and receive $(S_0^b - 2k) e^{rT}$ at $T$.

4. At $T$, pay $F_{0,T}$ and receive one stock. Return the stock to the broker.

The net cash flow after 1 through 4 is zero. Your profit at $T$ is:

$$S_0^b - 2k e^{rT}.$$

Arbitrage is possible if:

$$S_0^b - 2k e^{rT} > F_{0,T} \rightarrow F_{0,T} < (S_0^b - 2k) e^{rT}.$$

To avoid arbitrage, we need to have:

$$F_{0,T} = S_0^b - 2k e^{rT}.$$

Problem 5.15.

a. $k = 0$ and there’s no bid-ask spread (so $S_0^b = S_0^b = 800$).

So the non-arbitrage bound is:

$$800 e^{(0.05)} = F^- \leq F_{0,T} \leq F^+ = 800 e^{(0.05)};$$

$$\rightarrow 841.02 = F^- \leq F_{0,T} \leq F^+ = 845.23$$

Hence arbitrage is not profitable if $841.02 \leq F_{0,T} \leq 845.23$

b. $k = 1$ and there’s no bid-ask spread (so $S_0^b = S_0^b = 800$).

Please note that you can’t blindly copy the formula:

$$(S_0^b - 2k) e^{rT} = F^- \leq F_{0,T} \leq F^+ = (S_0^b + 2k) e^{rT}.$$

This is because the problem states that $k$ is incurred for longing or shorting a forward and that $k$ is not incurred for buying or selling an index. Given $k$ is incurred only once, the non-arbitrage bound is:

$$S_0^b - k e^{rT} = F^- \leq F_{0,T} \leq F^+ = (S_0^b + k) e^{rT}.$$

$$800 (1 - e^{(0.05)}) = F^- \leq F_{0,T} \leq F^+ = (800 + 1) e^{(0.05)};$$

$$\rightarrow 839.97 = F^- \leq F_{0,T} \leq F^+ = 846.29$$

c. Once again, you can’t blindly use the formula

$$(S_0^b - 2k) e^{rT} = F^- \leq F_{0,T} \leq F^+ = (S_0^b + 2k) e^{rT}.$$

The problem states that $k_1 = 1$ is incurred for longing or shorting a forward and $k_2 = 2.4$ is incurred for buying or selling an index. The non-arbitrage formula becomes:
\[ (S_0^k - k_1 - k_2) e^{rT} = F^- \leq F_{0,T} \leq F^+ = (S_0^k + k_1 + k_2) e^{rT} \]
\[ (800 - 1 - 2.4) e^{(0.055)1} = F^- \leq F_{0,T} \leq F^+ = (800 + 1 + 2.4) e^{(0.055)1} \]
\[ \rightarrow 837.44 = F^- \leq F_{0,T} \leq F^+ = 848.82 \]

d. Once again, you can’t blindly use the formula
\[ (S_0^k - 2k) e^{rT} = F^- \leq F_{0,T} \leq F^+ = (S_0^k + 2k) e^{rT} \]

The problem states that \( k_1 = 1 \) is incurred for longing or shorting a forward; \( k_2 = 2.4 \) is incurred twice, for buying or selling an index, once at \( t = 0 \) and the other at \( T \). The non-arbitrage formula becomes:
\[ (S_0^k - k_1 - k_2) e^{rT} - k_2 = F^- \leq F_{0,T} \leq F^+ = (S_0^k + k_1 + k_2) e^{rT} + k_2 \]
\[ (800 - 1 - 2.4) e^{(0.055)1} - 2.4 = F^- \leq F_{0,T} \leq F^+ = (800 + 1 + 2.4) e^{(0.055)1} + 2.4 \]
\[ \rightarrow 837.44 - 2.4 = F^- \leq F_{0,T} \leq F^+ = 848.82 + 2.4 \]
\[ \rightarrow 835.04 = F^- \leq F_{0,T} \leq F^+ = 851.22 \]

e. The non-arbitrage higher bound can be calculated as follows:

1. At \( t = 0 \) sell a forward contract. Incur cost \( k_1 = 1 \).

2. At \( t = 0 \) buy 1.003 index. This is why we need to buy 1.003 index. We pay 0.3\% of the index value to the broker. So if we buy one index, this index becomes 1.003 = 0.997 index after the fee. To have one index, we need to have \( \frac{1}{0.997} = \frac{1}{1 - 0.3\%} \approx 1 + 0.3\% = 1.003 \) (remember we need to deliver one index at \( T \) to the buyer in the forward). To verify, if we have \( \frac{1}{0.997} \) index, this will become \( \frac{1}{0.997} (1 - 0.3\%) = 1 \) index after the fee is deducted. Notice we use the Taylor series \( \frac{1}{1 - x} \approx 1 + x + x^2 + \ldots \) for a small \( x \)

3. At \( t = 0 \) borrow 1.003 \( S_0 + k_1 = 1.003(800) + 1 \). Repay this loan with \( (1.003 S_0 + k_1) e^{rT} \) at \( T \)

4. At \( T \) deliver the index to the buyer and receive \( F_{0,T} \). Pay the settlement fee 0.3\% \( S_0 \)

Your initial cost for doing 1 through 4 is zero. Your profit at \( T \) is:
\[ F_{0,T} - (1.003 S_0 + k_1) e^{rT} - 0.3\% S_0 \]

To avoid arbitrage, we need to have
\[ F_{0,T} - (1.003 S_0 + k_1) e^{rT} - 0.3\% S_0 \leq 0 \]
\[ F_{0,T} \leq (1.003 S_0 + k_1) e^{rT} + 0.3\% S_0 \]
\[ = (1.003 \times 800 + 1) e^{(0.055)1} + 0.003 \times 800 = 851.22 \]

The lower bound price can also be calculated as follows:
1. Buy low. At $t = 0$, enter a forward to buy 1.003 index (why 1.003 index will be explained later). Incur transaction cost $k_1 = 1$.

2. Sell high. At $t = 0$, sell $0.997 \times 800 - 1$. This is why we need to short sell 0.997 index. If we short sell one index, the broker charges us 0.3% of the index value and we’ll owe the broker $1 + 0.3\% = 1.003$ index. In order to owe the broker exactly one index, we need to borrow

$$
\frac{1}{1.003} \approx 1 - 0.003 = 0.997 \text{ index from the broker.}
$$

3. Lend $0.997 \times 800 - 1$ and receive $(0.997 \times 800 - 1) e^{rT}$ at $T$.

4. At $T$, pay $1.003 F_{0,T}$ and receive 1.003 index. Pay settlement fee 0.3% (1.003). After the settlement fee, we have $(1.003)(1 - 0.3\%) \approx 1$ index left. We return this index to the broker.

The net initial cash flow after 1 through 4 is zero. Your profit at $T$ is:

$(0.997 \times 800 - 1) e^{rT} - 1.003 F_{0,T}$

To avoid arbitrage,

$(0.997 \times 800 - 1) e^{rT} - 1.003 F_{0,T} \leq 0$

$$
\rightarrow F_{0,T} \geq \frac{(0.997 \times 800 - 1) e^{rT}}{1.003} = \frac{(0.997 \times 800 - 1) e^{(0.05)T}}{1.003} = 834.94
$$

The non-arbitrage bound is:

$\rightarrow 834.94 \leq F_{0,T} \leq 851.22$

Make sure you understand part $e$, which provides a framework for finding the non-arbitrage bound for complex problems. Once you understand this framework, you don’t need to memorize non-arbitrage bound formulas.

**Problem 5.16.**

Not on the syllabus. Ignore.

**Problem 5.17.**

Not on the syllabus. Ignore.

**Problem 5.18.**

Not on the syllabus. Ignore.

**Problem 5.19.**

Not on the syllabus. Ignore.

**Problem 5.20.**

a. $r_{91} = (100 - 93.23) \times \frac{1}{100} \times \frac{1}{4} \times \frac{91}{90} = 1.7113\%$

b. $\$10 (1 + 0.017113) = \$10.17113 \text{ (million)}$
Chapter 8

Swaps

Problem 8.1.

<table>
<thead>
<tr>
<th>time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>annual interest during [0, t]</td>
<td>6%</td>
<td>6.5%</td>
<td></td>
</tr>
<tr>
<td>fixed payment</td>
<td>R</td>
<td>R</td>
<td></td>
</tr>
<tr>
<td>floating payments</td>
<td>22</td>
<td>23</td>
<td></td>
</tr>
</tbody>
</table>

PV fixed payments = PV floating payments

\[
\frac{22}{1.06^1} + \frac{23}{1.065^2} = R \frac{1}{1.06} + R \frac{1}{1.065^2}, R = 22.483
\]

Problem 8.2.

a.

<table>
<thead>
<tr>
<th>time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>annual interest during [0, t]</td>
<td>6%</td>
<td>6.5%</td>
<td>7%</td>
<td></td>
</tr>
<tr>
<td>fixed payment</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td></td>
</tr>
<tr>
<td>floating payments</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td></td>
</tr>
</tbody>
</table>

PV fixed payments = PV floating payments

\[
\frac{20}{1.06} + \frac{21}{1.065^2} + \frac{22}{1.07^3} = R \frac{1}{1.06} + R \frac{1}{1.065^2} + R \frac{1}{1.07^3}, R = 20.952
\]

b. We are now standing at t = 1

\[1\] I recommend that initially you don’t memorize the complex formula \( R = \frac{\sum P(0, t_i) f_0(t_i)}{\sum P(0, t_i)} \).

Draw a cash flow diagram and set up the equation PV fixed payments = PV floating payments.

Once you are familiar with the concept, you can use the memorized formula.
Problem 8.3.

The dealer pays fixed and gets floating. His risk is that oil’s spot price may drop significantly. For example, if the spot price at \( t = 2 \) is $18 per barrel (as opposed to the expected $21 per barrel) and at \( t = 3 \) is $19 per barrel (as opposed to the expected $22 per barrel), the dealer has overpaid the swap. This is because the fixed swap rate \( R = 20.952 \) is calculated under the assumption that the oil price is $21 per barrel at \( t = 2 \) and $22 per barrel at \( t = 3 \).

To hedge his risk, the dealer can enter 3 separate forward contracts, agreeing at \( t = 0 \) to deliver oil to a buyer at $20 per barrel at \( t = 1 \), at $21 per barrel at \( t = 2 \), and at $22 per barrel at \( t = 3 \).

Next, let’s verify that the PV of the dealer’s locked-in net cash flow is zero.

\[
PV(\text{net cash flows}) = \frac{0.952}{1.06} + \frac{-0.048}{1.065} + \frac{-1.048}{1.07} = 0
\]

Problem 8.4.

The fixed payer overpaid 0.952 at \( t = 1 \). The implied interest rate in Year 2 (from \( t = 1 \) to \( t = 2 \)) is \( 1.065^2 - 1 = 0.070024 \). So the overpayment 0.952 at \( t = 1 \) will grow into \( 0.952(1 + 0.070024) = 1.0187 \) at \( t = 2 \). Then at \( t = 2 \), the fixed payer underpays 0.048 and his net overpayment is \( 1.0187 - 0.048 = 0.9707 \). The implied interest rate in Year 3 (from \( t = 2 \) to \( t = 3 \)) is \( 1.065^3 - 1 = 0.080071 \). So the fixed payer’s net overpayment 0.9707 at \( t = 2 \) will grow into \( 0.9707(1 + 0.080071) = 1.0484 \), which exactly offsets his underpayment 1.048 at \( t = 3 \), now the accumulative net payment after the 3rd payment is zero.
Problem 8.5.

5 basis points=5\%\% = 0.5\% = 0.0005

a. immediately after the swap contract is signed the interest rate rises 0.5%

\[
\begin{array}{|c|c|c|c|}
\hline
\text{time } t & 0 & 1 & 2 \\
\hline
\text{original annual interest during } [0, t] & 6\% & 6.5\% & 7\% \\
\hline
\text{updated annual interest during } [0, t] & 6.5\% & 7\% & 7.5\% \\
\hline
\text{fixed payment} & R & R & R \\
\hline
\text{floating payments} & 20 & 21 & 22 \\
\hline
\end{array}
\]

\[
\frac{20}{1.065} + \frac{21}{1.07} + \frac{22}{1.075^2} = \frac{R}{1.065} + \frac{R}{1.07} + \frac{R}{1.075^2}. \quad R = 20.949 < 20.952
\]

The fixed rate is worth 20.949, but the fixed payer pays 20.952. His loss is:

\[
\frac{20}{1.065^2} + \frac{21}{1.07^2} + \frac{22}{1.075^3} - \left( \frac{20}{1.065} + \frac{21}{1.07} + \frac{22}{1.075^2} \right) = 0.51063
\]

b. immediately after the swap contract is signed the interest rate falls 0.5%

\[
\begin{array}{|c|c|c|c|}
\hline
\text{time } t & 0 & 1 & 2 \\
\hline
\text{original annual interest during } [0, t] & 6\% & 6.5\% & 7\% \\
\hline
\text{updated annual interest during } [0, t] & 5.5\% & 6\% & 6.5\% \\
\hline
\text{fixed payment} & R & R & R \\
\hline
\text{floating payments} & 20 & 21 & 22 \\
\hline
\end{array}
\]

\[
\frac{20}{1.055} + \frac{21}{1.06} + \frac{22}{1.065^2} = \frac{R}{1.055} + \frac{R}{1.06} + \frac{R}{1.065^2}. \quad R = 20.955 > 20.952
\]

The fixed rate is worth 20.955, but the fixed payer pays only 20.952. The fixed payer’s gain is:

\[
\frac{20}{1.055^2} + \frac{21}{1.06^2} + \frac{22}{1.065^3} - \left( \frac{20}{1.055} + \frac{21}{1.06} + \frac{22}{1.065^2} \right) = 1.0293
\]
CHAPTER 8. SWAPS

Problem 8.6.

(1) calculate the per-barrel swap price for 4-quarter oil swap

<table>
<thead>
<tr>
<th>time t (quarter)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed payment</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
</tr>
<tr>
<td>floating payments</td>
<td>21</td>
<td>21.1</td>
<td>20.8</td>
<td>20.5</td>
<td></td>
</tr>
<tr>
<td>discounting factor</td>
<td>1.015^{-1}</td>
<td>1.015^{-2}</td>
<td>1.015^{-3}</td>
<td>1.015^{-4}</td>
<td></td>
</tr>
</tbody>
</table>

\[ PV \text{ of floating payments} = PV \text{ of fixed payments} \]
\[ = 21 (1.015^{-1}) + 21.1 (1.015^{-2}) + 20.8 (1.015^{-3}) + 20.5 (1.015^{-4}) \]
\[ = R (1.015^{-1} + 1.015^{-2} + 1.015^{-3} + 1.015^{-4}) \]
\[ R = 20.8533 \]

(2) calculate the per-barrel swap price for 8-quarter oil swap

<table>
<thead>
<tr>
<th>time t (quarter)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed payment</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td></td>
</tr>
<tr>
<td>floating payments</td>
<td>21</td>
<td>21.1</td>
<td>20.8</td>
<td>20.5</td>
<td>20.2</td>
<td>20</td>
<td>19.9</td>
<td>19.8</td>
<td></td>
</tr>
</tbody>
</table>

The discounting factor at \( t \) is 1.015^{-t} (i.e. $1 at \( t \) is worth 1.015^{-t} at \( t = 0 \))
\[ PV \text{ of floating payments} = PV \text{ of fixed payments} \]
\[ = 21 (1.015^{-1}) + 21.1 (1.015^{-2}) + 20.8 (1.015^{-3}) + 20.5 (1.015^{-4}) + 20.2 (1.015^{-5}) + 20 (1.015^{-6}) + 19.9 (1.015^{-7}) + 19.8 (1.015^{-8}) \]
\[ R = 20.4284 \]

(3) calculate the total cost of prepaid 4-quarter and 8-quarter swaps

<table>
<thead>
<tr>
<th>4-quarter swap</th>
<th>8-quarter swap</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 (1.015^{-1}) + 21.1 (1.03^{-1}) + 20.8 (1.045^{-1}) + 20.5 (1.06^{-1}) = 80.41902</td>
<td></td>
</tr>
<tr>
<td>21 (1.015^{-1}) + 21.1 (1.015^{-2}) + 20.8 (1.015^{-3}) + 20.5 (1.015^{-4}) + 20.2 (1.015^{-5}) + 20 (1.015^{-6}) + 19.9 (1.015^{-7}) + 19.8 (1.015^{-8}) = 152.92564</td>
<td></td>
</tr>
</tbody>
</table>

Total cost: 80.41902 + 152.92564 = 233.34462
Problem 8.7.

The calculation is tedious. I'll manually solve the swap rates for the first 4 quarter but give you all the swaps.

<table>
<thead>
<tr>
<th>quarter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>forward</td>
<td>21</td>
<td>21.1</td>
<td>20.8</td>
<td>20.5</td>
<td>20</td>
<td>19.9</td>
<td>19.8</td>
<td></td>
</tr>
<tr>
<td>DiscFactor</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
<tr>
<td>$R$</td>
<td>21</td>
<td>21.05</td>
<td>20.97</td>
<td>20.85</td>
<td>20.73</td>
<td>20.61</td>
<td>20.51</td>
<td>20.43</td>
</tr>
</tbody>
</table>

I'll manually solve for the first 4 swap rates.

<table>
<thead>
<tr>
<th>quarter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>forward price</td>
<td>21</td>
<td>21.1</td>
<td>20.8</td>
<td>20.5</td>
</tr>
<tr>
<td>fixed payment</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
</tr>
<tr>
<td>zero coupon bond price</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
</tr>
</tbody>
</table>

(1) if there's only 1 swap occurring at $t = 1$ (quarter)
PV fixed=PV float
$21 \times 0.9852 = R \times 0.9852$
$R = 21$

(2) if there are two swaps occurring at $t = 1$ and $t = 2$
PV fixed=PV float
$21 \times 0.9852 + 21.1 \times 0.9701 = R \times (0.9852 + 0.9701)$
$R = 21.0496 \approx \frac{21 + 21.1}{2} = 21.05$

(3) if there are 3 swaps occurring at $t = 1, 2, 3$
PV fixed=PV float
$21 \times 0.9852 + 21.1 \times 0.9701 + 20.8 \times 0.9546 = R \times (0.9852 + 0.9701 + 0.9546)$
$R = 20.9667 \approx \frac{21 + 21.1 + 20.8}{3} = 20.9667$

(4) if there are 4 swaps occurring at $t = 1, 2, 3, 4$
PV fixed=PV float
$21 \times 0.9852 + 21.1 \times 0.9701 + 20.8 \times 0.9546 + 20.5 \times 0.9388 = R \times (0.9852 + 0.9701 + 0.9546 + 0.9388)$
$R = 20.853636 \approx \frac{21 + 21.1 + 20.8 + 20.5}{4} = 20.85$

---

2Zero coupon bond price is also the discounting factor.

3If you run out of time in the exam, just take $R$ as the average floating payments. This is often very close to the correct answer.
By the way, please note that in the textbook Table 8.9, the gas swap prices are not in line with the forward price and discounting factors. This is because the swap prices in Table 8.9 are stand-alone prices made up by the author of Derivatives Markets so he can set up problems for you to solve:

<table>
<thead>
<tr>
<th>$t_i$ (quarter)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>2.25</td>
<td>2.4236</td>
<td>2.3503</td>
<td>2.2404</td>
<td>2.2326</td>
<td>2.2753</td>
<td>2.2583</td>
<td>2.2044</td>
</tr>
</tbody>
</table>

To avoid confusion, the author of Derivatives Markets should have used multiple separate tables instead of combining separate tables into one.

Problem 8.8.

<table>
<thead>
<tr>
<th>quarter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>forward price</td>
<td>20.8</td>
<td>20.5</td>
<td>20.2</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>fixed payment</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td>$R$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>zero coupon bond price</td>
<td>0.9546</td>
<td>0.9388</td>
<td>0.9231</td>
<td>0.9075</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If you run out of time, then $R = \frac{20.8 + 20.5 + 20.2 + 20}{4} = 20.375 \approx 20.38$

The precise calculation is:

PV fixed=PV float

$20.8 (0.9546) + 20.5 (0.9388) + 20.2 (0.9231) + 20 (0.9075) = R (0.9546 + 0.9388 + 0.9231 + 0.9075)$

$R = 20.38069 \approx 20.38$
CHAPTER 8. SWAPS

Problem 8.9.

If the problem didn’t give you \( R = 20.43 \), you can quickly estimate it as 
\[
(21 + 21.1 + 20.8 + 20.5 + 20.2 + 20 + 19.9 + 19.8)/8 = 20.4125
\]

Back to the problem. Please note that this problem implicitly assumes that 
the actual interest rates are equal to the expected interest rates implied in the 
zero-coupon bonds. If the actual interest rates turn out to be different than 
the rates implied by the zero-coupon bonds, then you’ll need to know the actual 
interest rates quarter-by-quarter to solve this problem. So for the sake of solving 
this problem, we assume that the interest rates implied by the zero-coupon 
bonds are the actual interest rates.

<table>
<thead>
<tr>
<th>quarter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>fwd price</td>
<td>21</td>
<td>21.1</td>
<td>20.8</td>
<td>20.5</td>
<td>20.2</td>
<td>20</td>
<td>19.9</td>
<td>19.8</td>
</tr>
<tr>
<td>fixed pay</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td></td>
</tr>
<tr>
<td>fixed-fwd</td>
<td>-0.57</td>
<td>-0.67</td>
<td>-0.37</td>
<td>-0.07</td>
<td>0.23</td>
<td>0.43</td>
<td>0.53</td>
<td>0.63</td>
</tr>
<tr>
<td>disct factor</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
</tbody>
</table>

Loan balance at \( t = 0 \) is 0
Loan balance at \( t = 1 \) is -0.57.

The implicit interest rate from \( t = 1 \) to \( t = 2 \) is solved by
\[
\frac{0.9852}{1 + x} = 0.9701. \text{ So } 1 + x = \frac{0.9852}{0.9701}. \text{ The } -0.57 \text{ loan will grow into } -0.57 \times \frac{0.9852}{0.9701} = -0.579 \text{ at } t = 2.
\]
The loan balance at \( t = 2 \) is \(-0.579 - 0.67 = -1.249\).

\(-1.249 \text{ will grow into } -1.249 \times \frac{0.9701}{0.9546} = -1.269 \text{ at } t = 3.\)
The loan balance at \( t = 3 \) is\(-1.269 - 0.37 = -1.639, \text{ which grows into } -1.639 \times \frac{0.9546}{0.9388} = -1.667 \text{ at } t = 4.\)
The loan balance at \( t = 4 \) is \(-1.667 - 0.07 = -1.737\)

\(-1.737 \text{ grows into } -1.737 \times \frac{0.9388}{0.9231} = -1.767 \text{ at } t = 5.\)
So the loan balance at \( t = 5 \) is\(-1.767 + 0.23 = -1.537\)

\(-1.537 \text{ grows into } -1.537 \times \frac{0.9231}{0.9075} = -1.563 \text{ at } t = 6.\)
The loan balance at \( t = 6 \) is \(-1.563 + 0.43 = -1.133\)

\(-1.133 \text{ grows into } -1.133 \times \frac{0.9075}{0.8919} = -1.153 \text{ at } t = 7.\)
The loan balance at \( t = 7 \) is\(-1.153 + 0.53 = -0.623\)

\(-0.623 \text{ grows into } -0.623 \times \frac{0.8919}{0.8763} = -0.634 \text{ at } t = 8.\)
So the loan balance at \( t = 8 \) is \(-0.634 + 0.63 = -0.004 \approx 0\)

<table>
<thead>
<tr>
<th>quarter</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>loan bal</td>
<td>0</td>
<td>-0.57</td>
<td>-1.249</td>
<td>-1.639</td>
<td>-1.737</td>
<td>-1.537</td>
<td>-1.133</td>
<td>-0.623</td>
<td>0</td>
</tr>
</tbody>
</table>
You can also work backward from $t = 8$ to $t = 0$. You know that the loan balance at $t = 8$ is zero; overall the fixed payer and the floating payer each have no gain and no loss if the expected yield curve turns out to the real yield curve.

<table>
<thead>
<tr>
<th>quarter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>fwd price</td>
<td>21</td>
<td>21.1</td>
<td>20.8</td>
<td>20.5</td>
<td>20.2</td>
<td>20</td>
<td>19.9</td>
<td>19.8</td>
</tr>
<tr>
<td>fixed pay</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
<td>20.43</td>
</tr>
<tr>
<td>fixed–fwd</td>
<td>-0.57</td>
<td>-0.67</td>
<td>-0.37</td>
<td>-0.07</td>
<td>0.23</td>
<td>0.43</td>
<td>0.53</td>
<td>0.63</td>
</tr>
<tr>
<td>discf factor</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
</tbody>
</table>

Since the loan balance at $t = 8$ is zero and the fixed payer overpays 0.634\textsuperscript{4} at $t = 8$, the loan balance at $t = 7$ must be $-0.634 \div 0.8919 = -0.623$. Similarly, the loan balance at $t = 7$ must be $(-0.623 - 0.53) \div 0.9075 = -1.133$. And the loan balance at $t = 6$ must be $(-1.133 - 0.43) \div 0.9231 = -1.537$. So on and so forth.

This method is less intuitive. However, if the problem asks you to only find the loan balance at $t = 7$, this backward method is lot faster than the forward method.

\textsuperscript{4}I use 0.634 instead of 0.63 to show you that the backward method produces the same correct answer as the forward method. If you use 0.63, you won’t be able to reproduce the answer calculated by the forward method due to rounding (because 0.63 is rounded from 0.634).
Problem 8.10.

The floating payer delivers 2 barrels at even numbered quarters and 1 barrel at odd quarters.

The cash flow diagram is:

<table>
<thead>
<tr>
<th>quarter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>fwd price</td>
<td>21</td>
<td>21.1 (2)</td>
<td>20.8</td>
<td>20.5 (2)</td>
<td>20.2</td>
<td>20 (2)</td>
<td>19.9</td>
<td>19.8 (2)</td>
</tr>
<tr>
<td>fixed pay</td>
<td>R</td>
<td>2R</td>
<td>R</td>
<td>2R</td>
<td>R</td>
<td>2R</td>
<td>R</td>
<td>2R</td>
</tr>
<tr>
<td>disc factor</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
</tbody>
</table>

\[
\text{PV float} = \text{PV fixed} \\
21 \times 0.9852 + 21.1 \times 2 \times 0.9701 + 20.8 \times 0.9546 + 20.5 \times 2 \times 0.9388 \\
+ 20.2 \times 0.9231 + 20 \times 2 \times 0.9075 + 19.9 \times 0.8919 + 19.8 \times 2 \times 0.8763 \\
= R \times (0.9852 + 2 \times 0.9701 + 0.9546 + 2 \times 0.9388) \\
+ R \times (0.9231 + 2 \times 0.9075 + 0.8919 + 2 \times 0.8763) \\
R = 20.40994
\]

Please note that the cash flow diagram is not:

<table>
<thead>
<tr>
<th>quarter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>fwd price</td>
<td>21</td>
<td>21.1 (2)</td>
<td>20.8</td>
<td>20.5 (2)</td>
<td>20.2</td>
<td>20 (2)</td>
<td>19.9</td>
<td>19.8 (2)</td>
</tr>
<tr>
<td>fixed pay</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>disc factor</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
</tbody>
</table>
Problem 8.11.

The key formula is the textbook equation 8.13:

\[ R = \frac{\sum_{i=1}^{n} P(0, t_i) f_0(t_i)}{\sum_{i=1}^{n} P(0, t_i)} \]

From Table 8.9, we get:

<table>
<thead>
<tr>
<th>( t_i ) (quarter)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>2.25</td>
<td>2.4236</td>
<td>2.3503</td>
<td>2.2404</td>
<td>2.2326</td>
<td>2.2753</td>
<td>2.2583</td>
<td>2.2044</td>
</tr>
<tr>
<td>( P(0, t_i) )</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
</tbody>
</table>

Notation:

- \( P(0, t_i) \) is the present value at \( t = 0 \) of $1 at the \( t_i \).
- \( R \) is the swap rate. For example, for a 4-quarter swap, the swap rate is 2.2404
- \( f_0(t_i) \) is the price of the forward contract signed at \( t_{i-1} \) and expiring at \( t_i \)

The 1-quarter swap rate is

\[ R(1) = \frac{P(0, t_1) f_0(t_1)}{P(0, t_1)} \quad \rightarrow \quad f_0(t_1) = R(1) = 2.2500 \]

The 2-quarter swap rate is:

\[ R(2) = \frac{P(0, t_1) f_0(t_1) + P(0, t_2) f_0(t_2)}{P(0, t_1) + P(0, t_2)} \]

\[ \rightarrow 2.4236 = \frac{0.9852(2.25) + 0.9701 f_0(t_2)}{0.9852 + 0.9701} \quad f_0(t_2) = 2.5999 \]

The 3-quarter swap rate is:

\[ R(3) = \frac{P(0, t_1) f_0(t_1) + P(0, t_2) f_0(t_2) + P(0, t_3) f_0(t_3)}{P(0, t_1) + P(0, t_2) + P(0, t_3)} \]

\[ \rightarrow 0.9546 = \frac{0.9852(2.25) + 0.9701(2.60) + 0.9546 f_0(t_3)}{0.9852 + 0.9701 + 0.9546} \]

\[ \rightarrow f_0(t_3) = 2.2002 \]

So on and so forth. The result is:

<table>
<thead>
<tr>
<th>( t_i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>2.25</td>
<td>2.4236</td>
<td>2.3503</td>
<td>2.2404</td>
<td>2.2326</td>
<td>2.2753</td>
<td>2.2583</td>
<td>2.2044</td>
</tr>
<tr>
<td>( P(0, t_i) )</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
<tr>
<td>( f_0(t_i) )</td>
<td>2.2500</td>
<td>2.5999</td>
<td>2.2002</td>
<td>1.8998</td>
<td>2.2001</td>
<td>2.4998</td>
<td>2.1501</td>
<td>1.8002</td>
</tr>
</tbody>
</table>
Problem 8.12.

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(8)$</td>
<td>2.2044</td>
<td>2.2044</td>
<td>2.2044</td>
<td>2.2044</td>
</tr>
<tr>
<td>$f_0(t_i)$</td>
<td>2.2500</td>
<td>2.5999</td>
<td>0.9546</td>
<td>0.9388</td>
</tr>
<tr>
<td>$P(0, t_i)$</td>
<td>0.9852</td>
<td>0.9701</td>
<td>0.9546</td>
<td>0.9388</td>
</tr>
<tr>
<td>$i(t_{i-1}, t_i)$</td>
<td>1.5022%</td>
<td>1.5565%</td>
<td>1.6237%</td>
<td>1.6830%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(8)$</td>
<td>2.2044</td>
<td>2.2044</td>
<td>2.2044</td>
<td>2.2044</td>
</tr>
<tr>
<td>$f_0(t_i)$</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
<tr>
<td>$P(0, t_i)$</td>
<td>0.9231</td>
<td>0.9075</td>
<td>0.8919</td>
<td>0.8763</td>
</tr>
<tr>
<td>$i(t_{i-1}, t_i)$</td>
<td>1.7008%</td>
<td>1.7190%</td>
<td>1.7491%</td>
<td>1.7802%</td>
</tr>
</tbody>
</table>

First, let's calculate the quarterly forward interest rate $i(t_{i-1}, t_i)$, which is the interest during $[t_{i-1}, t_i]$. 

By the way, please note the difference between $f_0(t_i)$ and $i(t_{i-1}, t_i)$. $f_0(t_i)$ is the price of a forward contract and $i(t_{i-1}, t_i)$ is the forward interest rate.

The interest rate during the first quarter is $i(t_0, t_1)$. This is the effective interest per quarter from $t = 0$ to $t = 0.25$ (year). Because $P(0, t_1)$ represents the present value of $1$ at $t = 0.25$

$$P(0, t_1) = \frac{1}{1 + i(t_0, t_1)}$$

$$0.9852 = \frac{1}{1 + i(t_0, t_1)}$$

$$i(t_0, t_1) = \frac{1}{0.9852} - 1 = 1.5022\%$$

The 2nd quarter interest rate $i(t_1, t_2)$ satisfies the following equation:

$$P(0, t_2) = \frac{1}{1 + i(t_0, t_1)} \times \frac{1}{1 + i(t_1, t_2)} = P(0, t_1)$$

$$0.9701 = \frac{1}{1 + i(t_1, t_2)}$$

$$i(t_1, t_2) = 1.5566\%$$

Similarly,

$$P(0, t_3) = \frac{1}{1 + i(t_0, t_1)} \times \frac{1}{1 + i(t_1, t_2)} \times \frac{1}{1 + i(t_2, t_3)} = P(0, t_2)$$

$$0.9546 = \frac{1}{1 + i(t_2, t_3)}$$

$$i(t_2, t_3) = 1.6237\%$$

Keep doing this, you should be able to calculate all the forward interest rates.
Next, let’s calculate the loan balance.

\[
\begin{array}{|c|c|c|c|c|}
\hline
t_i & 1 & 2 & 3 & 4 \\
\hline
R(8) & 2.2044 & 2.2044 & 2.2044 & 2.2044 \\
f_0(t_i) & 2.2500 & 2.5999 & 0.9546 & 0.9388 \\
f_0(t_i) - R(8) & 0.0456 & 0.3955 & -0.0042 & -0.3046 \\
\text{Loan balance} & 0.0456 & 0.4418 & 0.4444 & 0.1470 \\
\iota(t_{i-1}, t_i) & 1.5022\% & 1.5565\% & 1.6237\% & 1.6830\% \\
\hline
\end{array}
\]

Next, let’s calculate the loan balance.

\[
\begin{array}{|c|c|c|c|c|}
\hline
t_i & 5 & 6 & 7 & 8 \\
\hline
R(8) & 2.2044 & 2.2044 & 2.2044 & 2.2044 \\
f_0(t_i) & 0.9231 & 0.9075 & 0.8919 & 0.8763 \\
f_0(t_i) - R(8) & -0.0043 & 0.2954 & -0.0543 & -0.4042 \\
\text{Loan balance} & 0.1451 & 0.4430 & 0.3963 & -0.0009 \\
\iota(t_{i-1}, t_i) & 1.7008\% & 1.7190\% & 1.7491\% & 1.7802\% \\
\hline
\end{array}
\]

At \( t = 0 \), the loan balance is zero. A swap is a fair deal and no money changes hands.

At \( t = 1 \) (i.e. the end of the first quarter), the floating payer lends \( 2.25 - 2.2044 = 0.0456 \) to the fixed payer. Had the floating payer signed a forward contract at \( t = 0 \) agreeing to deliver the oil at \( t = 1 \), he would have received \( 2.25 \) at \( t = 1 \). However, by entering into an 8-quarter swap, the floating payer receives only \( 2.2044 \) at the \( t = 1 \). So the floating payer lends \( 2.25 - 2.2044 = 0.0456 \) to the fixed payer.

The \( 0.0456 \) at \( t = 1 \) grows to \( 0.0456 \times (1 + 0.015022) = 0.0463 \)

The total loan balance at \( t = 2 \) is \( 0.0463 + 0.3955 = 0.4418 \)

The loan balance \( 0.4418 \) at \( t = 2 \) grows into \( 0.4418 \times (1 + 0.015565) = 0.4487 \)
at \( t = 3 \)

The total loan balance at \( t = 3 \) is \( 0.4487 - 0.0042 = 0.4445 \)

I used Excel to do the calculation. Due to rounding, I got \( 0.4487 - 0.0042 = 0.4444 \) instead of \( 0.4445 \).

So on and so forth. The final loan balance at \( t = 8 \) is:

\( 0.3963 \times (1 + 0.017491) - 0.4042 = -0.0009 \approx 0 \)

The loan balance at the end of the swap \( t = 8 \) should be zero. We didn’t get zero due to rounding.

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Guo FM, fall 2009
CHAPTER 8. SWAPS

Problem 8.13.

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>$P(0, t_i)$</th>
<th>$i(t_{i-1}, t_i)$</th>
<th>$P(0, t_i) i(t_{i-1}, t_i)$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9701</td>
<td>1.5565%</td>
<td>0.0151</td>
<td>4.6941</td>
</tr>
<tr>
<td>1</td>
<td>0.9546</td>
<td>1.6237%</td>
<td>0.0155</td>
<td>4.6941</td>
</tr>
<tr>
<td>2</td>
<td>0.9388</td>
<td>1.6830%</td>
<td>0.0158</td>
<td>4.6941</td>
</tr>
<tr>
<td>3</td>
<td>0.9231</td>
<td>1.7008%</td>
<td>0.0157</td>
<td>4.6941</td>
</tr>
<tr>
<td>4</td>
<td>0.9075</td>
<td>1.7190%</td>
<td>0.0156</td>
<td>4.6941</td>
</tr>
<tr>
<td>5</td>
<td>0.8919</td>
<td>1.7491%</td>
<td>0.0156</td>
<td>4.6941</td>
</tr>
<tr>
<td>6</td>
<td>0.8763</td>
<td>1.7802%</td>
<td>0.0156</td>
<td>4.6941</td>
</tr>
<tr>
<td>Total</td>
<td>3.8487</td>
<td></td>
<td>0.0612</td>
<td></td>
</tr>
</tbody>
</table>

$R = \frac{\sum_{i=2}^{6} P(0, t_i) r(t_{i-1}, t_i)}{\sum_{i=2}^{6} P(0, t_i)} = \frac{0.0777}{4.6941} = 1.6553\%$


<table>
<thead>
<tr>
<th>$t_i$</th>
<th>$P(0, t_i)$</th>
<th>$i(t_{i-1}, t_i)$</th>
<th>$P(0, t_i) i(t_{i-1}, t_i)$</th>
<th>$P(0, t_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9852</td>
<td>1.5022%</td>
<td>0.0148</td>
<td>1.5022%</td>
</tr>
<tr>
<td>2</td>
<td>0.9701</td>
<td>1.5565%</td>
<td>0.0151</td>
<td>1.5565%</td>
</tr>
<tr>
<td>3</td>
<td>0.9546</td>
<td>1.6237%</td>
<td>0.0155</td>
<td>1.6237%</td>
</tr>
<tr>
<td>4</td>
<td>0.9388</td>
<td>1.6830%</td>
<td>0.0158</td>
<td>1.6830%</td>
</tr>
<tr>
<td>Total</td>
<td>3.8487</td>
<td></td>
<td>0.0612</td>
<td></td>
</tr>
</tbody>
</table>

Calculate the 4-quarter swap rate.

$R = \frac{\sum_{i=1}^{4} P(0, t_i) r(t_{i-1}, t_i)}{\sum_{i=1}^{4} P(0, t_i)} = \frac{0.0612}{3.8487} = 1.5901\%$

Calculate the 8-quarter swap rate.

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>$P(0, t_i)$</th>
<th>$i(t_{i-1}, t_i)$</th>
<th>$P(0, t_i) i(t_{i-1}, t_i)$</th>
<th>$P(0, t_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9852</td>
<td>1.5022%</td>
<td>0.0148</td>
<td>1.5022%</td>
</tr>
<tr>
<td>2</td>
<td>0.9701</td>
<td>1.5565%</td>
<td>0.0151</td>
<td>1.5565%</td>
</tr>
<tr>
<td>3</td>
<td>0.9546</td>
<td>1.6237%</td>
<td>0.0155</td>
<td>1.6237%</td>
</tr>
<tr>
<td>4</td>
<td>0.9388</td>
<td>1.6830%</td>
<td>0.0158</td>
<td>1.6830%</td>
</tr>
<tr>
<td>5</td>
<td>0.9231</td>
<td>1.7008%</td>
<td>0.0157</td>
<td>1.7008%</td>
</tr>
<tr>
<td>6</td>
<td>0.9075</td>
<td>1.7190%</td>
<td>0.0156</td>
<td>1.7190%</td>
</tr>
<tr>
<td>7</td>
<td>0.8919</td>
<td>1.7491%</td>
<td>0.0156</td>
<td>1.7491%</td>
</tr>
<tr>
<td>8</td>
<td>0.8763</td>
<td>1.7802%</td>
<td>0.0156</td>
<td>1.7802%</td>
</tr>
<tr>
<td>Total</td>
<td>7.4475</td>
<td></td>
<td>0.1237</td>
<td></td>
</tr>
</tbody>
</table>

$R = \frac{\sum_{i=1}^{8} P(0, t_i) r(t_{i-1}, t_i)}{\sum_{i=1}^{8} P(0, t_i)} = \frac{0.1237}{7.4475} = 1.6610\%$

Problem 8.15.

Not on the FM syllabus. Skip.

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CHAPTER 8. SWAPS

Problem 8.16.
Not on the FM syllabus. Skip.

Problem 8.17.
Not on the FM syllabus. Skip.

Problem 8.18.
Not on the FM syllabus. Skip.
Problem 1
You are given two funds:

- Fund $M$: $100 invested at $t = 0$ with the annual effective interest rate 6%
- Fund $N$: $100 invested at $t = 2$ (i.e. end of Year 2) with an annual simple interest $y$, where $y$ is a constant.
- Fund $M$ and $N$ have the same force of interest at the end of Year 10.

Calculate the value of Fund $N$ at the end of Year 10.

A 185  B 195  C 205  D 215  E 225

Problem 2
You are given:

- $i$ is the annual effective interest rate
- in $a$ years $1$ will grow into $10$
- in $b$ years $2$ will grow into $25$
- in $c$ years $3$ will grow into $60$

Calculate $(1 + i)^{a+2b-3c}$

A 0.1  B 0.2  C 0.3  D 0.4  E 0.5

Problem 3
- $7$ payments are made semiannually for the first year, the first payment due 6 months from today
- The payments increase by $4$ each year, starting from Year 2 and lasting forever
- The annual effective interest rate is 9%

Calculate the present value of this annuity.

A 1170  B 1190  C 1210  D 1230  E 1250
Problem 4

- A loan of 15,000 is made at time zero
- The annual effective interest rate is 10%
- 20 annual repayments are made. Each repayment is 1,400

Calculate the loan balance immediately after the 20-th annual repayment is made.

A 20,727    B 20,777    C 20,827    D 20,877    E 20,927
Problem 5

<table>
<thead>
<tr>
<th>Date</th>
<th>fund balance before deposit and withdrawal</th>
<th>deposit</th>
<th>withdrawal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2000</td>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>7/1/2000</td>
<td>110</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>9/1/2000</td>
<td>80</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>1/1/2001</td>
<td>X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The time weighted return during the year is 0.15 more than the dollar weighted return during the year.

Calculate X

A 12   B 17   C 22   D 27   E 32
Problem 6
Joe buys an S&P 500 index futures contract for 1500. The initial margins is 10%. The maintenance margin is 80%. The contract is settled weekly. The continuously compounded interest rate is 6% per year. If the price of the index drops to 1400 next week, calculate the amount that Joe has to pay.

A 17,260  B 17,360  C 17,460  D 17,560  E 17,660
Problem 7

- Liabilities: $100 per year for five years, the first payment due one year from today
- Asset #1: $100 per year at the of Year 3, 4, and 5 respectively. The asset price is $100.
- Asset #2: $50 at the end of Year 1 and 2 respectively. The asset price is $80.
- Asset #3: $50 per year for five years, the first cash flow occurring one year from today. The asset price is $150.
- Asset #4: $100 per year for five years, the first cash flow occurring one year from today. The asset price is $250.

Find the cheapest method to exactly match the liabilities.

A Buy some units of Asset #1 and some units Asset #2
B Buy some units of Asset #1
C Buy some units of Asset #3
D Buy some units of Asset #4
E Buy some units of Asset #2 and some units Asset #3
Problem 8
A bank offers 5% annual effective interest rate to 6-year CD deposit. In addition, it offers, at the end of Year 6, a 2% bonus interest payment of the initial deposit.
Calculate the equivalence annual effective yield of this CD.

A 4.86%  B 5.06%  C 5.26%  D 5.46%  E 5.66%

Problem 9
• Fund A accumulates $100 at a simple rate of discount 6% per year for 5 years
• Fund B accumulates $100 at a constant force of interest of 4% per year for 5 years

Calculate the sum of the fund value A and B at the end of Year 5.
A 213  B 226  C 239  D 252  E 265

Problem 10
• $100,000 loan is made
• Monthly interest-only repayments are made for 10 years, the first repayment occurring one month from today
• Level annual repayments of $9,456 are made for as long as needed to pay off the remaining balance of the loan, where the first annual repayment is made at the end of Year 11.
• The annual effective interest rate is 9.2%

Calculate the total number of repayments necessary to pay off the loan.
A 133  B 140  C 147  D 154  E 161
Problem 11
A loan amount $X$ is repaid by $N$ level annual repayments, the first repayment due in one year. You are given:

- The interest portion of the 5-th repayment is 10251.9
- The principal paid in the $(N - 8)$th repayment is 7361.83
- The interest paid in the $(N - 16)$th repayment is 7469.15

Calculate $X$
A 184,500  B 184,750  C 185,000  D 185,250  E 185,500

Problem 12
The Macaulay duration of a 2-year annuity due of $50 per year is 0.4884. Calculate the Macaulay duration of a 3-year annuity due of $100 per year.
A 0.72  B 0.97  C 1.22  D 1.47  E 1.72

Problem 13
- The current stock price is $110
- The annualized continuously compounded risk-free interest rate is 8.7%
- You buy a one-year to expiration call on the stock with strike price $120
- You simultaneously sell a one-year to expiration put on the same stock with the same strike price

Calculate the net premium you have to pay.
A $-2$  B 0  C 2  D 4  E 6

Problem 14
- The current stock price is 100
- The annual continuously compounded risk free interest rate is 6%
- The cost of carry is 2%

Calculate the annual continuously compounded dividend yield of the stock
A 2.0%  B 2.5%  C 3.0%  D 3.5%  E 4%
Problem 15

A company has liabilities of 500 in t = 1, 2, and 3 years. It have the option of purchasing either 1 year or 3 year zero coupon bonds, with prices of 760 and 650, respectively. Assume a flat interest rate of 10%. What mix of 1 year and 3 year zero coupon bonds must it purchase now to exactly match the duration of assets to duration of liabilities?

A $540 of the 1-year bond and $460 of the 3-year bond
B $570 of the 1-year bond and $490 of the 3-year bond
C $600 of the 1-year bond and $520 of the 3-year bond
D $630 of the 1-year bond and $550 of the 3-year bond
E $660 of the 1-year bond and $580 of the 3-year bond
Problem 16
Stock A and B both pay perpetual annual dividends, the first dividend due one year from today.

- The dividends of Stock A increase by 50g%
- The dividend of Stock B increase by −50g%
- The first dividend of stock A is half of the first dividend of Stock B
- The price of Stock A is twice the price of Stock B
- The annual effective interest rate is 10%

Calculate g.
A 0.04   B 0.06   C 0.08   D 0.10   E 0.12

Problem 17
You are given the following data about a special annuity:

- Quarterly payments of 10, 20, 30, and 40 are made per year for 20 years, the 1st payment due one quarter from today
- The interest rate is 8% nominal per year compounded quarterly

Calculate the PV of this special annuity.

A \(10 (Ia)_{\frac{3}{2}} \left( \frac{\dd}{20|8} \right)\)
B \(10 (Ia)_{\frac{3}{2}} \left( \frac{\dd}{20|8.24} \right)\)
C \(10 (I\dd)_{\frac{3}{2}} \left( \frac{\dd}{20|8} \right)\)
D \(10 (Ia)_{\frac{3}{2}} \left( \frac{\dd}{20|8} \right)\)
E \(10 (I\dd)_{\frac{3}{2}} \left( \frac{\dd}{20|8.24} \right)\)
**Problem 18**
A loan is repaid through level monthly payments for 10 years, the first payment due one month from the loan date.

The annual effective interest rate is 8%. The principal for the 6th payment is 60.

Calculate the principal for the 24th payment.

\[ A \ 49 \quad B \ 55 \quad C \ 61 \quad D \ 67 \quad E \ 73 \]

**Problem 19**
For the first two years the annual effective interest rate is 6%. Then for \( t > 2 \) the force of interest is \( \frac{1}{1 + t} \), where \( t \) is the number of years from today.

Calculate the effective annual discount rate over the 5 years.

\[ A \ 0.12 \quad B \ 0.15 \quad C \ 0.18 \quad D \ 0.21 \quad E \ 0.24 \]
Problem 20

A company wants to buy a mixture of bonds that will satisfy its liability and cost the least.

Company’s Liability

<table>
<thead>
<tr>
<th>time t (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>liability cash flow</td>
<td>100,000</td>
<td>200,000</td>
<td></td>
</tr>
</tbody>
</table>

Assets:

- Bond #1: 1-year bond with 7.5% annual coupon
- Bond #2: 2-year bond with 5.5% annual coupon
- Bond #3: 2-year zero coupon bond at 6%

The company also has the option of buying a 7% one year zero coupon bond starting at time 1. In order to obtain this bond, however, the company needs to pay 2% of the purchase price at time 0. Calculate the total cost of the mixture of the bonds that will satisfy its liability and cost the least.

A 269,980    B 270,080    C 270,180    D 270,280    E 270,380
Problem 21
Information about the asset and liability of a retirement fund:

<table>
<thead>
<tr>
<th></th>
<th>Asset</th>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>amount</td>
<td>250,000</td>
<td>250,000</td>
</tr>
<tr>
<td>duration</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>convexity</td>
<td>360</td>
<td>360</td>
</tr>
</tbody>
</table>

Assume a parallel shift of the interest rate. Which of the following statements is correct?

A The retirement fund loses no matter the interest goes up slightly or down slightly
B The retirement fund neither gains nor loses no matter the interest goes up slightly or down slightly
C The retirement fund gains if the interest goes up slightly but loses if the interest goes down slightly
D The retirement fund gains if the interest goes down slightly but loses if the interest goes up slightly
E The retirement fund gains no matter the interest goes up slightly or down slightly
Problem 22
John wants a 5000 loan immediately. He can get the loan by one of the two options:

- Option 1. A 5-year loan with interest and principal paid at the end of Year 5
- Option 2. A 3-year loan with interest and principal paid by a 2-year loan issued at the end of Year 3.

Calculate the effective interest rate of the 2-year loan such that John is indifferent with either option.

You are also given the following structure of spot interest rates:

<table>
<thead>
<tr>
<th>Term (Year)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<tr>
<td>spot rate</td>
<td>3.5%</td>
<td>4.5%</td>
<td>5.5%</td>
<td>6.5%</td>
<td>7.5%</td>
</tr>
</tbody>
</table>

A 9.37% B 9.77% C 10.17% D 10.57% E 10.97%

Problem 23
Which one of these positions will NOT benefit from an increase of the stock price?
A. long forward
B. short stock
C. short put
D. long call

Problem 24
Which of the followings is a correct statement about the difference between a futures contract and a forward contract
A. Futures contracts are settled at expiration but forward contracts are settled daily
B. Futures contracts are traded over the counter but forward contracts are exchange-traded
C. Futures contracts are less liquid than forward contracts
D. Futures contracts are more rigid in terms and conditions, but forward contracts are more flexible

Problem 25
Which of the followings is correct about Redington immunization?
A. The asset cash flows should match the liability cash flows.
B. It immunizes against all the interest rate changes
C. The duration of the asset should be greater than the duration of the liability
D. The convexity of the liability should be greater than the convexity of the asset
E. None of the above
Problem 26
You are given two equivalent options of buying a car:

- Option 1. Buy the car outright with cash for 30,000
- Option 2. Lease the car for three years with down payment 1000 and monthly payments of X for 3 years. At the end of the Year 3, the car can be returned for 2,000.

The annual effective interest rate is 8%. Calculate X.

A 805  B 830  C 855  D 880  E 905

Problem 27
You borrow a 1000 loan at an annual effective rate 10%. You accumulate a sinking fund at the end of Year 10. The value of the sinking fund is 600. The 8-th payment is 140. How much of the 8th payment goes to interest?

A 100  B 110  C 120  D 130  E 140

Problem 28
A loan is paid through annual repayments at the end of the year for 30 years. The annual payments are $5 for each of the first 10 years, $4 for each of the next 10 years, and $3 for each of the last 10 years. The interest portion of the 11st payment is one and half the interest portion of the 21-th payment. Calculate the interest portion of the 21-th payment.

A 0.76  B 0.82  C 0.88  D 0.94  E 1

Problem 29
Company FastGrow Incorporated can have 500M profit (M=million) or 350M loss with equal probability. If FastGrow uses hedging, its profit is 100M for sure after hedging.

FastGrow has tax rate 40%. However, it pays no tax and receives no tax credit if it incurs losses.

Calculate the difference (in millions) between FastGrow’s hedged after-tax profit and unhedged after-tax profit.

A 75  B 80  C 85  D 90  E 95

Problem 30

- The nominal interest rate is 6% per year
- The inflation rate is 2% per year

Calculate the real interest rate per year.

A 4%  B 6%  C 8%  D 10%  E 12%
Guo FM Mock Exam 1

Allotted time: 3 hours

<table>
<thead>
<tr>
<th>Problem</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
</tr>
<tr>
<td>4</td>
<td>A</td>
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<td>5</td>
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<td>6</td>
<td>C</td>
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<td>7</td>
<td>D</td>
</tr>
<tr>
<td>8</td>
<td>C</td>
</tr>
<tr>
<td>9</td>
<td>E</td>
</tr>
<tr>
<td>10</td>
<td>E</td>
</tr>
<tr>
<td>11</td>
<td>C</td>
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<td>29</td>
<td>C</td>
</tr>
<tr>
<td>30</td>
<td>A</td>
</tr>
</tbody>
</table>

If you correctly answered 22+ problems, you most likely passed this exam.
If you correctly answered 17-21 problems, you are on the borderline.
If you correctly answered fewer than 17 problems, you most likely failed this exam.
Problem 1
You are given two funds:

- Fund $M$: $100 invested at $t = 0$ with the annual effective interest rate 6%.
- Fund $N$: $100 invested at $t = 2$ (i.e. end of Year 2) with an annual simple interest $y$, where $y$ is a constant.
- Fund $M$ and $N$ have the same force of interest at the end of Year 10.

Calculate the value of Fund $N$ at the end of Year 10.

A 185  B 195  C 205  D 215  E 225

Solution
Let $A^M(t)$ and $A^N(t)$ represent the value of Fund $M$ and $N$ at time $t$ respectively.

$A^M(t) = 100 \times 1.06^t$

$A^N(t) = 0$ if $t < 2$ and $A^N(t) = 100 (1 + y(t - 2))$ if $t \geq 2$

Force of interest at time $t$ for Fund $M$:

$\delta^M(t) = \frac{d \ln A^M(t)}{dt} = \frac{d \ln (100 \times 1.06^t)}{dt} = \frac{d \ln 100}{dt} + \frac{d \ln 1.06^t}{dt} = \frac{1}{1.06^t} \times$

$\frac{d 1.06^t}{dt} = \frac{1}{1.06^t} \times 1.06^t \ln 1.06 = \ln 1.06$

Notice that $\ln 100$ is a constant. Hence $\frac{d \ln 100}{dt} = 0$.

Force of interest at time $t$ for Fund $N$ where $t > 2$

$\delta^N(t) = \frac{d \ln A^N(t)}{dt} = \frac{d \ln [100 (1 + y(t - 2))]}{dt} = \frac{d \ln (1 + y(t - 2))}{dt} = \frac{y}{1 + y(t - 2)}$

$\delta^M(10) = \delta^N(10)$

$\implies \ln 1.06 = \frac{y}{1 + 8y} \quad y = 0.10915$

$A^N(10) = 100 (1 + 0.10915 (10 - 2)) = 187.32$

Problem 2
You are given:

- $i$ is the annual effective interest rate
- in $a$ years $1$ will grow into $10$
- in $b$ years $2$ will grow into $25$
- in $c$ years $3$ will grow into $60$
Calculate \((1 + i)^{a+2b-3c}\)

\[
\begin{array}{c}
A 0.1 \\
B 0.2 \\
C 0.3 \\
D 0.4 \\
E 0.5 \\
\end{array}
\]

**Solution**

1. \( (1 + i)^a = 10 \)
2. \( (1 + i)^b = 20 \)
3. \( (1 + i)^c = 30 \)

\[
(1 + i)^{a+2b-3c} = (1 + i)^a (1 + i)^b (1 + i)^{-3c} = (1 + i)^a \left[ (1 + i)^b \right]^2 [(1 + i)^c]^{-3} = \\
10 \left( \frac{25}{2} \right)^2 \left( \frac{60}{3} \right)^{-3} = 0.195
\]

**Problem 3**

- \$7 payments are made semiannually for the first year, the first payment due 6 months from today
- The payments increase by \$4 each year, starting from Year 2 and lasting forever
- The annual effective interest rate is 9%

Calculate the present value of this annuity.

\[
\begin{array}{c}
A 1170 \\
B 1190 \\
C 1210 \\
D 1230 \\
E 1250 \\
\end{array}
\]

**Solution**

The cash flows can be broken down as 3 streams: \(A\), \(B\), and \(C\).

<table>
<thead>
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<th>(t) (year)</th>
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<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
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<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
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<th>(\infty)</th>
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<tbody>
<tr>
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<td>7</td>
<td>7</td>
<td>7</td>
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<td>7</td>
</tr>
<tr>
<td>(B)</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
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<td>...</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(C)</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>16</td>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(A + B + C)</td>
<td>7</td>
<td>7</td>
<td>11</td>
<td>11</td>
<td>15</td>
<td>15</td>
<td>19</td>
<td>19</td>
<td>23</td>
<td>23</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

\[ PV \text{ of } A \text{ at } t = 0: \quad 7\alpha_{0.5|j} = \frac{7}{j} \]

The 6-month interest rate is \( j = \sqrt{1 + i} - 1 = \sqrt{1.09} - 1 \)

\[ PV \text{ of } B \text{ at } t = 0.5 \text{ is } 4 (Ia)_{0.5|i} = 4 \left( \frac{\hat{d}_{0.5|i} - nv^n}{i} \right)_{n\to\infty} \text{ where } i = 9\% \]

as \( n \to \infty \), \( nv^n \to 0 \) because \( v^n \) approaches zero much faster than \( n \) approaches infinity

(Alternatively, you can use l’Hospital’s rule to prove that \( nv^n \to 0 \)).
\[\bar{a}_{ni} = \frac{1 - e^{-nt}}{d} \rightarrow \frac{1}{d} \]

Hence \((Ia)_{\bar{a}_{ni}} = \frac{1}{i} = \frac{1}{d} \left(\frac{1}{1 - \frac{1}{1+i}}\right) = \frac{1}{i} + \frac{1}{i^2}\]

PV of B at \(t = 0\) is \(\frac{4(Ia)_{\bar{a}_{ni}}}{1+j}\)

Similarly, PV of C at \(t = 1\) is \(4(Ia)_{\bar{a}_{ni}}\) and at \(t = 0\) is \(\frac{4(Ia)_{\bar{a}_{ni}}}{1+i}\)

Hence the PV of the total annuity at \(t = 0\) is:

\[
7a_{\bar{a}_{ni}} + \frac{4(Ia)_{\bar{a}_{ni}}}{1+j} + \frac{4(Ia)_{\bar{a}_{ni}}}{1+i} = \frac{7}{\sqrt{1 + 0.09} - 1} + 4 \times \frac{1}{\sqrt{1 + 0.09} - 1} + 4 \times \frac{1}{0.09 + 0.09^2} = 1168.38
\]

**Problem 4**

- A loan of 15,000 is made at time zero
- The annual effective interest rate is 10%
- 20 annual repayments are made. Each repayment is 1,400

Calculate the loan balance immediately after the 20-th annual repayment is made.

A 20,727  B 20,777  C 20,827  D 20,877  E 20,927

**Solution**

First, let’s calculate the number of repayments \(N\) needed to pay off the loan. Using BA II Plus, we enter:

\[PV = 15000 \quad 1/Y = 10 \quad PMT = -1400 \quad FV = 0\]

If we try to calculate \(N\), we’ll get an error message. What’s going on?

It turned out that the annual repayment is less than the interest accrued during the year. The annual interest accrued on the loan is \(15000 \times 0.1 = 1500\); the annual repayment is only 1400. If the annual repayment is less than the interest accrued during the year, then the loan balance will go up instead of going down; the future value of the loan is negative (i.e. the borrower needs to a final lump sum repayment). In other words, if you have \(PV = 15000 \quad 1/Y = 10 \quad PMT = -1400\), then \(FV < 0\); there’s no solution for \(PV = 15000 \quad 1/Y = 10 \quad PMT = -1400 \quad FV = 0\).

Here is how to solve the problem conceptually:

The loan balance immediately after the 20th repayment is
Here is how to solve the problem using BA II Plus:

\[
P V = 15000 \quad 1/Y = 10 \quad PMT = -1400 \quad N = 20
\]

You should get: \( FV = -20727 \)

**Problem 5**

<table>
<thead>
<tr>
<th>Date</th>
<th>fund balance before deposit and withdrawal</th>
<th>deposit</th>
<th>withdrawal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2000</td>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>7/1/2000</td>
<td>110</td>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>9/1/2000</td>
<td>80</td>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>1/1/2001</td>
<td>X</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The time weighted return during the year is 0.15 more than the dollar weighted return during the year.

Calculate \( X \)

\[
A \quad B \quad C \quad D \quad E
\]

**Solution**

Time weighted return:

\[
\frac{110 \times 80}{110 + 15} \times \frac{X}{80 - 60} = 1 + r_{\text{time}}
\]

Dollar weighted return:

\[
100 \left(1 + r_{\text{dollar}}\right) + 15 \left(1 + 0.5r_{\text{dollar}}\right) - 60 \left(1 + \frac{r_{\text{dollar}}}{3}\right) = X
\]

\[
r_{\text{dollar}} = \frac{X - 100 - 15 + 60}{100 + 15 \times 0.5 - 60 \times \frac{1}{3}} = \frac{X - 55}{87.5}
\]

\[
\frac{110 \times 80}{110 + 15} \times \frac{X}{80 - 60} - 1 = \frac{X - 55}{87.5} + 0.15
\]

**Problem 6**

Joe buys an S&P 500 index futures contract for 1500. The initial margins is 10\%. The maintenance margin is 80\%. The contract is settled weekly. The continuously compounded interest rate is 6\% per year. If the price of the index drops to 1400 next week, calculate the amount that Joe has to pay.

\[
A \quad B \quad C \quad D \quad E
\]

**Solution**

At time zero, Joe deposits 0.1 \times 1500 \times 250 = 37500 into his margin account. Please note there’s a scaling factor 250 (see Derivatives Markets Section 5.4).

At the end of Week 1, Joe’s loss 250 \times (1500 - 1400) = 25000 is marked to market. Now Joe’s margin account is 37500e^{0.06/52} - 25000 = 12543, which is
less than the maintenance margin requirement $37500 \times 0.8 = 30000$. Hence Joe needs to deposit $30000 - 12543 = 17457$ into the margin account.

**Problem 7**

- Liabilities: $100$ per year for five years, the first payment due one year from today
- Asset #1: $100$ per year at the end of Year 3, 4, and 5 respectively. The asset price is $100$.
- Asset #2: $50$ at the end of Year 1 and 2 respectively. The asset price is $80$.
- Asset #3: $50$ per year for five years, the first cash flow occurring one year from today. The asset price is $150$.
- Asset #4: $100$ per year for five years, the first cash flow occurring one year from today. The asset price is $250$.

Find the cheapest method to exactly match the liabilities.

- A Buy some units of Asset #1 and some units Asset #2
- B Buy some units of Asset #1
- C Buy some units of Asset #3
- D Buy some units of Asset #4
- E Buy some units of Asset #2 and some units Asset #3

**Solution**

<table>
<thead>
<tr>
<th>Cost at $t = 0$</th>
<th>time $t$ (Year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</tbody>
</table>

Liability can be matched by

- Asset #1 + 2×Asset #2. Cost: $100 + 2 \times 80 = 260$
- 2×Asset #3. Cost: $2 \times 150 = 300$
The least expensive way to exactly match the liability cash flow is to buy Asset #4.

**Problem 8**
A bank offers 5% annual effective interest rate to 6-year CD deposit. In addition, it offers, at the end of Year 6, a 2% bonus interest payment of the initial deposit.
Calculate the equivalence annual effective yield of this CD.

A 4.86%  B 5.06%  C 5.26%  D 5.46%  E 5.66%

**Solution**
\[
(1 + i)^6 = (1 + 0.05)^6 + 0.02 = 1.3601
\]
\[
i = 5.26\%
\]

**Problem 9**
- Fund A accumulates $100 at a simple rate of discount 6% per year for 5 years
- Fund B accumulates $100 at a constant force of interest of 4% per year for 5 years

Calculate the sum of the fund value A and B at the end of Year 5.

A 213  B 226  C 239  D 252  E 265

**Solution**
Value of Fund A at \(t = 5\):
\[
100 \left(1 - 0.06 \times 5\right)^{-1} = 142.86
\]
Value of Fund B at \(t = 5\):
\[
100e^{0.04 \times 5} = 122.14
\]
The total fund value at \(t = 5\):
\[
142.86 + 122.14 = 265.0
\]

**Problem 10**
- $100,000 loan is made
- Monthly interest-only repayments are made for 10 years, the first repayment occurring one month from today
- Level annual repayments of $9,456 are made for as long as needed to pay off the remaining balance of the loan, where the first annual repayment is made at the end of Year 11.
- The annual effective interest rate is 9.2%

Calculate the total number of repayments necessary to pay off the loan.

A 133  B 140  C 147  D 154  E 161
100,000 = 9456e^{r \cdot 9.2\%} = 9456 \times \frac{1 - 1.092^{-n}}{0.092} \quad n = 41

So the total number of repayments is 12 \times 10 + 41 = 161

**Problem 11**

A loan amount $X$ is repaid by $N$ level annual repayments, the first repayment due in one year. You are given:

- The interest portion of the 5-th repayment is 10251.9
- The principal paid in the $(N - 8)$th repayment is 7361.83
- The interest paid in the $(N - 16)$th repayment is 7469.15

Calculate $X$

$A$ 184,500 $B$ 184,750 $C$ 185,000 $D$ 185,250 $E$ 185,500

**Solution**

$Y$ is the level annual repayment.

\[ Yv^{N+1-(N-8)} = 7361.83 \quad Yv^9 = 7361.83 \]
\[ Y(1 - v^{N+1-(N-16)}) = 7469.15 \quad Y(1 - v^{17}) = 7469.15 \]

\[ \frac{v^9}{1 - v^{17}} = \frac{7361.83}{7469.15} = 0.98563 \]
\[ v^9 = 0.98563 - 0.98563v^{17} \quad v^9 + 0.98563v^{17} - 0.98563 = 0 \]

The above equation corresponds to the following diagram:

<table>
<thead>
<tr>
<th>time t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>...</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flow</td>
<td>$-0.98563$</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0.98563</td>
</tr>
</tbody>
</table>

(If you set the PV of the above cash flow to zero, you’ll get $v^9 + 0.98563v^{17} - 0.98563 = 0$)

Use BA II Plus CF worksheet to solve for the interest rate. Enter

$CF0 = -0.98563$

$C01 = 0 \quad F01 = 8$ (to account for zero cash flow at $t = 1, 2, ..., 8$)

$C02 = 1 \quad F02 = 1$ (to account for $1$ at $t = 9$)

$C03 = 0 \quad F03 = 7$ (to account for zero cash flow at $t = 10, 11, ..., 16$)

$C04 = 1 \quad F04 = 1$ (to account for 0.98563 at $t = 17$)

You should get: IRR=5.75 (so the interest rate is 5.75%)

\[ Y \times 1.0575^{-9} = 7361.83 \quad \rightarrow Y = 12176 \]

The interest portion of the 5-th repayment is:
Problem 12

The Macaulay duration of a 2-year annuity due of $50 per year is 0.4884. Calculate the Macaulay duration of a 3-year annuity due of $100 per year.

\[ D_{MAC} = \sum \frac{tCF(t)v^t}{\sum CF(t)v^t} \]

2-year annuity due of $50 per year

<table>
<thead>
<tr>
<th>Time t (year)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flow</td>
<td>$50</td>
<td>$50</td>
</tr>
</tbody>
</table>

\[
\frac{50}{50(1+v)} = 0.4884 \quad v = 0.95465
\]

3-year annuity due of $100 per year

<table>
<thead>
<tr>
<th>Time t (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flow</td>
<td>$100</td>
<td>$100</td>
<td>$100</td>
</tr>
</tbody>
</table>

\[
D_{MAC} = \frac{100(v + 2v^2)}{100(1 + v + v^2)} = \frac{0.95465 + 2(0.95465^2)}{1 + 0.95465 + 0.95465^2} = 0.96907
\]

Problem 13

- The current stock price is $110
- The annualized continuously compounded risk-free interest rate is 8.7%
- You buy a one-year to expiration call on the stock with strike price $120
- You simultaneously sell a one-year to expiration put on the same stock with the same strike price

Calculate the net premium you have to pay.

\[ A \quad B \quad C \quad D \quad E \]

Solution
\[ C + PV(K) = P + S_0 \]
The net premium is \( C - P = S_0 - PV(K) = 110 - 120e^{-0.087} = 0 \)

**Problem 14**

- The current stock price is 100
- The annual continuously compounded risk free interest rate is 6%
- The cost of carry is 2%

Calculate the annual continuously compounded dividend yield of the stock

\[ A \ 2.0\% \quad B \ 2.5\% \quad C \ 3.0\% \quad D \ 3.5\% \quad E \ 4\% \]

**Solution**

The cost of carry is the difference between the risk-free rate and the dividend yield (refer to Derivatives Market Chapter 5)

\[ 6\% - x = 2\% \quad x = 4\% \]

The annual continuously compounded dividend yield of the stock is 4%

**Problem 15**

A company has liabilities of 500 in \( t = 1, 2, \) and 3 years. It have the option of purchasing either 1 year or 3 year zero coupon bonds, with prices of 760 and 650, respectively. Assume a flat interest rate of 10%. What mix of 1 year and 3 year zero coupon bonds must it purchase now to exactly match the duration of assets to duration of liabilities?

\[ A \ 540 \text{ of the 1-year bond and } 460 \text{ of the 3-year bond} \]
\[ B \ 570 \text{ of the 1-year bond and } 490 \text{ of the 3-year bond} \]
\[ C \ 600 \text{ of the 1-year bond and } 520 \text{ of the 3-year bond} \]
\[ D \ 630 \text{ of the 1-year bond and } 550 \text{ of the 3-year bond} \]
\[ E \ 660 \text{ of the 1-year bond and } 580 \text{ of the 3-year bond} \]

**Solution**

Duration of liability:

\[ \frac{500 \left( v + 2v^2 + 3v^3 \right)}{500 \left( v + v^2 + v^3 \right)} = \frac{1.1^{-1} + 2 \times 1.1^{-2} + 3 \times 1.1^{-3}}{1.1^{-1} + 1.1^{-2} + 1.1^{-3}} = 1.9366 \]

The company invests \( x \) portion in the 1-year bond and \( (1 - x) \) portion in the 3-year bond. We set the PV of the mixture equal to the PV of the liability; we set the duration of the mixture equal to the duration of the liability.

Since the 1-year bond and the 3-year bond have duration of 1 and 3 respectively, the duration of the mixture is \( x + (1 - x) 3 \).

\[ x + (1 - x) 3 = 1.9366 \quad x = 0.532 \]

The PV of the mixture is equal to PV of the liabilities:

\[ 500 \left( 1.1^{-1} + 1.1^{-2} + 1.1^{-3} \right) = 1243.4 \]
Hence we spend $1243.4 \times 0.532 = 661.49$ on the 1-year bond (i.e. we buy $661.49$ worth of the 1-year bond, which represents \( \frac{661.49}{760} = 0.87038 \) fraction of the 1-year bond).

We spend $1243.4 \times (1 - 0.532) = 581.91$ on the 3-year bond (i.e. we buy $581.91$ worth of the 3-year bond, which represents \( \frac{581.91}{650} = 0.89525 \) fraction of the 3-year bond).

**Problem 16**

Stock A and B both pay perpetual annual dividends, the first dividend due one year from today.

- The dividends of Stock A increase by $50 \%$
- The dividend of Stock B increase by $-50 \%$
- The first dividend of stock A is half of the first dividend of Stock B
- The price of Stock A is twice the price of Stock B
- The annual effective interest rate is $10 \%$

Calculate $g$.

\[
A \quad 0.04 \quad B \quad 0.06 \quad C \quad 0.08 \quad D \quad 0.10 \quad E \quad 0.12
\]

**Solution**

<table>
<thead>
<tr>
<th>$t$ (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A’s dividend</td>
<td>$a$</td>
<td>$a(1 + 0.5g)$</td>
<td>$a(1 + 0.5g)^2$</td>
<td>$a(1 + 0.5g)^3$</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>B’s dividend</td>
<td>$2a$</td>
<td>$2a(1 - 0.5g)$</td>
<td>$2a(1 - 0.5g)^2$</td>
<td>$2a(1 - 0.5g)^3$</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

PV of A:

\[
\frac{a}{i - 0.5g} + a(1 + 0.5g) \frac{v^2}{1 - (1 + 0.5g)} + a(1 + 0.5g)^2 \frac{v^3}{1 - (1 + 0.5g)} + \ldots = \frac{a}{(1 + i) - (1 + 0.5g)} = \frac{a}{i - 0.5g}
\]

PV of B:

\[
\frac{2a}{i + 0.5g} + 2a(1 - g) \frac{v^2}{1 - (1 - 0.5g)} + 2a(1 - g)^2 \frac{v^3}{1 - (1 - 0.5g)} + \ldots = \frac{2a}{(1 + i) - (1 - 0.5g)} = \frac{2a}{i + 0.5g}
\]

\[
\frac{a}{i - 0.5g} = \frac{4a}{i + 0.5g} \quad \frac{i + 0.5g}{i - 0.5g} = 4 \quad g = \frac{6}{5} \quad i = \frac{6}{5} \times 0.1 = 0.12
\]

**Problem 17**

You are given the following data about a special annuity:
Quarterly payments of 10, 20, 30, and 40 are made per year for 20 years, the 1st payment due one quarter from today

The interest rate is 8% nominal per year compounded quarterly

Calculate the PV of this special annuity.

A 10 \( (Ia)_{1|2\%} \left( \hat{a}_{20|8\%} \right) \)
B 10 \( (Ia)_{1|2\%} \left( \hat{a}_{20|8.24\%} \right) \)
C 10 \( (Ia)_{1|2\%} \left( \hat{a}_{20|8\%} \right) \)
D 10 \( (Ia)_{1|2\%} \left( \hat{a}_{20|8\%} \right) \)
E 10 \( (Ia)_{1|2\%} \left( \hat{a}_{20|8.24\%} \right) \)

Solution
The quarterly effective interest rate is \( \frac{8\%}{4} = 2\% \)
The annual effective interest rate is \( 1.02^4 - 1 = 8.24\% \)

<table>
<thead>
<tr>
<th>t (quarter)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>...</th>
<th>77</th>
<th>78</th>
<th>79</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flow</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>...</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>

The above cash flow diagram can be simplified as:

<table>
<thead>
<tr>
<th>t (quarter)</th>
<th>0</th>
<th>4</th>
<th>...</th>
<th>8</th>
<th>76</th>
</tr>
</thead>
<tbody>
<tr>
<td>t (Yr)</td>
<td>0</td>
<td>1</td>
<td>...</td>
<td>2</td>
<td>19</td>
</tr>
<tr>
<td>cash flow</td>
<td>10 ( (Ia)_{1</td>
<td>2%} )</td>
<td>10 ( (Ia)_{1</td>
<td>2%} )</td>
<td>10 ( (Ia)_{1</td>
</tr>
</tbody>
</table>

The PV of this cash flow is 10 \( (Ia)_{1|2\%} \left( \hat{a}_{20|8.24\%} \right) \)

**Problem 18**
A loan is repaid through level monthly payments for 10 years, the first payment due one month from the loan date.

The annual effective interest rate is 8%. The principal for the 6th payment is 60.

Calculate the principal for the 24th payment.

A 49  B 55  C 61  D 67  E 73

Solution
The monthly discounting factor is \( v = 1.08^{-1/12} \)

Let \( X \) represent the level monthly payment.

principal for the 6th payment is:

\[ X v^{12(6-1)} = 60 \]

principal for the 24th payment:
\[ Xv^{121-24} = Xv^{121-6}v^{-18} = 60 \times (1.08^{-1/12})^{-18} = 67.34 \]

**Problem 19**
For the first two years the annual effective interest rate is 6%. For \( t > 2 \) force of interest is \( \frac{1}{1+t} \), where \( t \) is the number of years from today.

Calculate the effective annual discount rate over the 5 years.

\[ A \: \: 0.12 \quad B \: \: 0.15 \quad C \: \: 0.18 \quad D \: \: 0.21 \quad E \: \: 0.24 \]

**Solution**

\[
(1 - d)^{-5} = 1.06^2 e^{\int_2^5 \frac{1}{1+t} \, dt} = \int_2^5 \frac{1}{1+t} \, dt = [\ln (1+t)]_2^5 = \ln 6 - \ln 3 = \ln 2 \\
(1 - d)^{-5} = 1.06^2 e^{\ln 2} = 2 \times 1.06^2 \\
d = 1 - (2 \times 1.06^2)^{-1/5} = 0.14951
\]

**Problem 20**

A company wants to buy a mixture of bonds that will satisfy its liability and cost the least.

Company’s Liability

<table>
<thead>
<tr>
<th>time t (year)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>liability</td>
<td>100,000</td>
<td>200,000</td>
<td></td>
</tr>
<tr>
<td>cash flow</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Assets:

- Bond #1: 1-year bond with 7.5% annual coupon
- Bond #2: 2-year bond with 5.5% annual coupon
- Bond #3: 2-year zero coupon bond at 6%

The company also has the option of buying a 7% one year zero coupon bond starting at time 1. In order to obtain this bond, however, the company needs to pay 2% of the purchase price at time 0. Calculate the total cost of the mixture of the bonds that will satisfy its liability and cost the least.

\[ A \: \: 269,980 \quad B \: \: 270,080 \quad C \: \: 270,180 \quad D \: \: 270,280 \quad E \: \: 270,380 \]

**Solution**

1-year bond with 7.5% annual coupon means that if you buy $100 worth of the bond at \( t = 0 \), the bond will pay you 107.5 at \( t = 1 \).

2-year bond with 5.5% annual coupon means that if you buy $100 worth of the bond at \( t = 0 \), the bond will pay you 5.5 at \( t = 1 \) and 105.5 at \( t = 2 \).
2-year zero coupon bond at 6% means that the bond doesn’t pay any coupon and that it is discounted at 6%. If you want the bond to pay you $100 at t=2, you need to buy $100 \times 1.06^{-2} = 89$ worth of the bond at $t = 0$.

To ensure that it has 100,000 at $t = 1$, the company can buy $100000 \times 1.075^{-1} = 93023$ worth of Bond #1 at $t = 0$.

To ensure that it has 200,000 at $t = 2$, the company can do one the following two things:

1. buy $200000 \times 1.06^{-2} = 177,999$ worth of Bond #3.
2. buy the special bond so it will pay 200000 at $t = 2$. The cost at $t=1$ is $200000 \times 1.07^{-1}$.

The cost at $t=0$ is $200000 \times 1.07^{-1} \times 1.075^{-1} \times 1.02 = 177,353$

Option 2 is cheaper. So the least expensive way to match the liability cash flow is to buy a mixture of Bond #1 and the special bond. The total cost is $93023 + 177353 = 270,376$

Please note that Bond #2 is not needed to solve the problem; Bond #2 can be replicated with some mixture of Bond #1 and Bond #3.

Problem 21
Information about the asset and liability of a retirement fund:

<table>
<thead>
<tr>
<th></th>
<th>Asset</th>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>amount</td>
<td>250,000</td>
<td>250,000</td>
</tr>
<tr>
<td>duration</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>convexity</td>
<td>360</td>
<td>360</td>
</tr>
</tbody>
</table>

Assume a parallel shift of the interest rate. Which of the following statements is correct?

A. The retirement fund loses no matter the interest goes up slightly or down slightly
B. The retirement fund neither gains nor loses no matter the interest goes up slightly or down slightly
C. The retirement fund gains if the interest goes up slightly but loses if the interest goes down slightly
D. The retirement fund gains if the interest goes down slightly but loses if the interest goes up slightly
E. The retirement fund gains no matter the interest goes up slightly or down slightly

Solution

$P'(r)$ – the PV of an asset (or liability), which is a function of the interest rate $r$

Let
• $D_{MAC}$ represent the Macaulay duration
• $C$ represent the convexity
• $A$ represent the asset
• $L$ represent the liability
• $S$ represent the surplus. $S = A - L$

If $S$ increases when $r$ changes, the retirement fund benefits from the change in the interest rate $r$.
If $S$ decreases when $r$ changes, the retirement fund loses from the change in the interest rate $r$.
If $S$ is the same when $r$ changes, the retirement fund neither gains nor loses from the change in the interest rate $r$.

\[
\frac{\Delta P}{P} \approx \frac{1}{P} \frac{dP}{dr} \Delta r + \frac{1}{2} \frac{d^2P}{dr^2} (\Delta r)^2 = -\frac{1}{1+r} \times D_{MAC} \times \Delta r + \frac{1}{2} \times C \times (\Delta r)^2
\]
\[
\Delta P \approx -\frac{P}{1+r} \times D_{MAC} \times \Delta r + \frac{P}{2} \times C \times (\Delta r)^2
\]
\[
\Delta P^A \approx -\frac{P^A}{1+r} \times D^A_{MAC} \times \Delta r + \frac{P^A}{2} \times C^A \times (\Delta r)^2
\]
\[
\Delta P^L \approx -\frac{P^L}{1+r} \times D^L_{MAC} \times \Delta r + \frac{P^L}{2} \times C^L \times (\Delta r)^2
\]
Since $P^A = P^L$ and $C^A = C^L$
\[
\Delta S = \Delta P^A - \Delta P^L \approx -\frac{P^A}{1+r} \times \Delta r \times (D^A_{MAC} - D^L_{MAC}) = -\frac{P^A}{1+r} \times \Delta r \times (16 - 12)
\]
(16 - 12) = $\frac{P^A}{1+r} \times \Delta r \times 4$

If $\Delta r > 0$, then $\Delta S < 0$.
If $\Delta r < 0$, then $\Delta S > 0$.
The retirement fund gains if the interest goes down slightly but loses if the interest goes up slightly.

**Problem 22**

John wants a 5000 loan immediately. He can get the loan by one of the two options:

• Option 1. A 5-year loan with interest and principal paid at the end of Year 5
• Option 2. A 3-year loan with interest and principal paid by a 2-year loan issued at the end of Year 3.
Calculate the effective interest rate of the 2-year loan such that John is indifferent with either option.
You are also given the following structure of spot interest rates:

<table>
<thead>
<tr>
<th>Term (Year)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>spot rate</td>
<td>3.5%</td>
<td>4.5%</td>
<td>5.5%</td>
<td>6.5%</td>
<td>7.5%</td>
</tr>
</tbody>
</table>

A 9.37%   B 9.77%   C 10.17%   D 10.57%   E 10.97%

**Solution**
John’s repayment at \( t = 5 \) under Option 1:
\[ 5000 \times 1.075^5 \]

John’s repayment at \( t = 5 \) under Option 2:
\[ 5000 \times 1.055^3 \times (1 + x)^2 \]
\[ 5000 \times 1.075^5 = 5000 \times 1.055^3 \times (1 + x)^2 \quad x = 10.57\% \]

Alternative solution:
To avoid arbitrage, the effective interest rate of the 2-year loan must be the forward rate from \( t = 3 \) to \( t = 5 \) implied by the term structure of the spot rates.
\[ 1.055^3 \times (1 + x)^2 = 1.075^5 \quad x = 10.57\% \]

**Problem 23**
Which one of these positions will NOT benefit from an increase of the stock price?
A long forward
B short stock
C short put
D long call

**Solution**
If you sell a stock short and the stock price goes up, you have to buy it back later at a higher price. So you won’t benefit from the increase of the stock price.

**Problem 24**
Which of the followings is a correct statement about the difference between a futures contract and a forward contract
A. Futures contracts are settled at expiration but forward contracts are settled daily
B. Futures contracts are traded over the counter but forward contracts are exchange-traded
C. Futures contracts are less liquid than forward contracts
D. Futures contracts are more rigid in terms and conditions, but forward contracts are more flexible
Solution

D. Forward contracts are private agreements between two parties and can have flexible terms and conditions to meet the two parties’ needs.

Problem 25
Which of the followings is correct about Redington immunization?
A. The asset cash flows should match the liability cash flows.
B. It immunizes against all the interest rate changes
C. The duration of the asset should be greater than the duration of the liability
D. The convexity of the liability should be greater than the convexity of the asset
E. None of the above

Solution

E

Problem 26

You are given two equivalent options of buying a car:

• Option 1. Buy the car outright with cash for 30,000

• Option 2. Lease the car for three years with down payment 1000 and monthly payments of $X$ for 3 years. At the end of the Year 3, the car can be returned for 2,000.

The annual effective interest rate is 8%. Calculate $X$.

A 805  B 830  C 855  D 880  E 905

Solution

The PV of option 1 is 30000.
The PV of option 2 is $1000 + x a_{36}^{10} + 2000 \times 1.08^{-3}$. $j = 1.08^{1/12} - 1 = 0.6434$

$30000 = 1000 + x a_{36}^{10} + 2000 \times 1.08^{-3}$

$1000 + x \times \frac{1 - 1.08^{-3}}{0.006434} + 2000 \times 1.08^{-3} = 30000$

$x = 855.47$

Problem 27

You borrow a 1000 loan at an annual effective rate 10%. You accumulate a sinking fund at the end of Year 10. The value of the sinking fund is 600. The 8-th payment is 140. How much of the 8-th payment goes to interest?
Solution
The information is less than perfect, but you need to make most of it.

In a typical sinking fund, the borrower pays the interest year by year. He then separately sets up a sinking fund to repay the principal.

The 8-th payment is greater than the interest accrued: $1000 \times 0.1 = 100$. So we assume that the 100 goes toward the interest and the remaining 40 is used to reduce the principal. Please note that the sinking fund balance 600 is less than the principal 1000. This confirms that the annual sinking fund payment is greater than the accrued interest; the excess of the annual sinking fund payment over the accrued interest is used to pay down the principal.

Problem 28
A loan is paid through annual repayments at the end of the year for 30 years. The annual payments are $5 for each of the first 10 years, $4 for each of the next 10 years, and $3 for each of the last 10 years. The interest portion of the 11st payment is one and half the interest portion of the 21-th payment. Calculate the interest portion of the 21-th payment.

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The interest portion of the 11th payment is:

\[ I_{11} = i \times OB_{10} = i \left( v^{10} a_{10|} + v^{20} a_{10|} \right) \]

\( OB_{10} \) is the outstanding balance immediately after the 10th payment is made.

The interest portion of the 21-th payment

\[ I_{21} = i \times OB_{20} = i \times 3a_{10|} \]

\( OB_{20} \) is the outstanding balance immediately after the 20-th payment is made.

\[ i \left( v^{10} a_{10|} + v^{20} a_{10|} \right) = 1.5 \times i \times 3a_{10|} \quad \quad 4v^{10} + 3v^{20} = 4.5 \]

Solving this quadratic equation, we get

\[ x = 0.7278 \]

\[ I_{21} = i \times 3a_{10|} = i \times 3 \times \frac{1 - v^{10}}{i} = 3 \left( 1 - v^{10} \right) = 3 \left( 1 - 0.7278 \right) = 0.8166 \]

Problem 29
Company FastGrow Incorporated can have 500M profit (M=million) or 350M loss with equal probability. If FastGrow uses hedging, its profit is 100M for sure after hedging.
FastGrow has tax rate 40%. However, it pays no tax and receives no tax credit if it incurs losses.

Calculate the difference (in millions) between FastGrow’s hedged after-tax profit and unhedged after-tax profit.

A 75    B 80    C 85    D 90    E 95

**Solution**

The tax rate is 40% if positive profit and 0% if zero or negative profit

After tax profit if hedged (100% probability of getting 75 before tax):

\[100 \times (1 - 0.4) = 60\]

After tax profit if unhedged:

\[0.5 (500 \times (1 - 0.4) + (-350) (1 - 0)) = -25\]

The difference between hedged profit and unhedged profit is: \(60 - (-25) = 85\)

(Please refer to Derivatives Markets Section 4.3 for more examples of how to calculate the after tax profit.)

**Problem 30**

- The nominal interest rate is 6% per year
- The inflation rate is 2% per year

Calculate the real interest rate per year.

A 4%    B 6%    C 8%    D 10%    E 12%

**Solution**

\[1 + r_{\text{Real}} = \frac{1 + r_{\text{Nominal}}}{1 + r_{\text{Inflation}}} = \frac{1.06}{1.02} = 1.0392\]

\[r_{\text{Real}} = 3.92\%\]